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Regeneration estimation in partially stable two class retrial queue*

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Abstract. We consider two-class retrial queueing system with constant retrial rate fed by Poisson input and apply regenerative confidence estimation for mean number of customers in the stable orbit, while the other orbit intimately grows. The simulation results illustrate that partially stable case providing accurate confidence estimation, even the stability conditions, related for the whole system, are violated.

Keywords: retrial queue, constant retrial rate, stability, partial stability, regeneration estimation

AMS Subject Classification: 60, 62

1. Introduction

The paper deals with a single server retrial rate queuing system under constant retrial rate policy. The model admits two classes of customers, arrivals join the system according to Poisson input. The service times are independent and identically distributed among the corresponding class. If the server is busy at arrival instant, the new customer joins the orbit associated with its class and then try to occupy the server after class-dependent exponentially distributed retrial time according to FIFO discipline.

Retrial systems have a huge sphere of modern applications. For instance, such models successfully describe various call centers [16, 19] or the work of computer networks and internet protocols [7, 8]. The applications of retrial models to wireless

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technologies are presented in [9, 15]. Retrial queuing systems are widely studied in literature, it is worth mentioning the basic books and surveys [1, 2, 10, 18].

We consider two class retrial system in partially stable mode: the first class orbit is stochastically bounded and the second class orbit infinitely grows in probability. To define partially stability conditions we rely on the preliminary results obtained in [6] and developed in [5]. The basic goal of the paper is to construct confidence interval for mean number of customers in the first orbit in case of partially stable mode. We apply regenerative method of confidence estimation. Generally, the regenerative method is applicable if the system under consideration is stable. The novelty of the present research is the following: we use regenerative approach to obtain confidence interval in case the only orbit is stochastically bounded while stability conditions for the whole system are violated.

The paper is organized as follow. Section 2 contains the detailed description of the system under consideration. Section 3 presents the concept of partial stability and known conditions for the partially stable mode. Next in Section 4 we briefly discuss the regenerative method of confidence estimation. Section 5 containes simulation results for partially stable model. We compare obtained confidence intervals with the results for corresponding single orbit retrial system in a stable mode. Section 6 concludes the paper.

2. Description of the model

We consider a single-server bufferless retrial system under *constant retrial rate* policy denoted by system Σ. The model admits two classes of customers. Namely arrivals form the superposition of two Poisson inputs with corresponding rates λ_i, where $i = 1, 2$ defines the class number. Thus the total input rate is the following: $\lambda = \lambda_1 + \lambda_2$. We define the sequence of arrival instants by $\{t_n, n \geq 1\}$. Note that interarrival times $\tau_n = t_{n+1} - t_n$ are independent and exponentially distributed with a rate λ. Let τ define the generic interarrival time, thus $\mathsf{E}\tau = 1/\lambda$.

Next we assume that class-i service times are independent, generally distributed and stochastically equivalent to $S^{(i)}$ with corresponding mean $1/\mu_i$. Define the marginal load coefficient by

$$\rho_i = \lambda_i/\mu_i.$$

Thus the total load coefficient is obtained as

$$\rho = \rho_1 + \rho_2.$$

If the class-i arrival, that meets the server busy, joins the corresponding infinite-capacity virtual *orbit* and then tries to occupy the server after an exponential time with a rate α_i. We define the auxiliary load coefficient associated with class-i orbit customers by

$$\hat{\rho}_i = \alpha_i/\mu_i.$$

The total orbits load coefficient is the following

$$\hat{\rho} = \hat{\rho}_1 + \hat{\rho}_2.$$

Consider $N^{(i)}(t)$ – the number of customers at orbit i at time instant t. The total number of customers in the system Σ is defined by the following process

$$X(t) = \nu(t) + N^{(1)}(t) + N^{(2)}(t), \quad t \geq 0, \tag{2.1}$$

where $\nu(t) \in \{0, 1\}$ represents the number of customers on service. Thus the only reason for unstable behavior of the system is the infinite growth of orbits size. (Note that the term "size" actually means the number of customers on the orbit, while the configuration of the system admits the infinite number of waiting places for orbit calls).

Constant retrial rate policy implies that the orbit rates α_i are fixed and do not depend on the processes $N^{(i)}(t)$. Unlike the classical retrial models, where the intensity of orbit customers increases proportionally to its number. Thus in classical multi-orbit case, the behavior of one (at instance, class-i_0) orbit affects to other orbit(s). Namely when the load of class-i_0 customers increases, the corresponding orbit size grows, and the server attack in more intensive. This implies more load to the other orbits and the growth of their sizes. Thus in classical retrial models instability of one orbit leads to the instability of other orbits. Such a property does not hold for constant retrial rate model, considered in present paper: the orbit size does not affect the intensity of orbit customers, and one orbit can infinitely grow, while the other is stable. In such a case the phenomenon of *partial stability* arises.

3. Partial stability: preliminary results

In this section we refer to the known results related to the conditions of partially stable regime in two-class retrial model with constant retrial rate. First we briefly discuss the stability concept. Note that all considered continuous-time processes are assumed to be defined at instant t^-. Each instant t_n when the new arrival joins into totally empty system ($X(t_n) = 0$) the model starts over in stochastic sense or *regenerates*. From this point of view the process $X(t)$ is called a *regenerative process*. The regenerative process is called *positive recurrent* if regeneration period has finite mean. In zero-delayed case positive recurrence implies that the system possesses have *stationary regime* [3]. Actually the positive recurrence of the process X means that starting from the arbitrary instant t the system becomes empty in a finite time. In this case we define that the system is *stable*. From this point of view the stability is equivalent to the positive recurrence. Detailed description of the regeneration approach to the stability analysis could be found in [12–14, 17].

By *partial stability* (of class-1 orbit) we define the case when class-1 orbit size process stays tight while class-2 orbit increases unlimited in probability. Note the process $N^{(1)}$ is tight [17] if for any $\delta > 0$ exists a finite constant $C \geq 0$ such that

$$\inf_t \mathsf{P}\big(\mathsf{N}^{(1)}(\mathsf{t}) \leq \mathsf{C}\big) \geq 1 - \delta. \tag{3.1}$$

Namely we obtain that only the first orbit is stochastically bounded. (Obviously that the symmetric case for partial stability of class-2 orbit is defined in analogical terms.)

Consider the absolutely continuous distribution function F with density f, defined for all x such that $1 - F(x) > 0$. Next define the failure rate by $r(x) := f(x)/\big(1 - F(x)\big)$. We say that the distribution F belongs to a special sub-class $\mathcal{D}$ if $\inf_{x\geq 0} r(x) > 0$.

The conditions of partially stable regime were firstly formulated in [6] for the multi-class retrial model, where service time distributions belongs The partial stability conditions for the model Σ considered in present paper (when class-1 orbit is tight) was obtained in [5] via load coefficients as follows:

$$\hat{\rho}_1 > \rho_1(\rho + \hat{\rho}), \tag{3.2}$$

$$\rho > \hat{\rho}_2/(\rho_2 + \hat{\rho}_2). \tag{3.3}$$

Note that to obtained the conditions (3.2), (3.3) the authors in [5] had analyzed two-dimensional Markov Chain

$$\mathbf{Y} = \big\{Y_k^{(1)}, Y_k^{(2)}\big\}, \quad k \geq 1,$$

associated with corresponding numbers of customers in the first and in the second orbit just after the departure instants (k defines the actual number of departures from the system after its service completion). The Markov property holds for the random sequences $\{Y_k^{(i)}, k \geq 1\}$, $i = 1, 2$ because input stream is assumed to be Poisson.

Relying on the technique presented in [11], it is possible to show that under conditions (3.2), (3.3) the Markov Chain $\mathbf{Y}$ is *transient*. Such a transient case is illustrated by the stability of the first orbit dynamics and the infinite growth of the second one, see [5] for details. Moreover under assumption that service time distributions belong to the sub-class $\mathcal{D}$ the conditions (3.2), (3.3) coincide with the partial stability conditions from [6]. Note that positive recurrence of $\mathbf{Y}$ implies the stability for the model Σ and corresponds to the positive recurrence of the basic process X.

Next our goal is to explore the behavior of the model under consideration when (3.2), (3.3) hold true. In this case we can expect that after some finite instant the second orbit is not empty and the total load to the server is equivalent to the load in the single-orbit retrial system, where class-2 customers arrive with a rate $\lambda_2 + \alpha_2$ and are lost in case the server is busy at arrival instants. Then we construct the auxiliary process in original two-orbit system Σ as follows:

$$X^{(1)}(t) = \nu(t) + N^{(1)}(t), \quad t \geq 0$$

and its discrete analogue $X_n^{(1)} = X^{(1)}(t_n^-)$, $n \geq 1$. Next consider the sequence

$$\beta_k = \min_n\{n > \beta_{k-1} : X_n^{(1)} = 0\}, \quad k \geq 1,\ \beta_0 = 0,$$

which defines the numbers of arrivals to the system when the server is idle and the first orbit is empty. Thus $\{\beta_k\}$ represents the regeneration points of the process

$\{X_n^{(1)}\}$. Next we define the sequence of independent and identically distributed (iid) regeneration cycles length in discrete time (with a generic length B) by

$$B_k = \beta_k - \beta_{k-1}, \quad n \geq 1.$$

Note that under conditions (3.2), (3.3) the process $\{X(t)\}$ (and the whole system) does not regenerate at instants t_{β_k}, while the process $X^{(1)}$ is positive recurrent. Partial stability conditions consider solely the tightness of the first orbit size process (3.1), which allows to show that with a positive probability the process $X^{(1)}$ reaches the zero value in a finite time, hence $\mathsf{E}B < \infty$.

In case the positive recurrence we can apply regeneration method (RM) for the system under consideration. RM is a powerful tool in stochastic analysis, in the next section rely on the regeneration confidence estimation to bound the dynamics of the first orbit size in partially stable regime.

4. Regenerative estimation

Recall the regenerative process $X_n^{(1)}$, which is the positive recurrent under conditions (3.2), (3.3). Note that in this case the orbit size process $N_n^{(1)}$ also regenerates with regeneration points $\{\beta_k\}$. In present section we construct the interval estimators for the mean value of the process $N_n^{(1)}$. Consider iid accumulated numbers of customers in the first orbit over the k-th regeneration cycle by

$$Z_k = \sum_{j=\beta_{(k-1)}}^{(\beta_k)-1} N_j^{(1)}, \quad k \geq 1.$$

By the results from regeneration theory and in case of positive recurrence, the following limit exists:

$$r_k := \frac{\sum_{j=1}^k Z_j}{\sum_{j=1}^k B_j} \to \frac{\mathsf{E}Z}{\mathsf{E}B} =: r, \qquad k \to \infty, \tag{4.1}$$

where Z is a generic element of a sequence $\{Z_k, \, k \geq 1\}$.

Note, that r_k coincides with an average number of customers in the first orbit within interval $[0, \, t_{\beta_k})$:

$$r_k = \frac{1}{\beta_k} \sum_{j=1}^{\beta_k} N_j^{(1)}.$$

Actually, the result (4.1) means that with a growth of cycle number, time average value of regenerative process converges to the ratio of mean cumulative value over cycle to mean cycle length. Namely, in case of positive recurrence, the behavior of regenerative process could is described by its cycle characteristics.

By Proposition 4.1 from [4] the estimator r_k satisfies the following Central Limit Theorem

$$\sqrt{k}(r_k - r) \Rightarrow \mathbb{N}(0,\, \sigma^2), \qquad n \to \infty, \tag{4.2}$$

where

$$\sigma^2 = \frac{E[Z - rB]^2}{(EB)^2}$$

and $\mathbb{N}(0,\, \sigma^2)$ is a normal distribution with zero mean. Hence, if limit (4.1) exists, then weak convergence (4.2) holds and implies the following $100(1-\gamma)\%$ confidence interval:

$$r \in \Big[r_k - \Delta_k,\, r_k + \Delta_k\Big], \tag{4.3}$$

with the accuracy

$$\Delta_k = \frac{z_\gamma \overline{\sigma}_k}{\sqrt{k}}.$$

Note, that γ is a given reliability and

$$\overline{\sigma}_k^2 = \frac{k^2}{k-1} \frac{\sum_{i=1}^k \big(Z_i - r_k B_i\big)^2}{\big(\sum_{i=1}^k B_i\big)^2}.$$

(The value z_γ defines $(1 - \gamma/2)$-quantile of the standard normal law.)

4.1. Single-orbit system

The sequence $\{\beta_k\}$ does not detect regenerations of the whole sequence Σ, and we analyze the positive recurrent process $X^{(1)}$ to obtain the confidence interval (4.3).

Next we construct an additional single orbit retrial model denoted by $\hat{\Sigma}$ as follows: the input stream is fed by Poisson process with a total rate $\lambda_1 + \lambda_2 + \alpha_2$, the new arrival belongs to class 1 with a probability

$$\frac{\lambda_1}{\lambda_1 + \lambda_2 + \alpha_2}.$$

Service times are iid, class-dependent and stochastically equivalent to the corresponding service times $\{S_n^{(i)}\}$ previously defined for a model Σ. If class-1 arrival met the busy server it joins the orbit with a constant retrial rate α_1, while the second class arrival in this case leaves the system. We can expect that such new system is not less loaded than original system Σ: in case the second orbit is empty, the server is attacked by the Poisson input with a rate $\lambda_1 + \lambda_2$ and the customers from the orbit 1 (if any). In case $N_n^{(2)} > 0$ both models Σ and $\hat{\Sigma}$ behave equivalently in the sense of server load. The convergence of the first orbit size process in Σ to the orbit size in $\hat{\Sigma}$ is illustrated in [5].

The model $\hat{\Sigma}$ strictly regenerates when arrivals join into totally empty system. Denote the generic regeneration cycle length by $\hat{B}$. Stability condition for such a model is defined as follows, see [6]:

$$\hat{\rho}_1 > \rho_1(\rho + \hat{\rho})$$

and coincides with (3.2).

Thus partially stable regime in the original model Σ implies the positive recurrence of corresponding single orbit system: $\mathsf{E}\hat{\mathsf{B}} < \infty$, and we can apply the regenerative method of confidence estimation for mean number of orbit customers in $\hat{\Sigma}$. Define by $\hat{r}$ the mean orbit size, by $\hat{r}_k$ and $\hat{\Delta}_k$ the corresponding estimators obtained with the regenerative method for the system $\hat{\Sigma}$ exactly as in (4.3). (Note k defines the number of regeneration cycles in $\hat{\Sigma}$).

Next our goal is to validate the accuracy of interval $[r_k \pm \Delta_k]$, comparing it with $[\hat{r}_k \pm \hat{\Delta}_k]$ under assumption that conditions (3.2) and (3.3) hold true. Note that (3.3) does not influence to the stability of $\hat{\Sigma}$ and the regenerative estimation is applicable even if (3.3) is violated, but in this case original model Σ does not converge to $\hat{\Sigma}$ the comparison of obtained intervals have no sense. Note that under conditions

$$\begin{aligned} \hat{\rho}_1 &> \rho_1(\rho + \hat{\rho}), \\ \rho &\leq \hat{\rho}_2/(\rho_2 + \hat{\rho}_2) \end{aligned}$$

the model Σ is strictly stable, see [5].

5. Simulations

We assume exponential distributions of service times and fix the following values:

$$\lambda_1 = 4,\ \lambda_2 = 1, \quad \mu_1 = 8,\ \mu_2 = 4,$$

thus

$$\rho_1 = 0.5,\ \rho_2 = 0.25,\ \rho = 0.75.$$

5.1. Partial stability region

We define $\alpha_1 = 20$, $\alpha_2 = 2$, which implies

$$\hat{\rho}_1 = 3.125,\ \hat{\rho}_2 = 0.500, \hat{\rho} = 3.625.$$

Note that initial values of parameters were arbitrary chosen inside the partly stable region to provide the fulfilness of conditions (3.2) and (3.3). Next we consider $n = 100\,000$ arrivals and simulate both systems Σ and $\hat{\Sigma}$. All the experiments were implemented in RStudio development environment. We obtained $k_1 = 6083$ regenerations in the original system, $k_2 = 6020$ regeneration in the single orbit system. Average orbit sizes as follows: $r_{k_1} = 3.66$, $\hat{r}_{k_2} = 3.68$. The comparison of confidence intervals obtained by regeneration method is presented on Figure 1. The results for both systems almost coincide: $\Delta_{k_1} \approx \hat{\Delta}_{k_2} = 0.36$.

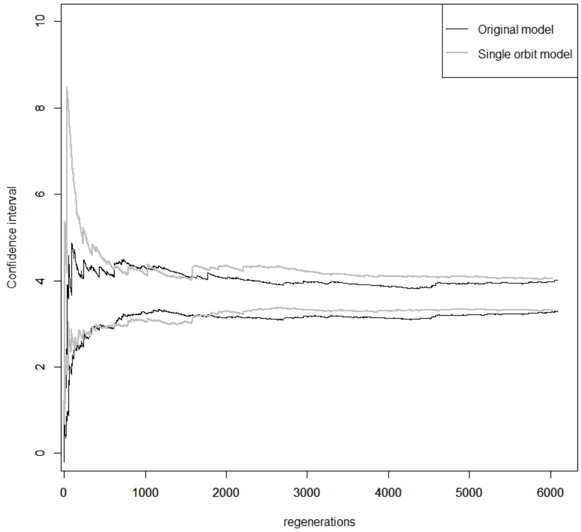

Figure 1. Mean orbit size in Σ and $\hat{\Sigma}$, $\alpha_1 = 20$, $\alpha_2 = 2$.

5.2. Orbit-2 stability border

Next we set the value $\alpha_2 = 2.8$, while $\alpha_1 = 20$. Thus in comparison with the first example we decrease the difference between two parts of the inequality (3.3) and go closer to the border of stability region for the system Σ. Note that in this case the input stream in $\hat{\Sigma}$ is more intensive. We obtain $k_1 = 4184$, $k_2 = 3957$, $r_{k_1} = 4.32$, $\hat{r}_{k_2} = 4.79$. The accuracy difference is more notable: $\Delta_{k_1} = 0.38$, $\hat{\Delta}_{k_2} = 0.45$. Confidence intervals for the considered parameters are presented on Figure 2.

With the growth of the second orbit rate the difference between two models become more significant.

5.3. Instability border

In this example we define $\alpha_1 = 11$, $\alpha_2 = 2$. Thus we touch on the condition (3.2) and move closer to the instability border for the model $\hat{\Sigma}$. (Note that in case the condition (3.2) is violated and (3.3) holds, both orbits in Σ go to infinity, see [5].)

We obtained rare (in comparison with previous cases) regenerations $k_1 = 1162$, $k_2 = 1081$. Note that all simulations are based on $n = 100\,000$ arrivals. Less number of regeneration cycles provide less accurate intervals $r_{k_1} = 18.93$, $\Delta_{k_1} = 3.89$, $\hat{r}_{k_2} = 21.91$, $\hat{\Delta}_{k_2} = 3.91$.

Remind that in all presented examples the conditions (3.2) and (3.3) hold. We started from $\alpha_1 = 20$, $\alpha_2 = 2$ and then explored the cases $\alpha_2 \uparrow$ and $\alpha_1 \downarrow$. Namely in examples *B* and *C* we decreased the differences in two parts of inequalities (3.2)

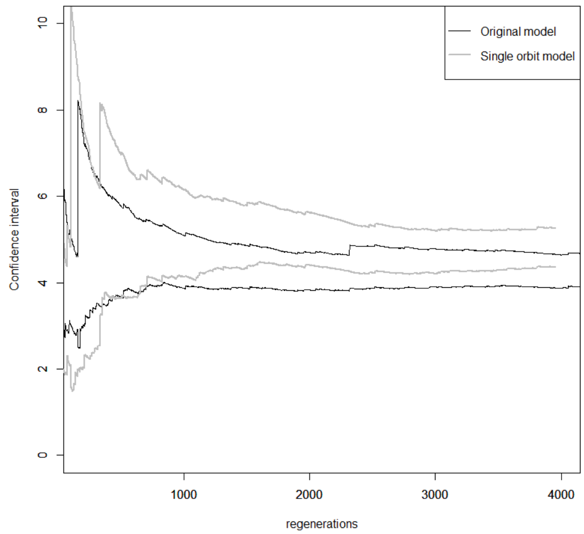

Figure 2. Mean orbit size in Σ and $\hat{\Sigma}$, $\alpha_1 = 20$, $\alpha_2 = 2.8$.

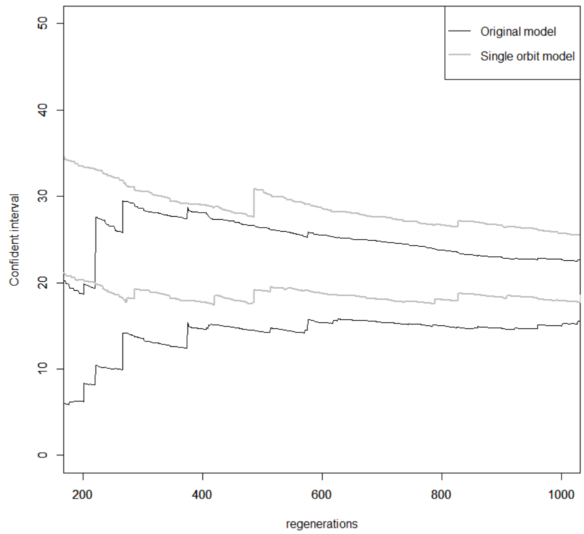

Figure 3. Mean orbit size in Σ and $\hat{\Sigma}$, $\alpha_1 = 11$, $\alpha_2 = 2$.

and (3.3), respectively. Note that in cases $\alpha_2 \downarrow$ and $\alpha_1 \uparrow$ the model Σ converges to $\hat{\Sigma}$ and confidence intervals obtained for mean orbit sizes for both system almost coincide (as on Figure 1).

6. Conclusion

In this paper we study two-class retrial model with constant retrial rates in partially stable regime. In spite of the model under consideration is not stable, we analyze the positive recurrent class-1 orbit size process and apply the regenerative method to construct confidence interval for mean number of class-1 orbit customers. The simulation results correspond with confidence intervals obtained for strictly stable single-orbit model. Thus we illustrate that partially stable case allows to provide accurate confidence estimation.

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