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Addendum and corrigenda to the paper "Infinitary superperfect numbers"

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Abstract

We shall give an elementary proof for Lemma 2.4 and correct some errors in Table 1 of the author's paper of the title. Moreover, we shall extend this table up to integers below 2^{32}.

Keywords: Odd perfect numbers, infinitary superperfect numbers, unitary divisors, infinitary divisors, the sum of divisors

MSC: 11A05, 11A25

In p. 215, Lemma 2.4 of the author's paper "Infinitary superperfect numbers", this journal **47** (2017), 211–218, it is stated that, if $p^2+1 = 2q^m$ with $m \geq 2$, then a) m must be a power of 2 and, b) for any given prime q, there exists at most one such m. Here the author owed the former to an old result of Størmer [5] that the equation $x^2 + 1 = 2y^m$ with m odd has only one positive integer solution $(x, y) = (1, 1)$ and the latter to Ljunggren's result [2] that the equation $x^2 + 1 = 2y^n$ has only two positive integer solutions $(x, y) = (1, 1)$ and $(239, 13)$. However, Ljunggren's proof is quite difficult. Steiner and Tzanakis [3] gave a simpler proof, which uses lower bounds for linear forms in logarithms and is still analytic.

We note that the latter fact on $p^2 + 1 = 2q^m$ mentioned above can be proved in a more elementary way. In his earlier paper [4], Størmer proved that, if x, y, A, t are positive integers such that $x^2 + 1 = 2A, y^2 + 1 = 2A^{2^t}$ and $x \pm y \equiv 0 \pmod{A}$, then $(x, y, A, 1) = (3, 7, 5, 2)$ or $(5, 239, 13, 2)$. We can easily see that if A is prime and $x^2 + 1 \equiv y^2 + 1 \equiv 0 \pmod{A}$, then we must have $x \pm y \equiv 0 \pmod{A}$. Now the latter fact for $p^2 + 1 = 2q^m$ mentioned above immediately follows. Moreover, the above-mentioned result for $x^2 + 1 = 2y^m$ with m odd had also already been proved in [4]. Hence, the above statement follows from results in [4]. The most advanced method used in [4] is classical arithmetic in Gaussian integers.

Moreover, we can prove the latter fact on $p^2+1=2q^m$ in a completely elementary way. Applying Théorème 1 of Størmer [5] to $x^2-2q^2y^2=-1$, we see that if $(x,y)=(x_0,y_0)$ is a solution of $x^2-2q^2y^2=-1$ and y_0 is a power of q, then (x_0,y_0) must be the smallest solution of $x^2-2q^2y^2=-1$. Hence, for any give prime q, $x^2+1=2q^{2^t}$ can have at most one positive integer solution (x,t).

Anothor elementary way is to use a theorem of Carmichael [1] (a simpler proof is given by Yabuta [6]). Let $(x,y)=(x_1,y_1)$ be the smallest solution of $x^2-2y^2=-1$ with y divisible by q. Carmichael's theorem applied to the Pell sequence implies that, if $(x,y)=(x_2,y_2)$ is another solution of $x^2-2y^2=-1$ with y divisible by q, then y_2 must have a prime factor other than q. Hence, y_2 cannot be a power of q.

Corrigenda to p. 213, Table 1, the right row for N:

- The fifth column should be $856800=2^5\cdot 3^2\cdot 5^2\cdot 7\cdot 17$.
- The sixth column should be $1321920=2^6\cdot 3^5\cdot 5\cdot 17$.
- The twelfth column should be $30844800=2^7\cdot 3^4\cdot 5^2\cdot 7\cdot 17$.
- Moreover, we extended our search limit to 2^{32} and found four more integers N dividing $\sigma_\infty(\sigma_\infty(N))$:

N	k
$1304784000=2^7\cdot 3^2\cdot 5^3\cdot 13\cdot 17\cdot 41$	7
$1680459462=2^9\cdot 3^3\cdot 11\cdot 43\cdot 257$	5
$4201148160=2^8\cdot 3^3\cdot 5\cdot 11\cdot 43\cdot 257$	6
$4210315200=2^6\cdot 3^5\cdot 5^2\cdot 7^2\cdot 13\cdot 17$	8

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