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# On structure of the family of regularly distributed sets with respect to the union*

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**Abstract**

Let $0 \leq q \leq 1$ and $\mathbb{N}$ denotes the set of all positive integers. In this paper we will be interested in the family $\mathcal{U}(x^q)$ of all regularly distributed set $X \subset \mathbb{N}$ whose ratio block sequence is asymptotically distributed with distribution function $g(x) = x^q$; $x \in (0,1]$, and we will study the structure of this family with respect to the union.

*Keywords:* Ideals of sets of positive integers, distribution functions, block sequences, exponent of convergence

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## 1. Introduction

In the whole paper we assume $X = \{x_1 < x_2 < \cdots < x_n < \cdots\} \subset \mathbb{N}$ where $\mathbb{N}$ denotes the set of all positive integers.

The following sequence derived from $X$

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1.1}$$

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is called *the ratio block sequence* of the set (sequence) $X$.

It is formed by the blocks $X_1, X_2, \dots, X_n, \dots$ where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \quad n = 1, 2, \dots$$

is called the *n-th block*. This kind of block sequences was introduced by O. Strauch and J. T. Tóth [12] and they studied the set $G(X_n)$ of its distribution functions. Further, we will be interested in ratio block sequences of type (1.1) possessing an asymptotic distribution function, i.e. $G(X_n)$ is a singleton (see definitions in the next section).

By means of these distribution functions in [13] was defined the next families of subsets of $\mathbb{N}$. For $0 \le q \le 1$ we denote $\mathcal{U}(x^q)$ the family of all regularly distributed set $X \subset \mathbb{N}$ whose ratio block sequence is asymptotically distributed with distribution function $g(x) = x^q;\ x \in (0, 1]$.

Further in [13] the following interesting results can be seen, that $\lambda$ the exponent of convergence is closely related to distributional properties of sets of positive integers. More precisely, for each $q \in [0, 1]$ the family $\mathcal{I}_{\le q}$ of all sets $A \subset \mathbb{N}$ such that $\lambda(A) \le q$ is identical with the family $\mathcal{I}(x^q)$ of all sets $A \subset \mathbb{N}$ which are covered by some regularly distributed set $X \in \mathcal{U}(x^q)$.

The *exponent of convergence* of a set $A \subset \mathbb{N}$ is defined by

$$\lambda(A) = \inf\Big\{s \in (0, \infty) : \sum_{n \in \mathbb{N}} a_n^{-s} < \infty\Big\},$$

where $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$.

In this paper we will be interested in the family $\mathcal{U}(x^q)$ and study the structure of this family respect to the union.

The rest of our paper is organized as follows. In Section 2 and Section 3 we recall some known definitions, notations and theorems, which will be used and extended. In Section 4 our new results are presented.

## 2. Definitions

The following basic definitions are from papers [9, 12, 14].

- For each $n \in \mathbb{N}$ consider the *step distribution function*

  $$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$

  for $x \in [0, 1)$, and for $x = 1$ we define $F(X_n, 1) = 1$.

- A non-decreasing function $g\colon [0, 1] \to [0, 1]$, $g(0) = 0$, $g(1) = 1$ is called a *distribution function* (abbreviated d.f.). We shall identify any two d.f.s coinciding at common points of continuity.

- A d.f. $g(x)$ is a d.f. of the sequence of blocks $X_n$, $n = 1, 2, \dots$, if there exists an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that
$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$
a.e. on $[0, 1]$. This is equivalent to the weak convergence, i.e., the preceding limit holds for every point $x \in [0, 1]$ of continuity of $g(x)$.

- Denote by $G(X_n)$ the set of all d.f.s of $X_n$, $n = 1, 2, \dots$. The set of distribution functions of ratio block sequences was studied in [1–7, 9–12].

  If $G(X_n) = \{g(x)\}$ is a singleton, the d.f. $g(x)$ is also called the *asymptotic distribution function* of $X_n$.

- Let $\lambda$ be the convergence exponent function on the power set $2^{\mathbb{N}}$ of $\mathbb{N}$, i.e. for $A \subset \mathbb{N}$ put
$$\lambda(A) = \inf\Big\{t > 0 : \sum_{a\in A} \frac{1}{a^t} < \infty\Big\}.$$
If $q > \lambda(A)$ then $\sum_{a\in A} \frac{1}{a^q} < \infty$ and if $q < \lambda(A)$ then $\sum_{a\in A} \frac{1}{a^q} = \infty$. In the case when $q = \lambda(A)$, the series $\sum_{a\in A} \frac{1}{a^q}$ can be either convergent or divergent.

  From [8, p. 26, Exercises 113, 114], it follows that the set of all possible values of $\lambda$ forms the whole interval $[0, 1]$, i.e. $\{\lambda(A) : A \subset \mathbb{N}\} = [0, 1]$ and if $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ then $\lambda(A)$ can be calculated by
$$\lambda(A) = \limsup_{n\to\infty} \frac{\log n}{\log a_n}.$$
Evidently the exponent of convergence $\lambda$ is a monotone set function, i.e. $\lambda(A) \le \lambda(B)$ for $A \subset B \subset \mathbb{N}$ and also $\lambda(A \cup B) = \max\{\lambda(A), \lambda(B)\}$ holds for all $A, B \subset \mathbb{N}$.

- By means of $\lambda$ the following sets were defined (see [14]):
$$\begin{aligned} \mathcal{I}_{<q} &= \{A \subset \mathbb{N} : \lambda(A) < q\} \quad \text{for} \quad 0 < q \le 1, \\ \mathcal{I}_{\le q} &= \{A \subset \mathbb{N} : \lambda(A) \le q\} \quad \text{for} \quad 0 \le q \le 1 \quad \text{and} \\ \mathcal{I}_0 &= \{A \subset \mathbb{N} : \lambda(A) = 0\}. \end{aligned}$$
Obviously $\mathcal{I}_{\le 0} = \mathcal{I}_0$ and $\mathcal{I}_{\le 1} = 2^{\mathbb{N}}$.

  For a finite set $A \subset \mathbb{N}$ we have $\lambda(A) = 0$. Consequently, $\mathcal{F}in = \{A \subset \mathbb{N} : A \text{ is finite}\} \subset \mathcal{I}_0$. Families $\mathcal{I}_{<q}, \mathcal{I}_{\le q}$ are related for $0 < q < q' < 1$ by following inclusions (see [14, Theorem 1]),
$$\mathcal{F}in \subsetneq \mathcal{I}_0 \subsetneq \mathcal{I}_{<q} \subsetneq \mathcal{I}_{\le q} \subsetneq \mathcal{I}_{<q'} \subsetneq \mathcal{I}_{<1},$$
and the difference of successive sets is infinite, so equality does not hold in any of the inclusions.

- Let $\mathcal{I} \subset 2^{\mathbb{N}}$. Then $\mathcal{I}$ is called an *ideal* of subsets of positive integers, if $\mathcal{I}$ is additive (if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$), hereditary (if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$), $\mathcal{I} \supseteq \mathcal{F}in$ and $\mathbb{N} \notin \mathcal{I}$.

# 3. Overwiew of known results

In this section we mention known results related to the topic of this paper and some other ones we use in the proofs of our theorems. In the whole part in (S1)–(S7) we assume $X = \{x_1 < x_2 < \cdots < x_n < \cdots\} \subset \mathbb{N}$.

(S1) We will use step function

$$c_0(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } 0 < x \le 1. \end{cases}$$

Assume that $G(X_n)$ is singleton, i.e., $G(X_n) = \{g(x)\}$. Then either $g(x) = c_0(x)$ for $x \in [0, 1]$; or $g(x) = x^q$ for $x \in [0, 1]$ and some fixed $0 < q \le 1$.

[12, Theorem 8.2]

The result (S1) provides motivation to introduce the following families of subsets of $\mathbb{N}$( see [13]):

$$\mathcal{U}(c_0(x)) = \{X \subset \mathbb{N} : G(X_n) = \{c_0(x)\}\},$$
$$\mathcal{I}(c_0(x)) = \{A \subset \mathbb{N} : \exists X \in \mathcal{U}(c_0(x)), A \subset X\},$$

and for $0 < q \le 1$

$$\mathcal{U}(x^q) = \{X \subset \mathbb{N} : G(X_n) = \{x^q\}\},$$
$$\mathcal{I}(x^q) = \{A \subset \mathbb{N} : \exists X \in \mathcal{U}(x^q), A \subset X\}.$$

Obviously,

$$\mathcal{U}(c_0(x)) \subsetneq \mathcal{I}(c_0(x)), \quad \mathcal{U}(x^q) \subsetneq \mathcal{I}(x^q).$$

Sets $X$ from $\mathcal{U}(c_0(x))$ are characterized by (S4) and sets belonging to $\mathcal{U}(x^q)$ are characterized by (S2) and (S5). In [13, Theorem 1 and Example 1] is proved that the family $\mathcal{U}(c_0(x))$ is additive, i.e. it is closed with respect to finite unions and does not form an ideal as it is not hereditary, i.e. there exists sets $C \in \mathcal{U}(c_0(x))$ and $B \subset C$ such that $B \notin \mathcal{U}(c_0(x))$. On the other hand the family $\mathcal{I}(c_0(x))$ is an ideal (see [13, Theorem 2]). For these families the following statements hold.

(S2) Let $0 < q \le 1$ be a real number. Then

$$X \in \mathcal{U}(x^q) \iff \forall\, k \in \mathbb{N} : \lim_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\frac{1}{q}}.$$

[6, Theorem 1]

(S3) Let $0 < q \leq 1$ be a real number and $X \in \mathcal{U}(x^q)$. Then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1.$$

[4, Remark 3]

(S4) We have

$$X \in \mathcal{U}(c_0(x)) \iff \lim_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 0.$$

[12, Theorem 7.1]

(S5) Let $0 < q \leq 1$ be a real number. Then

$$X \in \mathcal{U}(x^q) \iff \lim_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = \frac{q}{q+1}.$$

[3, Theorem 1]

(S6) Let $X \in \mathcal{U}(c_0(x))$. Then

$$\lim_{n\to\infty} \frac{\log n}{\log x_n} = 0 \text{ (i.e. } \lambda(X) = 0).$$

[3, Theorem 2]

(S7) Let $0 < q \leq 1$ be a real number and $X \in \mathcal{U}(x^q)$. Then

$$\lim_{n\to\infty} \frac{\log n}{\log x_n} = q \text{ (therefore } \lambda(X) = q).$$

[3, Theorem 3]

(S8) Let $0 < q \leq 1$. Then each of the families $\mathcal{I}_0$, $\mathcal{I}_{<q}$ and $\mathcal{I}_{\leq q}$ forms an admissible ideal, except for $\mathcal{I}_{\leq 1}$.

[14, Theorem 1]

(S9) Let $0 < q \leq 1$. Then each of the families $\mathcal{I}(c_0(x))$, $\mathcal{I}(x^q)$ forms an admissible ideal and $\mathcal{I}(c_0(x)) = \mathcal{I}_0$, $\mathcal{I}(x^q) = \mathcal{I}_{\leq q}$.

[13, Theorem 5 and Theorem 7]

Given $t \geq 1$, define the counting function of $X \subset \mathbb{N}$ as

$$X(t) = \#\{x \leq t : x \in X\}.$$

(S10) Let $0 < q \leq 1$, $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$ and $Y = \{y_1 < y_2 < \cdots\} \subset \mathbb{N}$. Let $g(x) \in \{c_0(x), x^q\}$ be fixed and assume that

$$Y \in \mathcal{U}(g(x)) \quad \text{and} \quad \lim_{t\to\infty} \frac{X(t)}{Y(t)} = 0.$$

Then

$$X \cup Y \in \mathcal{U}(g(x)).$$

[13, Theorem 4]

# 4. Results

In this section we will study the structure of the family $\mathcal{U}(x^q)$ respect to the union of its elements. We show that there exist such sets $X, Y \in \mathcal{U}(x^q)$ that $X \cup Y \notin \mathcal{U}(x^q)$, but on the other hand, if $X, Y \in \mathcal{U}(x^q)$ (hence $\lambda(X) = q$ and $\lambda(Y) = q$) then necessary $\lambda(X \cup Y) = q$, thus

$$X \cup Y \in \mathcal{I}_{\leq q} \setminus \mathcal{I}_{<q} = \mathcal{I}(x^q) \setminus \mathcal{I}_{<q} \subsetneq \mathcal{I}(x^q).$$

This follows from the (S7), (S9) and the fact that $\lambda(X \cup Y) = \max\{\lambda(X), \lambda(Y)\}$.

**Theorem 4.1.** *Let $0 < q \leq 1$. Then the family $\mathcal{U}(x^q)$ does not form an ideal as it is not additive, i.e. it is not closed with respect to finite unions.*

*Proof.* It is sufficent to show that there exist sets $X, Y \in \mathcal{U}(x^q)$ such that $X \cup Y \notin \mathcal{U}(x^q)$. Let $0 < q \leq 1$ and $X = \{x_1 < x_2 < \cdots < x_n < \cdots\} \subset \mathbb{N}$ be such that $x_{n+1} > x_n + 1$ for every $n \in \mathbb{N}$ and $X \in \mathcal{U}(x^q)$. For example, it will be like that $x_n = \lfloor 2n^{\frac{1}{q}} \rfloor$ (as usual, $\lfloor x \rfloor$ is the integer part of the real $x$). From (S2) it is clear that $X \in \mathcal{U}(x^q)$.

Then $x_n = 2n^{\frac{1}{q}} - \varepsilon(n)$ for some $0 \leq \varepsilon(n) < 1$, and by Lagrange's Mean Value Theorem for $f(x) = 2x^{\frac{1}{q}}$ on $[n, n+1]$ we get that $x_{n+1} > x_n + 1$ for all $n$.

Define the set $Y = \{y_1 < y_2 < \cdots < y_n < \cdots\}$ such that $y_1 = x_1$ and for $n \geq 2$

$$y_n = \begin{cases} x_n - 1, & \text{if } n \in \left(2^{2k}, 2^{2k+1}\right], \quad k = 0, 1, 2, \dots, \\ x_n, & \text{if } n \in \left(2^{2k+1}, 2^{2k+2}\right], \quad k = 0, 1, 2, \dots. \end{cases}$$

We show that $Y \in \mathcal{U}(x^q)$. Since $x_n - 1 \leq y_n \leq x_n$ then for every $k \in \mathbb{N}$

$$\frac{x_{kn} - 1}{x_{kn}} \frac{x_{kn}}{x_n} = \frac{x_{kn} - 1}{x_n} \leq \frac{y_{kn}}{y_n} \leq \frac{x_{kn}}{x_n - 1} = \frac{x_n}{x_n - 1} \frac{x_{kn}}{x_n}.$$

From this according to (S2) for each $k \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \frac{y_{kn}}{y_n} = \lim_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\frac{1}{q}},$$

thus $Y \in \mathcal{U}(x^q)$.

Further let

$$X \cup Y = \{z_1 < z_2 < \cdots < z_n < \cdots\}.$$

We now show that $X \cup Y \notin \mathcal{U}(x^q)$, i.e. according to (S5)

$$\lim_{n\to\infty} \frac{1}{nz_n} \sum_{i=1}^{n} z_i \neq \frac{q}{q+1}.$$

Let $n_k$ $(k = 1, 2, \dots)$ be such that $z_{n_k} = x_{2^{2k+1}}$. Then

$$n_k = 2^{2k+1} + \sum_{i=0}^{k} (2^{2i+1} - 2^{2i}) = 2^{2k+1} + \sum_{i=0}^{k} 2^{2i}$$

$$= 2^{2k+1} + \frac{2^{2k+2}-1}{2^2-1} = \frac{5}{3}2^{2k+1} - \frac{1}{3}. \tag{4.1}$$

We estimate the following means

$$\begin{aligned}
\frac{1}{n_k z_{n_k}} \sum_{i=1}^{n_k} z_i \geq \frac{1}{n_k z_{n_k}} &\left( \sum_{i=1}^{2^{2k+1}} x_i + \sum_{i=2^{2k}+1}^{2^{2k+1}} y_i \right) \\
&= \frac{1}{n_k x_{2^{2k+1}}} \left( \sum_{i=1}^{2^{2k+1}} x_i + \sum_{i=1}^{2^{2k+1}} y_i - \sum_{i=1}^{2^{2k}} y_i \right) \\
&= \frac{2^{2k+1}}{n_k} \frac{1}{2^{2k+1} x_{2^{2k+1}}} \sum_{i=1}^{2^{2k+1}} x_i \\
&\quad + \frac{2^{2k+1}}{n_k} \frac{y_{2^{2k+1}}}{x_{2^{2k+1}}} \frac{1}{2^{2k+1} y_{2^{2k+1}}} \sum_{i=1}^{2^{2k+1}} y_i \\
&\quad - \frac{2^{2k}}{n_k} \frac{y_{2^{2k}}}{x_{2^{2k+1}}} \frac{1}{2^{2k} y_{2^{2k}}} \sum_{i=1}^{2^{2k}} y_i.
\end{aligned} \tag{4.2}$$

Since $X, Y \in \mathcal{U}(x^q)$ then by (S5) we give

$$\lim_{k\to\infty} \frac{1}{2^{2k+1} x_{2^{2k+1}}} \sum_{i=1}^{2^{2k+1}} x_i = \lim_{k\to\infty} \frac{1}{2^{2k+1} y_{2^{2k+1}}} \sum_{i=1}^{2^{2k+1}} y_i$$
$$= \lim_{k\to\infty} \frac{1}{2^{2k} y_{2^{2k}}} \sum_{i=1}^{2^{2k}} y_i = \frac{q}{q+1}.$$

From definition of the set $Y$ and (S2) it follows

$$\lim_{k\to\infty} \frac{y_{2^{2k}}}{x_{2^{2k+1}}} = \lim_{k\to\infty} \frac{x_{2^{2k}}}{x_{2^{2k+1}}} = \lim_{k\to\infty} \frac{x_{2^{2k}}}{x_{2.2^{2k}}} = \frac{1}{2^{\frac{1}{q}}} \leq \frac{1}{2}.$$

Furthermore we have

$$\lim_{k\to\infty} \frac{y_{2^{2k+1}}}{x_{2^{2k+1}}} = \lim_{k\to\infty} \frac{x_{2^{2k+1}-1}}{x_{2^{2k+1}}} = 1,$$

and (4.1) implies

$$\lim_{k\to\infty} \frac{2^{2k+1}}{n_k} = \frac{3}{5}, \quad \lim_{k\to\infty} \frac{2^{2k}}{n_k} = \frac{3}{10}.$$

Then from estimation (4.2) by previously statements we obtain

$$\liminf_{k\to\infty} \frac{1}{n_k z_{n_k}} \sum_{i=1}^{n_k} z_i \geq \left( \frac{3}{5} + \frac{3}{5} \cdot 1 - \frac{3}{10} \cdot \frac{1}{2} \right) \frac{q}{q+1} = \frac{21}{20} \frac{q}{q+1} > \frac{q}{q+1},$$

which it means that $X \cup Y \notin \mathcal{U}(x^q)$. $\square$

However, if we choose such sets $X, Y \in \mathcal{U}(x^q)$ that $X \cap Y \in \mathcal{I}_0$, then holds already the following.

**Theorem 4.2.** *Let $0 < q \leq 1$ and sets $X, Y \in \mathcal{U}(x^q)$ are such that $X \cap Y \in \mathcal{I}_0$. Then $X \cup Y \in \mathcal{U}(x^q)$.*

*Proof.* Let $0 < q \leq 1$, $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$, $Y = \{y_1 < y_2 < \cdots\} \subset \mathbb{N}$. Assume that $X, Y \in \mathcal{U}(x^q)$. According to (S5) and (S3) we have

$$\frac{1}{nx_n}\sum_{i=1}^{n} x_i \to \frac{q}{q+1} \quad \text{and} \quad \frac{1}{ny_n}\sum_{i=1}^{n} y_i \to \frac{q}{q+1} \quad \text{as} \quad n \to \infty, \tag{4.3}$$

and

$$\frac{x_{k+1}}{x_k} \to 1 \quad \text{and} \quad \frac{y_{k+1}}{y_k} \to 1 \quad \text{as} \quad n \to \infty. \tag{4.4}$$

Let $X \cap Y = \{y_{i_1}, y_{i_2}, \ldots, y_{i_n}, \ldots\}$. We denote

$$A(X \cap Y, y_n) = \sum_{y_{n_i} \in [1, y_n]} y_{n_i}.$$

Further, let $X \cup Y = \{z_1 < z_2 < \cdots < z_m < \cdots\}$ and choose sufficiently large $m \in \mathbb{N}$. Let $z_m \in X \cup Y$. If $z_m = y_n$ then

$$x_k \leq y_n < x_{k+1} \text{ and } y_{i_l} \leq y_n < y_{i_{l+1}},$$

for some $k, l \in \mathbb{N}$.

Thus $m = X \cup Y(y_n)$, $X \cap Y(y_n) = l$ and $m = k + n - l$. Then we estimate the value

$$\begin{aligned}
\frac{1}{mz_m}\sum_{i=1}^{m} z_i &= \frac{1}{k+n-l}\frac{1}{y_n}\left(\sum_{i=1}^{n} y_i + \sum_{i=1}^{k} x_i - A(X \cap Y, y_n)\right) \qquad (4.5)\\
&= \frac{n}{k+n-l}\frac{1}{ny_n}\sum_{i=1}^{n} y_i + \frac{k}{k+n-l}\frac{x_k}{y_n}\frac{1}{kx_k}\sum_{i=1}^{k} x_i - \frac{A(X \cap Y, y_n)}{(k+n-l)y_n}\\
&= \frac{k+n}{k+n-l}\frac{1}{ny_n}\sum_{i=1}^{n} y_i + \frac{k}{k+n-l}\left(\frac{x_k}{y_n}\frac{1}{kx_k}\sum_{i=1}^{k} x_i - \frac{1}{ny_n}\sum_{i=1}^{n} y_i\right) - \frac{A(X \cap Y, y_n)}{(k+n-l)y_n}.
\end{aligned}$$

On the other hand

$$\frac{k+n}{k+n-l} = 1 - \frac{X \cap Y(y_n)}{X \cup Y(y_n)},$$

$$0 \leq \frac{A(X \cap Y, y_n)}{(k+n-l)y_n} \leq \frac{X \cap Y(y_n).y_n}{(k+n-l)y_n} = \frac{X \cap Y(y_n)}{X \cup Y(y_n)} \leq \frac{X \cap Y(y_n)}{X(y_n)},$$

and as $m \to \infty$, also $k \to \infty$ and $n \to \infty$. Since from Theorem 4.3 we have

$$\frac{X \cap Y(n)}{X(n)} \to 0 \quad \text{as} \quad n \to \infty,$$

then holds

$$\frac{k+n}{k+n-l} \to 1, \quad \frac{A(X \cap Y, y_n)}{(k+n-l)y_n} \to 0 \quad \text{as} \quad m \to \infty.$$

Furthermore from (4.4) and condition $x_k \le y_n < x_{k+1}$ we obtain

$$\frac{x_k}{y_n} \to 1 \quad \text{as} \quad m \to \infty.$$

Then by (4.3), (4.5) and from the fact, that $\frac{k}{k+n-l}$ is bounded we have

$$\frac{1}{mz_m}\sum_{i=1}^{m} z_i \to \frac{q}{q+1} \quad \text{as} \quad m \to \infty,$$

thus $X \cup Y \in \mathcal{U}(x^q)$.

The proof in the case $z_m = x_k$ and $y_n \le x_k \le y_{n+1}$ is similar. □

In the following theorems we will deal with sets X, Y for which $X \in \mathcal{U}(g_1(x))$ $Y \in \mathcal{U}(g_2(x))$ where $g_1(x) \ne g_2(x)$ and $g_1(x), g_2(x) \in \{c_0(x), x^q\}$.

**Theorem 4.3.** *Let $0 < q \le 1$ and sets $X \in \mathcal{U}(c_0(x))$(it can also be $X \in \mathcal{I}_0$), $Y \in \mathcal{U}(x^q)$. Then*

$$\lim_{n\to\infty} \frac{X(n)}{Y(n)} = 0.$$

*Proof.* Let $0 < q \le 1$, $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$, $Y = \{y_1 < y_2 < \cdots\} \subset \mathbb{N}$. Assume that $X \in \mathcal{U}(c_0(x))$ and $Y \in \mathcal{U}(x^q)$. Then by (S6) and (S7) for sufficiently large $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ we have

$$x_n > n^k \quad \text{and} \quad y_n < n^{\frac{1}{q}+\frac{1}{k}}.$$

Therefore

$$0 \le \frac{X(n)}{Y(n)} < \frac{n^{\frac{1}{k}}}{n^{\frac{qk}{q+k}}} = n^{\frac{1}{k}-\frac{qk}{q+k}},$$

where the exponent for sufficiently large $k$ is negative, since $\frac{1}{k} - \frac{qk}{q+k} \to -q$ as $k \to \infty$. From this and previous estimation follows $\frac{X(n)}{Y(n)} \to 0$ as $n \to \infty$. □

Note that the previous Theorem 4.3 holds even if for the sets $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$, $Y = \{y_1 < y_2 < \cdots\} \subset \mathbb{N}$ we assume that

$$\lim_{n\to\infty} \frac{\log n}{\log x_n} = 0 \text{ (i.e. } X \in \mathcal{I}_0) \quad \text{and} \quad \lim_{n\to\infty} \frac{\log n}{\log y_n} = q.$$

On the other hand we have.

**Corollary 4.4.** *Let $0 < q \le 1$ and sets $X \in \mathcal{U}(c_0(x))$, $Y \in \mathcal{U}(x^q)$. Then*

$$X \cup Y \in \mathcal{U}(x^q).$$

*Proof.* This is a direct corollary of Theorem 4.3 and (S10). □

**Theorem 4.5.** *Let* $0 < q_1 < q_2 \leq 1$ *and sets* $X \in \mathcal{U}(x^{q_1})$, $Y \in \mathcal{U}(x^{q_2})$. *Then*

$$\lim_{n\to\infty} \frac{X(n)}{Y(n)} = 0.$$

*Proof.* Let $0 < q_1 < q_2 \leq 1$, $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$, $Y = \{y_1 < y_2 < \cdots\} \subset \mathbb{N}$. Assume that $X \in \mathcal{U}(x^{q_1})$ and $Y \in \mathcal{U}(x^{q_2})$. Then by (S7) for sufficiently large $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$x_n > n^{\frac{1}{q_1} - \frac{1}{k}} \quad \text{and} \quad y_n < n^{\frac{1}{q_2} + \frac{1}{k}}.$$

Therefore

$$0 \leq \frac{X(n)}{Y(n)} < \frac{n^{\frac{q_1 k}{q_1 + k}}}{n^{\frac{q_2 k}{q_2 + k}}} = n^{\frac{q_1 k}{q_1+k} - \frac{q_2 k}{q_2 + k}},$$

where the exponent for sufficiently large $k$ is negative, since $\frac{q_1 k}{q_1+k} - \frac{q_2 k}{q_2+k} \to q_1 - q_2$ as $k \to \infty$. From this and previous estimation follows $\frac{X(n)}{Y(n)} \to 0$ as $n \to \infty$. □

Note that the previous Theorem 4.5 holds even if for the sets $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$, $Y = \{y_1 < y_2 < \cdots\} \subset \mathbb{N}$ we assume that

$$\lim_{n\to\infty} \frac{\log n}{\log x_n} = q_1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\log n}{\log y_n} = q_2.$$

**Corollary 4.6.** *Let* $0 < q_1 < q_2 \leq 1$ *and sets* $X \in \mathcal{U}(x^{q_1})$, $Y \in \mathcal{U}(x^{q_2})$. *Then*

$$X \cup Y \in \mathcal{U}(x^{q_2}).$$

*Proof.* This is a direct corollary of Theorem 4.5 and result (S10). □

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