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The structure of the unit group of the group algebra $F(C_3 \times D_{10})$

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Abstract

Let D_n be the dihedral group of order n. The structure of the unit group $U(F(C_3 \times D_{10}))$ of the group algebra $F(C_3 \times D_{10})$ over a finite field F of characteristic 3 is given in [10]. In this article, the structure of $U(F(C_3 \times D_{10}))$ is obtained over any finite field F of characteristic $p \neq 3$.

Keywords: Group rings, unit groups, dihedral groups, cyclic groups

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1. Introduction

Let $U(FG)$ be the group of invertible elements of the group algebra FG of a group G over a field F. The study of units and their properties is one of the most challenging problems in the theory of group rings. Explicit calculations in $U(FG)$ are usually difficult, even when G is fairly small and F is a finite field. The results obtained in this direction are also useful for the investigation of the Lie properties of group rings, the isomorphism problem and other open questions in this area, see [2].

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For a normal subgroup H of G, the natural homomorphism $G \to G/H$ can be extended to an F-algebra homomorphism from $FG \to F(G/H)$ defined by $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g gH$, $a_g \in F$. The kernel of this homomorphism, denoted by $\Delta(G,H)$, is the ideal generated by $\{h-1 : h \in H\}$ in FG and $FG/\Delta(G,H) \cong F(G/H)$.

Let $J(FG)$ be the Jacobson radical of FG and let $V = 1 + J(FG)$. The F-algebra $FG/J(FG)$ is semisimple whenever G is a finite group. It is known from the Wedderburn structure theorem that

$$FG/J(FG) \cong \bigoplus_{i=1}^{r} M(n_i, K_i)$$

where r is the number of non-isomorphic irreducible FG modules, $n_i \in \mathbb{N}$ and K_i's are finite dimensional division algebras over F. In this context two results by Ferraz [3, Theorem 1.3 and Prop 1.2] (stated at the end of this section) are very useful in determining the Wedderburn decomposition of $FG/J(FG)$.

If FG is semisimple, then $J(FG) = 0$ and by [8, Prop 3.6.11],

$$FG \cong F(G/G') \oplus \Delta(G, G')$$

where $F(G/G')$ is the sum of all the commutative simple components of FG, whereas $\Delta(G, G')$ is the sum of all the non-commutative simple components of FG. We conclude that, if FG is semisimple, then

$$FG \cong F(G/G') \oplus \bigoplus_{i=1}^{l} M(n_i, K_i).$$

Now, if $\dim_F(Z(FG)) = r$ and if the number of commutative simple components is s, then $l \leq r - s$.

In what follows, D_n is the dihedral group of order n, C_n is the cyclic group of order n, F^n is the direct sum of n copies of F, F_n is the extension of F of degree n, $M(n, F)$ is the algebra of all $n \times n$ matrices over F, $GL(n, F)$ is the general linear group of degree n over F, $Z(FG)$ is the center of FG, $[g]$ is the conjugacy class of $g \in G$ and T_p is the set of all p-elements of G including 1.

Let F be a field of characteristic $p > 0$ and let G be a finite group. An element $g \in G$ is p-regular, if $p \nmid o(g)$. Let t be the l.c.m. of the orders of p-regular elements of G and let ω be a primitive t-th root of unity over the field F. Then

$$A = \{r \mid \omega \to \omega^r \text{ is an automorphism of } F(\omega) \text{ over } F\}.$$

Let γ_g be the sum of all conjugates of $g \in G$. If g is a p-regular element, then the cyclotomic F-class of γ_g is

$$S_F(\gamma_g) = \{\gamma_{g^r} \mid r \in A\}.$$

Many authors [1, 4, 5, 7, 9–12] have studied the structure of $U(FG)$ for a finite group G and for a finite field F. The structure of $U(F(C_3 \times D_{10}))$ for $p = 3$ is given in [10]. In this article, we provide an explicit description for the Wedderburn decomposition of $FG/J(FG)$, $G = C_3 \times D_{10}$ and F a finite field of characteristic $p \neq 3$, using the theory developed by Ferraz [3] and with the help of this description we obtain the structure of $U(F(C_3 \times D_{10}))$.

Lemma 1.1 ([3, Proposition 1.2]). *Let K be a field and let G be a finite group. The number of simple components of $KG/J(KG)$ is equal to the number of cyclotomic K-classes in G.*

Lemma 1.2 ([3, Theorem 1.3]). *Let K be a field and let G be a finite group. Suppose that $Gal(K(\omega)/K)$ is cyclic. Let s be the number of cyclotomic K-classes in G. If $R_1, R_2, \ldots, R_s$ are the simple components of $Z(KG/J(KG))$ and $P_1, P_2, \ldots, P_s$ are the cyclotomic K-classes of G, then with a suitable re-ordering of indices, $|P_i| = [R_i : K]$.*

2. Structure of $U(F(C_3 \times D_{10}))$

Theorem 2.1. *Let F be a finite field of characteristic p with $|F| = q = p^n$ and let $G = C_3 \times D_{10}$.*

1. *If $p = 2$, then $U(FG) \cong$*

$$\begin{cases} C_2^{3n} \rtimes \left(C_{2^n-1}^3 \times GL(2,F)^6\right), & \text{if } q \equiv 1, 4 \bmod 15; \\ C_2^{3n} \rtimes \left(C_{2^n-1} \times C_{2^{2n}-1} \times GL(2,F_2)^3\right), & \text{if } q \equiv 2, -7 \bmod 15. \end{cases}$$

2. *If $p = 5$, then*

$$U(FG) \cong V \rtimes \begin{cases} C_{5^n-1}^6, & \text{if } q \equiv 1 \bmod 6; \\ C_{5^n-1}^2 \times C_{5^{2n}-1}^2, & \text{if } q \equiv -1 \bmod 6. \end{cases}$$

where $V \cong (C_5^{15n} \rtimes C_5^{6n}) \rtimes C_5^{3n}$ and $Z(V) \cong C_5^{9n}$.

3. *If $p > 5$, then $U(FG) \cong$*

$$\begin{cases} C_{p^n-1}^6 \times GL(2,F)^6, & \text{if } q \equiv 1, -11 \bmod 30; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^2 \times GL(2,F)^2 \times GL(2,F_2)^2, & \text{if } q \equiv -1, 11 \bmod 30; \\ C_{p^n-1}^6 \times GL(2,F_2)^3, & \text{if } q \equiv 7, 13 \bmod 30; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^2 \times GL(2,F_2)^3, & \text{if } q \equiv -7, -13 \bmod 30. \end{cases}$$

Proof. Let $G = \langle x, y, z \mid x^2 = y^5 = z^3 = 1, xy = y^4x, xz = zx, yz = zy\rangle$. The conjugacy classes in G are:

$$\begin{aligned} [z^i] &= \{z^i\} \text{ for } i = 0, 1, 2; \\ [yz^i] &= \{y^{\pm 1} z^i\} \text{ for } i = 0, 1, 2; \\ [y^2 z^i] &= \{y^{\pm 2} z^i\} \text{ for } i = 0, 1, 2; \\ [xz^i] &= \{xz^i, xy^{\pm 1} z^i, xy^{\pm 2} z^i\} \text{ for } i = 0, 1, 2. \end{aligned}$$

1. $p = 2$. Clearly, $\widehat{T_2} = 1 + x\widehat{y}$.

   Let $\alpha = \sum_{k=0}^{1}\sum_{j=0}^{2}\sum_{i=5(j+3k)}^{5(j+3k)+4} a_i x^k y^{i-5(j+3k)} z^j$. If $\alpha\widehat{T_2} = 0$, then we have

$$\alpha + \sum_{k=0}^{1}\sum_{j=0}^{2}\sum_{i=5(j+3k)}^{5(j+3k)+4} a_i x^{k+1}\widehat{y} z^j = 0.$$

   For $k = 0, 1, 2$ and $i = 0, 1, 2, 3, 4$ this yields the following equations:

$$a_{5k+i} + \sum_{j=0}^{4} a_{5k+j+15} = 0,$$
$$a_{5k+15+i} + \sum_{j=0}^{4} a_{5k+j} = 0.$$

   After simplification we get, $a_{5k} = a_{5k+i} = a_{5k+i+15}$ for $i = 0, 1, 2, 3, 4$ and $k = 0, 1, 2$. Hence

$$\operatorname{Ann}(\widehat{T_2}) = \left\{\sum_{i=0}^{2} \beta_i (1+x)\widehat{y} z^i \mid \beta_i \in F\right\}.$$

   Since $z, \widehat{y} \in Z(FG)$, $\operatorname{Ann}^2(\widehat{T_2}) = 0$ and $\operatorname{Ann}(\widehat{T_2}) \subseteq J(FG)$. Thus by [12, Lemma 2.2], $J(FG) = \operatorname{Ann}(\widehat{T_2})$ and $\dim_F(J(FG)) = 3$. Hence $V \cong C_2^{3n}$ and by [6, Lemma 2.1],

$$U(FG) \cong C_2^{3n} \rtimes U(FG/J(FG)).$$

   Now it only remains to find the Wedderburn decomposition of $FG/J(FG)$.

   As [1], [y], [y^2], [z], [z^2], [yz], [yz^2], [y^2z], and [y^2z^2] are the 2-regular conjugacy classes of G, $t = 15$ and $\dim_F(FG/J(FG)) = 27$. Now the following cases occur:

   (a) If $q \equiv 1, 4 \bmod 15$, then $|S_F(\gamma_g)| = 1$ for $g = 1$, y, y^2, z, z^2, yz, yz^2, y^2z, y^2z^2. Consquently, [3, Theorem 1.3], yields nine components in the decomposition of $FG/J(FG)$. In view of the dimension requirements, the only possibility is:

$$FG/J(FG) \cong F^3 \oplus M(2, F)^6.$$

   (b) If $q \equiv 2, -7 \bmod 15$, then $|S_F(\gamma_g)| = 1$ for $g = 1$ and $|S_F(\gamma_g)| = 2$ for $g = y$, z, yz, yz^2. So, due to the dimension restrictions, we have

$$FG/J(FG) \cong F \oplus F_2 \oplus M(2, F_2)^3.$$

2. $p = 5$. If $K = \langle y \rangle$, then $G/K \cong H \cong \langle x, z \rangle \cong C_6$. Thus from the ring epimorphism $\eta : FG \to FH$, given by

$$\eta\left(\sum_{j=0}^{2}\sum_{i=0}^{4} y^i z^j (a_{i+5j} + a_{i+5j+15}x)\right) = \sum_{j=0}^{2}\sum_{i=0}^{4} z^j (a_{i+5j} + a_{i+5j+15}x),$$

we get a group epimorphism $\phi\colon U(FG) \to U(FH)$ and $\ker\phi \cong 1 + J(FG) = V$. Further, we have the inclusion map $i\colon U(FH) \to U(FG)$ such that $\phi i = 1_{U(FH)}$. Thus $U(FG) \cong V \rtimes U(FC_6)$.

The structure of $U(FC_6)$ is given in [11, Theorem 4.1].

If $v = \sum_{j=0}^{2}\sum_{i=0}^{4} y^i z^j (a_{i+5j} + a_{i+5j+15}x) \in U(FG)$, then $v \in V$ if and only if $\sum_{i=0}^{4} a_i = 1$ and $\sum_{i=0}^{4} a_{i+5k} = 0$ for $k = 1, 2, 3, 4, 5$. Hence

$$V = \left\{1 + \sum_{j=0}^{2}\sum_{i=1}^{4}(y^i - 1)z^j(b_{i+4j} + b_{i+4j+12}x) \mid b_i \in F\right\}$$

and $|V| = 5^{24n}$. Since, $J(FG)^5 = 0$, $V^5 = 1$.

Now we show that $V \cong (C_5^{15n} \rtimes C_5^{6n}) \rtimes C_5^{3n}$. The proof is split into the following steps:

Step 1: Let $R = \{1 + ay(1-y)^3x \mid a \in F\} \subseteq V$. Then $R \cong C_5^n$.

If

$$r_1 = 1 + ay(1-y)^3x \in R$$

and

$$r_2 = 1 + by(1-y)^3x \in R$$

where $a, b \in F$, then

$$r_1 r_2 = 1 + (a+b)y(1-y)^3x \in R.$$

Therefore, R is an abelian subgroup of V of order 5^n. Hence $R \cong C_5^n$.

Step 2: $|C_V(R)| = 5^{21n}$, where $C_V(R) = \{v \in V \mid r^v = r \text{ for all } r \in R\}$.

Let

$$r = 1 + ay(1-y)^3x \in R$$

and

$$v = 1 + \sum_{j=0}^{2}\sum_{i=1}^{4}(y^i - 1)z^j(b_{i+4j} + b_{i+4j+12}x) \in V$$

where $a, b_i \in F$. Then $v = 1 + v_1 + v_2x$, $v_1 = \sum_{j=0}^{2}\sum_{i=1}^{4} b_{i+4j}(y^i - 1)z^j$ and $v_2 = \sum_{j=0}^{2}\sum_{i=1}^{4} b_{i+4j+12}(y^i - 1)z^j$. So $v^{-1} = v^4 = 1 + 4v_1 + 4v_2x$ mod $(y-1)^2FG$. Thus

$$r^v = 1 + v^{-1}ay(1-y)^3xv = r + 2a\widehat{y}\sum_{j=0}^{2}\sum_{i=1}^{4} ib_{i+4j}z^jx.$$

Thus $r^v = r$ if and only if $\sum_{i=1}^{4} ib_{i+4j} = 0$ for $j = 0, 1, 2$. Hence

$$C_V(R) = \Big\{1 + \sum_{j=0}^{2}\sum_{i=1}^{3}[(y^i - 1) + i(y^4 - 1)]c_{i+3j}z^j + \sum_{j=0}^{2}\sum_{i=1}^{4}(y^i - 1)c_{i+4j+9}z^j x \mid c_i \in F\Big\}$$

and $|C_V(R)| = 5^{21n}$.

Step 3: $C_V(R) \cong C_5^{15n} \rtimes C_5^{6n}$.

Consider the sets

$$S = \{1 + y^3(y-1)^2[yb_1 + y(y+2)b_2 + b_3 + (yb_4 + (y+1)^2 b_5)x]\}$$

and

$$T = \{1 + y^3(y-1)[(y-1)(yc_1 + (y+1)^2 c_2) + (yc_3 + (y^2 + y + 1)c_4)x]\}$$

where $b_{1+j} = \sum_{i=0}^{2} p_{i+3j}z^i$ for $j = 0, 1, 2, 3, 4$ and $c_{1+j} = \sum_{i=0}^{2} q_{i+3j}z^i$ for $j = 0, 1, 2, 3$. With some computation it can be shown that S and T are abelian subgroups of $C_V(R)$. So $S \cong C_5^{15n}$ and $T \cong C_5^{12n}$.

Now, let

$$s = 1 + y^3(y-1)^2[yb_1 + y(y+2)b_2 + b_3 + (yb_4 + (y+1)^2 b_5)x] \in S$$

and

$$t = 1 + y^3(y-1)[(y-1)(yc_1 + (y+1)^2 c_2) + (yc_3 + (y^2 + y + 1)c_4)x] \in T.$$

Then

$$s^t = 1 + y^3(y-1)^2\{yb_1 + y(y+2)b_2 + b_3 + k_1 y^3(1-y) + [yb_4 + (y+1)^2 b_5 + (y-1)^2(k_2 + k_3)]x\} \in S$$

where

$$k_1 = (c_4 + 2c_3)(b_4 - b_5), k_2 = (c_4 + 2c_3)(b_2 - b_3)$$
$$k_3 = 2(c_4^2 - c_3c_4 - c_3^2)(b_4 - b_5).$$

Let

$$U = S \cap T = \{1 + y^3(y-1)^2[yc_1 + (y+1)^2 c_2]\}$$

where $c_{1+j} = \sum_{i=0}^{2} q_{i+3j}z^i$ for $j = 0, 1$. Thus $U \cong C_5^{6n}$. So for some subgroup $W \cong C_5^{6n}$ of T, $T = U \times W$ and $W \cap S = 1$. Hence $C_V(R) \cong S \rtimes W \cong C_5^{15n} \rtimes C_5^{6n}$.

Step 4: Let $M = \{1+\sum_{j=0}^{2} r_j z^j y(y+1)^2(1-y)(1+x) \mid r_i \in F\} \subseteq V$. Then $M \cong C_5^{3n}$.

Let

$$m_1 = 1 + \sum_{j=0}^{2} r_j z^j y(y+1)^2(1-y)(1+x) \in M$$

and

$$m_2 = 1 + \sum_{j=0}^{2} s_j z^j y(y+1)^2(1-y)(1+x) \in M$$

where $r_j, s_j \in F$. Then

$$m_1 m_2 = 1 + \sum_{j=0}^{2} (r_j + s_j) z^j y(y+1)^2(1-y)(1+x) \in M.$$

Therefore, M is an abelian subgroup of V of order 5^{3n}. Hence, $M \cong C_5^{3n}$.

Step 5: $V \cong C_V(R) \rtimes M$.

Let

$$\begin{aligned} a = 1 &+ \sum_{j=0}^{2}\sum_{i=1}^{3} [(y^i - 1) + i(y^4 - 1)] c_{i+3j} z^j \\ &+ \sum_{j=0}^{2}\sum_{i=1}^{4} (y^i - 1) c_{i+4j+9} z^j x \in C_V(R) \end{aligned}$$

and let

$$b = 1 + \sum_{j=0}^{2} r_j z^j y(y+1)^2(1-y)(1+x) \in M$$

where $c_i, r_i \in F$. Then

$$\begin{aligned} a^b = 1 &+ \sum_{j=0}^{2}\sum_{i=1}^{3} [(y^i - 1) + i(y^4 - 1)] c_{i+3j} z^j \\ &+ \sum_{j=0}^{2}\sum_{i=1}^{4} (y^i - 1) c_{i+4j+9} z^j x + (k_1 + k_2 x) \in C_V(R) \end{aligned}$$

where

$$\begin{aligned} k_1 = &\sum_{j=0}^{2} r_j z^j \{\sum_{k=0}^{2} (c_{10+4k} - c_{11+4k} - c_{12+4k} + c_{13+4k}) z^k \\ &+ 3\sum_{j=0}^{2} r_j z^j \sum_{i=1}^{4}\sum_{k=0}^{2} i(c_{i+4k+9} z^k)\} y(1-y)^3 \end{aligned}$$

and

$$k_2 = 2\sum_{j=0}^{2} r_j z^j \{\sum_{k=0}^{2}(c_{2+3k} - c_{3+3k})z^k(1-y)$$
$$- \sum_{j=0}^{2} r_j z^j \sum_{i=1}^{4}\sum_{k=0}^{2} ic_{i+4k+9}z^k\}y(1-y)^3 - 2\sum_{j=0}^{2} r_j z^j \sum_{i=0}^{4} d_i y^i$$

with

$$d_0 = \sum_{j=0}^{2}(4c_{10+4j} + 4c_{11+4j} + c_{12+4j} + c_{13+4j})z^j,$$
$$d_1 = \sum_{j=0}^{2}(4c_{10+4j} + 3c_{11+4j} + 3c_{12+4j})z^j,$$
$$d_2 = \sum_{j=0}^{2}(4c_{11+4j} + 3c_{12+4j} + 3c_{13+4j})z^j,$$
$$d_3 = \sum_{j=0}^{2}(2c_{10+4j} + 2c_{11+4j} + c_{12+4j})z^j,$$
$$d_4 = \sum_{j=0}^{2}(2c_{11+4j} + 2c_{12+4j} + c_{13+4j})z^j.$$

Clearly, $C_V(R) \cap M = 1$. Therefore, $V = C_V(R) \rtimes M$.

In the sequel, we show that $Z(V) \cong C_5^{9n}$.

If $v = 1 + \sum_{j=0}^{2}\sum_{i=1}^{4}(y^i - 1)z^j(b_{i+4j} + b_{i+4j+12}x) \in C_V(y) = \{v \in V \mid vy = yv\}$, then

$$vy - yv = \sum_{i=1}^{4}\sum_{j=0}^{2} y(1-y^i)(y^3-1)b_{i+4j+12}z^j x.$$

Thus $v \in C_V(y)$ if and only if $b_i = b_{i+j}$ for $j = 1, 2, 3$ and $i = 13, 17, 21$. Hence

$$C_V(y) = \left\{1 + \sum_{j=0}^{2}\sum_{i=1}^{4}(y^i - 1)c_{i+4j}z^j + \widehat{y}\sum_{j=0}^{2} c_{j+13}z^j x \mid c_i \in F\right\}.$$

Since $Z(V) \subseteq C_V(y)$,

$$Z(V) = \{s \in C_V(y) \mid sv = vs \text{ for all } v \in V\}.$$

Let $u = 1 + \sum_{j=0}^{2} \sum_{i=1}^{4} (y^i - 1)c_{i+4j} z^j + \widehat{y}x \sum_{j=0}^{2} c_{j+13} z^j \in C_V(y)$. Since $v = 1 + (y-1)zx \in V$ and $\widehat{y} \in Z(FG)$, $vu - uv = 0$ yields

$$(y-1)\sum_{i=1}^{4}\sum_{j=0}^{2}(y^i - y^{-i})c_{i+4j} z^{j+1} x = 0.$$

Thus $c_i = c_{i+3}$ for $i = 1, 5, 9$ and $c_j = c_{j+1}$ for $j = 2, 6, 10$ and $u = 1 + y^4(y-1)^2 \sum_{j=0}^{2} d_{1+j} z^j + y^3(y^2-1)^2 \sum_{j=0}^{2} d_{4+j} z^j + \widehat{y} \sum_{j=0}^{2} d_{7+j} z^j x$. Clearly $u \in Z(V)$.

We conclude that $Z(V) = \{1 + y^4(y-1)^2 \sum_{j=0}^{2} d_{1+j} z^j + y^3(y^2-1)^2 \sum_{j=0}^{2} d_{4+j} z^j + \widehat{y} \sum_{j=0}^{2} d_{7+j} z^j x \mid d_i \in F\} \cong C_5^{9n}$.

3. If $p > 5$, then $J(FG) = 0$. Thus FG is semisimple and $t = 30$. As $G/G' \cong C_6$, we have

$$FG \cong FC_6 \oplus \bigoplus_{i=1}^{l} M(n_i, K_i).$$

Since $\dim_F(Z(FG)) = 12$, $l \leq 6$. Now we have the following cases:

(a) If $q \equiv 1, -11 \bmod 30$, then $|S_F(\gamma_g)| = 1$ for all $g \in G$. Therefore by [11, Theorem 4.1] and [3, Prop 1.2 and Theorem 1.3],

$$FG \cong F^6 \oplus \bigoplus_{i=1}^{6} M(n_i, F)$$

and $\sum_{i=1}^{6} n_i^2 = 24$. Clearly $n_i = 2$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Hence,

$$FG \cong F^6 \oplus M(2, F)^6.$$

(b) If $q \equiv -1, 11 \bmod 30$, then $|S_F(\gamma_g)| = 1$ for $g = 1, x, y, y^2$ and $|S_F(\gamma_g)| = 2$ for $g = z, xz, yz, y^2z$. In this case $FC_6 \cong F^2 \oplus F_2^2$, thus dimension constraints yield

$$n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2 = 24.$$

We get $n_1 = n_2 = n_3 = n_4 = 2$. Hence,

$$FG \cong F^2 \oplus F_2^2 \oplus M(2, F)^2 \oplus M(2, F_2)^2.$$

(c) If $q \equiv 7, 13 \bmod 30$, then $T = \{1, 7, 13, 19\} \bmod 30$. Thus $|S_F(\gamma_g)| = 1$ for $g = 1, x, z, z^2, xz, xz^2$ and $|S_F(\gamma_g)| = 2$ for $g = y, yz, yz^2$. Therefore,

$$2(n_1^2 + n_2^2 + n_3^2) = 24.$$

We get $n_1 = n_2 = n_3 = 2$. Hence,

$$FG \cong F^6 \oplus M(2, F_2)^3.$$

(d) If $q \equiv -7, -13 \bmod 30$, then $T = \{1, 17, 19, 23\} \bmod 30$. Thus $|S_F(\gamma_g)| = 1$ for $g = 1, x$ and $|S_F(\gamma_g)| = 2$ for $g = y, z, xz, yz, yz^2$. Hence,

$$FG \cong F^2 \oplus F_2^2 \oplus M(2, F_2)^3.$$ □

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