Generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric $\phi$-symmetric connection

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Abstract

Jin [13] introduced the notion of non-metric $\phi$-symmetric connection on semi-Riemannian manifolds and studied lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a non-metric $\phi$-symmetric connection [12]. We study further the geometry of this subject. In this paper, we study generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric $\phi$-symmetric connection.

Keywords: non-metric $\phi$-symmetric connection, generic lightlike submanifold, indefinite trans-Sasakian structure

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1. Introduction

The notion of non-metric $\phi$-symmetric connection on indefinite almost contact manifolds or indefinite almost complex manifolds was introduced by Jin [12, 13]. Here we quote Jin's definition in itself as follows:

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A linear connection \( \nabla \) on a semi-Riemannian manifold \((\bar{M}, \bar{g})\) is called a non-metric \( \phi \)-symmetric connection if it and its torsion tensor \( \bar{T} \) satisfy

\[
(\nabla_X \bar{g})(Y, Z) = -\theta(Y) \phi(X, Z) - \theta(Z) \phi(X, Y), \quad (1.1)
\]
\[
\bar{T}(X, Y) = \theta(Y) JX - \theta(X) JY, \quad (1.2)
\]

where \( \phi \) and \( J \) are tensor fields of types \((0, 2)\) and \((1, 1)\) respectively, and \( \theta \) is an 1-form associated with a smooth vector field \( \zeta \) by \( \theta(X) = \bar{g}(X, \zeta) \). Throughout this paper, we denote by \( \bar{X}, \bar{Y} \) and \( \bar{Z} \) the smooth vector fields on \( \bar{M} \).

In case \( \phi = \bar{g} \) in (1.1), the above non-metric \( \phi \)-symmetric connection reduces to so-called the quarter-symmetric non-metric connection. Quarter-symmetric non-metric connection was introduced by S. Golad [7], and then, studied by many authors [2, 4, 19, 20]. In case \( \phi = \bar{g} \) in (1.1) and \( J = I \) in (1.2), the above non-metric \( \phi \)-symmetric connection reduces to so-called the semi-symmetric non-metric connection. Semi-symmetric non-metric connection was introduced by Ageshe and Chafle [1] and later studied by many geometers.

The notion of generic lightlike submanifolds on indefinite almost contact manifolds or indefinite almost complex manifolds was introduced by Jin-Lee [14] and later, studied by Duggal-Jin [6], Jin [9, 10] and Jin-Lee [16] and several geometers. We cite Jin-Lee’s definition in itself as follows:

A lightlike submanifold \( M \) of an indefinite almost contact manifold \( \bar{M} \) is said to be generic if there exists a screen distribution \( S(TM) \) on \( M \) such that

\[
J(S(TM)\perp) \subset S(TM), \quad (1.3)
\]

where \( S(TM)\perp \) is the orthogonal complement of \( S(TM) \) in the tangent bundle \( T\bar{M} \) on \( \bar{M} \), i.e., \( T\bar{M} = S(TM) \oplus_{\text{orth}} S(TM)\perp \). The geometry of generic lightlike submanifolds is an extension of that of lightlike hypersurfaces and half lightlike submanifolds of codimension 2. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

The notion of trans-Sasakian manifold, of type \((\alpha, \beta)\), was introduced by Oubina [18]. If \( \bar{M} \) is a semi-Riemannian manifold with a trans-Sasakian structure of type \((\alpha, \beta)\), then \( \bar{M} \) is called an indefinite trans-Sasakian manifold of type \((\alpha, \beta)\). Indefinite Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of indefinite trans-Sasakian manifolds such that

\[
\alpha = 1, \beta = 0; \quad \alpha = 0, \beta = 1; \quad \alpha = \beta = 0, \quad \text{respectively.}
\]

In this paper, we study generic lightlike submanifolds \( M \) of an indefinite trans-Sasakian manifold \( M = (\bar{M}, J, \zeta, \theta, \bar{g}) \) with a non-metric \( \phi \)-symmetric connection, in which the tensor field \( J \) in (1.2) is identical with the indefinite almost contact structure tensor field \( J \) of \( \bar{M} \), the tensor field \( \phi \) in (1.1) is identical with the fundamental 2-form associated with \( J \), that is,

\[
\phi(\bar{X}, \bar{Y}) = \bar{g}(J\bar{X}, \bar{Y}), \quad (1.4)
\]
and the 1-form $\theta$, defined by (1.1) and (1.2), is identical with the structure 1-form $\theta$ of the indefinite almost contact metric structure $(J, \zeta, \theta, \bar{g})$ of $\bar{M}$.

Remark 1.1. Denote $\nabla$ by the unique Levi-Civita connection of $(\bar{M}, \bar{g})$ with respect to the metric $\bar{g}$. It is known [13] that a linear connection $\nabla$ on $\bar{M}$ is non-metric $\phi$-symmetric connection if and only if it satisfies

$$\nabla_X Y = \tilde{\nabla}_X Y + \theta(Y) \bar{J}X. \quad (1.5)$$

For the rest of this paper, by the non-metric $\phi$-symmetric connection we shall mean the non-metric $\phi$-symmetric connection defined by (1.5).

2. Non-metric $\phi$-symmetric connections

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an indefinite trans-Sasakian manifold if there exist (1) a structure set \{J, $\zeta$, $\theta$, $\bar{g}$\}, where $J$ is a (1,1)-type tensor field, $\zeta$ is a vector field and $\theta$ is a 1-form such that

$$J^2X = -X + \theta(X)\zeta, \quad \theta(\zeta) = 1, \quad \theta(X) = \epsilon \bar{g}(X, \zeta), \quad (2.1)$$

$$\theta \circ J = 0, \quad \bar{g}(JX, JY) = \bar{g}(X, Y) - \epsilon \theta(X)\theta(Y),$$

(2) two smooth functions $\alpha$ and $\beta$, and a Levi-Civita connection $\tilde{\nabla}$ such that

$$(\tilde{\nabla}_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \epsilon \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \epsilon \theta(Y)JX\},$$

where $\epsilon$ denotes $\epsilon = 1$ or $-1$ according as $\zeta$ is spacelike or timelike respectively. \{J, $\zeta$, $\theta$, $\bar{g}$\} is called an indefinite trans-Sasakian structure of type $(\alpha, \beta)$.

In the entire discussion of this article, we shall assume that the vector field $\zeta$ is a spacelike one, i.e., $\epsilon = 1$, without loss of generality.

Let $\nabla$ be a non-metric $\phi$-symmetric connection on $(\bar{M}, \bar{g})$. Using (1.5) and the fact that $\theta \circ J = 0$, the equation in the item (2) is reduced to

$$\nabla_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \theta(Y)X\} \quad (2.2)$$

$$+ \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)JX\} + \theta(Y)\{X - \theta(X)\zeta\}.$$

Replacing $Y$ by $\zeta$ to (2.2) and using $J\zeta = 0$ and $\theta(\tilde{\nabla}_X \zeta) = 0$, we obtain

$$\nabla_X \zeta = -(\alpha - 1)JX + \beta\{X - \theta(X)\zeta\}. \quad (2.3)$$

Let $(M, g)$ be an $m$-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $(\bar{M}, \bar{g})$ of dimension $(m + n)$. Then the radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$ on $M$ is a subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$, of rank $r \leq \min\{m, n\}$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $\text{Rad}(TM)$.
in $TM$ and $TM^\perp$ respectively, which are called the screen distribution and the co-screen distribution of $M$, such that

$$TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. Also denote by (2.1) the $i$-th equation of (2.1). We use the same notations for any others. Let $X, Y, Z$ and $W$ be the vector fields on $M$, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots \in \{1, \ldots, r\}, \quad a, b, c, \ldots \in \{r + 1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to $TM$ in $T\bar{M}_M$ and $TM^\perp$ in $S(TM)^\perp$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)|_{\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of $M$, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\text{Rad}(TM)|_{\mathcal{U}}$. Then we have

$$T\bar{M} = TM \oplus tr(TM) = \{\text{Rad}(TM) \oplus tr(TM)\} \oplus_{\text{orth}} S(TM)$$

$$= \{\text{Rad}(TM) \oplus ltr(TM)\} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp).$$

We say that a lightlike submanifold $M = (M, g, S(TM), S(TM^\perp))$ of $\bar{M}$ is

1. $r$-lightlike submanifold if $1 \leq r < \min\{m, n\};$

2. co-isotropic submanifold if $1 \leq r = n < m;$

3. isotropic submanifold if $1 \leq r = m < n;$

4. totally lightlike submanifold if $1 \leq r = m = n.$

The above three classes (2)~(4) are particular cases of the class (1) as follows:

$$S(TM^\perp) = \{0\}, \quad S(TM) = \{0\}, \quad S(TM) = S(TM^\perp) = \{0\}$$

respectively. The geometry of $r$-lightlike submanifolds is more general than that of the other three types. For this reason, we consider only $r$-lightlike submanifolds $M$, with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, F_{r+1}, \ldots, F_m, E_{r+1}, \ldots, E_n\},$$

where $\{F_{r+1}, \ldots, F_m\}$ and $\{E_{r+1}, \ldots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^\perp)$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

In the sequel, we shall assume that $\zeta$ is tangent to $M$. Călin [5] proved that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$ which we assumed in this paper. Let $P$
be the projection morphism of $TM$ on $S(TM)$. Then the local Gauss-Weingarten formulae of $M$ and $S(TM)$ are given respectively by

$$\nabla_X Y = \nabla_X Y + \sum_{i=1}^{r} h^i(X,Y) N_i + \sum_{a=r+1}^{n} h^a(X,Y) E_a,$$  \hspace{1cm} (2.4)

$$\bar{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{a=r+1}^{n} \rho_{ia}(X) E_a;$$  \hspace{1cm} (2.5)

$$\bar{\nabla}_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \lambda_{ai}(X) N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) E_b;$$  \hspace{1cm} (2.6)

$$\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^{r} h^i(X,PY) \xi_i,$$  \hspace{1cm} (2.7)

$$\nabla_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X) \xi_j;$$  \hspace{1cm} (2.8)

where $\nabla$ and $\nabla^*$ are induced linear connections on $M$ and $S(TM)$ respectively, $h^i$ and $h^a$ are called the local second fundamental forms on $M$, $h^*_i$ are called the local second fundamental forms on $S(TM)$. $A_{N_i}$, $A_{E_a}$ and $A^*_{\xi_i}$ are called the shape operators, and $\tau_{ij}$, $\rho_{ia}$, $\lambda_{ai}$ and $\sigma_{ab}$ are 1-forms.

Let $M$ be a generic lightlike submanifold of $\tilde{M}$. From (1.3) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions $H_o$ and $H$ with respect to $J$, i.e., $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{\text{orth}} J(S(TM^\perp)) \oplus_{\text{orth}} H_o,$$

$$H = Rad(TM) \oplus_{\text{orth}} J(Rad(TM)) \oplus_{\text{orth}} H_o.$$  \hspace{1cm} (2.9)

In this case, the tangent bundle $TM$ on $M$ is decomposed as follows:

$$TM = H \oplus J(ltr(TM)) \oplus_{\text{orth}} J(S(TM^\perp)).$$  \hspace{1cm} (2.9)

Consider local null vector fields $U_i$ and $V_i$ for each $i$, local non-null unit vector fields $W_a$ for each $a$, and their 1-forms $u_i$, $v_i$ and $w_a$ defined by

$$U_i = -JN_i, \hspace{1cm} V_i = -J\xi_i, \hspace{1cm} W_a = -JE_a;$$  \hspace{1cm} (2.10)

$$u_i(X) = g(X,V_i), \hspace{1cm} v_i(X) = g(X,U_i), \hspace{1cm} w_a(X) = \epsilon_a g(X,W_a).$$  \hspace{1cm} (2.11)

Denote by $S$ the projection morphism of $TM$ on $H$ and by $F$ the tensor field of type $(1,1)$ globally defined on $M$ by $F = J \circ S$. Then $JX$ is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X) N_i + \sum_{a=r+1}^{n} w_a(X) E_a.$$  \hspace{1cm} (2.12)
Applying $J$ to (2.12) and using (2.1) and (2.10), we have

$$F^2X = -X + \theta(X)\zeta + \sum_{i=1}^{r} u_i(X)Ui + \sum_{a=r+1}^{n} w_a(X)Wa.$$  

(2.13)

In the following, we say that $F$ is the structure tensor field on $M$.

3. Structure equations

Let $\tilde{M}$ be an indefinite trans-Sasakian manifold with a non-metric $\phi$-symmetric connection $\tilde{\nabla}$. In the following, we shall assume that $\zeta$ is tangent to $M$. Călin [5] proved that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$ which we assumed in this paper. Using (1.1), (1.2), (1.4), (2.4) and (2.12), we see that

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \{h^i(Y, X)\eta_i(Z) + h^i(X, Z)\eta_i(Y)\} - \theta(Y)\phi(X, Z) - \theta(Z)\phi(X, Y),$$

(3.1)

$$T(X, Y) = \theta(Y)FX - \theta(X)FY,$$

(3.2)

$$h^i(Y, X) - h^i(X, Y) = \theta(Y)u_i(X) - \theta(X)u_i(Y),$$

(3.3)

$$h^a_s(Y, X) - h^a_s(X, Y) = \theta(Y)w_a(X) - \theta(X)w_a(Y),$$

(3.4)

$$\phi(X, \xi_i) = u_i(X), \quad \phi(X, N_i) = v_i(X), \quad \phi(X, E_a) = w_a(X),$$

(3.5)

$$\phi(X, V_i) = 0, \quad \phi(X, U_i) = -\eta_i(X), \quad \phi(X, W_a) = 0,$$

for all $i$ and $a$, where $\eta_i$’s are 1-forms such that $\eta_i(X) = \bar{g}(X, N_i)$.

From the facts that $h^i(Y, X) = \bar{g}(\tilde{\nabla}_X Y, \xi_i)$ and $\epsilon_a h^a_s(Y, X) = \bar{g}(\tilde{\nabla}_X Y, E_a)$, we know that $h^i$ and $h^a_s$ are independent of the choice of $S(TM)$. Applying $\tilde{\nabla}_X$ to $\bar{g}(\xi_i, \xi_j) = 0, \bar{g}(\xi_i, E_a) = 0, \bar{g}(N_i, N_j) = 0, \bar{g}(N_i, E_a) = 0$ and $\bar{g}(E_a, E_b) = \epsilon\delta_{ab}$ by turns and using (1.1) and (2.4) ~ (2.6), we obtain

$$h^i(X, \xi_j) + h^j(X, \xi_i) = 0, \quad h^a_s(X, \xi_j) = -\epsilon_a \lambda_{ai}(X),$$

$$\eta_j(A_{N_i} X) + \eta_i(A_{N_j} X) = 0, \quad \eta_i(A_{E_a} X) = \epsilon_a \rho_{ia}(X),$$

(3.6)

$$\epsilon_b \sigma_{ab} + \epsilon_a \sigma_{ba} = 0; \quad h^i_b(X, \xi_j) = 0, \quad h^i_a(X, \xi_k) = 0, \quad A^i_{a\xi_i} = 0.$$

Definition 3.1. We say that a lightlike submanifold $M$ of $\tilde{M}$ is

(1) irrotational [17] if $\tilde{\nabla}_X \xi_i \in \Gamma(TM)$ for all $i \in \{1, \cdots, r\}$,

(2) solenoidal [15] if $A_{Wa}$ and $A_{Ni}$ are $S(TM)$-valued for all $\alpha$ and $i$.

From (2.4) and (3.1) 2, the item (1) is equivalent to

$$h^i_j(X, \xi_i) = 0, \quad h^a_s(X, \xi_i) = \lambda_{ai}(X) = 0.$$
By using (3.1)$_4$, the item (2) is equivalent to
\[ \eta_j(A_{N_i} X) = 0, \quad \rho_{ia}(X) = \eta_i(A_{E_a} X) = 0. \]

The local second fundamental forms are related to their shape operators by
\[ h^\ell_i(X, Y) = g(A_{\xi_i}^*, X, Y) + \theta(Y)u_i(X) - \sum_{k=1}^r h^\ell_{ik}(X, \xi_k)\eta_k(Y), \quad (3.7) \]
\[ \epsilon_a h^a_s(X, Y) = g(A_{E_a}, X, Y) + \theta(Y)w_a(X) - \sum_{k=1}^r \lambda_{ak}(X)\eta_k(Y), \quad (3.8) \]
\[ h^s_i(X, PY) = g(A_{N_i}, X, PY) + \theta(PY)v_i(X). \quad (3.9) \]

Replacing $Y$ by $\zeta$ to (2.4) and using (2.3), (2.12), (3.7) and (3.8), we have
\[ \nabla_X \zeta = -(\alpha - 1)FX + \beta(X - \theta(X))\zeta, \quad (3.10) \]
\[ \theta(A_{\xi_i}^* X) = -\alpha u_i(X), \quad h^\ell_i(X, \zeta) = -(\alpha - 1)u_i(X), \quad (3.11) \]
\[ \theta(A_{E_a} X) = -(\epsilon_a(\alpha - 1) + 1)w_a(X), \quad (3.12) \]
\[ h^a_a(X, \zeta) = -(\alpha - 1)w_a(X). \]

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, N_i)$ and using (2.3), (2.5) and (3.9), we have
\[ \theta(A_{N_i} X) = -\alpha v_i(X) + \beta \eta_i(X), \quad (3.13) \]
\[ h^*_i(X, \zeta) = -(\alpha - 1)v_i(X) + \beta \eta_i(X). \]

Applying $\bar{\nabla}_X$ to (2.10)$_{1,2,3}$ and (2.12) by turns and using (2.2), (2.4) $\sim$ (2.8), (2.10) $\sim$ (2.12) and (3.7) $\sim$ (3.9), we have
\[ h^\ell_j(X, U_i) = h^\ell_i(X, V_j), \quad \epsilon_a h^*_i(X, W_a) = h^*_a(X, U_i), \]
\[ h^\ell_j(X, V_i) = h^\ell_i(X, V_j), \quad \epsilon_a h^\ell_i(X, W_a) = h^a_a(X, V_i), \quad (3.14) \]
\[ \epsilon_b h^*_b(X, W_a) = \epsilon_a h^*_a(X, W_b), \]
\[ \nabla_X U_i = F(A_{N_i} X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{a=r+1}^n \rho_{ia}(X)W_a \]
\[ - \{\alpha \eta_i(X) + \beta v_i(X)\}\zeta, \]
\[ \nabla_X V_i = F(A_{\xi_i}^* X) - \sum_{j=1}^r \tau_{ji}(X)V_j + \sum_{j=1}^r h^\ell_j(X, \xi_i)U_j \]
\[ - \sum_{a=r+1}^n \epsilon_a \lambda_{ai}(X)W_a - \beta u_i(X)\zeta, \]
\[ \nabla_X W_a = F(A_{E_a} X) + \sum_{i=1}^r \lambda_{ai}(X)U_i + \sum_{b=r+1}^n \sigma_{ab}(X)W_b \quad (3.17) \]
\[
(\nabla_X F)(Y) = \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X \quad (3.18)
\]
\[
- \sum_{i=1}^{r} h_i^f(X,Y)U_i - \sum_{a=r+1}^{n} h_a^*(X,Y)W_a \\
+ \{\alpha g(X,Y) + \beta \bar{g}(JX,Y) - \theta(X)\theta(Y)\} \zeta \\
- (\alpha - 1)\theta(Y)X - \beta \theta(Y)FX,
\]
\[
(\nabla_X u_i)(Y) = - \sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) - \sum_{a=r+1}^{n} w_a(Y)\lambda_{ai}(X) \quad (3.19)
\]
\[
- h_i^f(X, FY) - \beta \theta(Y)u_i(X),
\]
\[
(\nabla_X v_i)(Y) = \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) \quad (3.20)
\]
\[
+ \sum_{j=r+1}^{r} u_j(Y)\eta_i(A_{N_j}X) - g(A_{N_i}X, FY) \\
- (\alpha - 1)\theta(Y)\eta_i(X) - \beta \theta(Y)v_i(X).
\]

**Theorem 3.2.** There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric \(\phi\)-symmetric connection such that \(\zeta\) is tangent to \(M\) and \(F\) satisfies the following equation:

\[
(\nabla_X F)Y = (\nabla_Y F)X, \quad \forall X, Y \in \Gamma(TM).
\]

**Proof.** Assume that \((\nabla_X F)Y - (\nabla_Y F)X = 0\). From (3.18) we obtain

\[
\sum_{i=1}^{r} \{u_i(Y)A_{N_i}X - u_i(X)A_{N_i}Y\} \quad (3.21)
\]
\[
+ \sum_{a=r+1}^{n} \{w_a(Y)A_{E_a}X - w_a(X)A_{E_a}Y\} - 2\beta\bar{g}(X, JY)\zeta \\
+ \{\theta(X)u_i(Y) - \theta(Y)u_i(X)\}U_i + \{\theta(X)w_a(Y) - \theta(Y)w_a(X)\}W_a \\
+ (\alpha - 1)\{\theta(X)Y - \theta(Y)X\} + \beta\{\theta(X)FY - \theta(Y)FX\} = 0.
\]

Taking the scalar product with \(\zeta\) and using (3.12)\(_1\) and (3.13)\(_1\), we have

\[
\alpha \sum_{i=1}^{r} \{u_i(Y)v_i(X) - u_i(X)v_i(Y)\} \\
= \beta \sum_{i=1}^{r} \{u_i(Y)\eta_i(X) - u_i(X)\eta_i(Y)\} - 2\beta\bar{g}(X, JY).
\]
Taking $X = V_j$, $Y = U_j$ and $X = \xi_j$, $Y = U_j$ to this equation by turns, we obtain $\alpha = 0$ and $\beta = 0$, respectively. Taking $X = \xi_i$ to (3.21), we have

$$\theta(X)\xi_i + \sum_{j=1}^{r} u_j(X)A_{N_j}\xi_i + \sum_{a=r+1}^{n} w_a(X)A_{E_a}\xi_i = 0.$$ 

Taking $X = U_k$ and $X = W_k$ to this equation, we have

$$A_{N_k}\xi_i = 0, \quad A_{E_b}\xi_i = 0.$$ 

Therefore, we get $\theta(X)\xi_i = 0$. It follows that $\theta(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction to $\theta(\zeta) = 1$. Thus we have our theorem. \hfill \Box

**Corollary 3.3.** There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric $\phi$-symmetric connection such that $\zeta$ is tangent to $M$ and $F$ is parallel with respect to the connection $\nabla$.

**Theorem 3.4.** Let $M$ be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a non-metric $\phi$-symmetric connection such that $\zeta$ is tangent to $M$. If $U_i$s or $V_i$s are parallel with respect to $\nabla$, then $\alpha = \beta = 0$, i.e., $\bar{M}$ is an indefinite cosymplectic manifold. Furthermore, if $U_i$ is parallel, $M$ is solenoidal and $\tau_{ij} = 0$, if $V_i$ is parallel, $M$ is irrotational and $\tau_{ij} = 0$.

**Proof.** (1) If $U_i$ is parallel with respect to $\nabla$, then, taking the scalar product with $\zeta$, $V_j$, $W_a$, $U_j$ and $N_j$ to (3.15) such that $\nabla_X U_i = 0$ respectively, we get

$$\alpha = \beta = 0, \quad \tau_{ij} = 0, \quad \rho_{ia} = 0, \quad \eta_j(A_{N_i}X) = 0, \quad h_i^*(X, U_j) = 0. \quad (3.22)$$

As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. As $\rho_{ia} = 0$ and $\eta_j(A_{N_i}X) = 0$, $M$ is solenoidal.

(2) If $V_i$ is parallel with respect to $\nabla$, then, taking the scalar product with $\zeta$, $U_j$, $V_j$, $W_a$ and $N_j$ to (3.16) with $\nabla_X V_i = 0$ respectively, we get

$$\beta = 0, \quad \tau_{ji} = 0, \quad h_j^\ell(X, \xi_i) = 0, \quad \lambda_{ai} = 0, \quad h_i^\ell(X, U_j) = 0. \quad (3.23)$$

As $h_j^\ell(X, \xi_i) = 0$ and $\lambda_{ai} = 0$, $M$ is irrotational.

As $h_i^\ell(X, U_j) = 0$, we get $h_i^\ell(\zeta, U_j) = 0$. Taking $X = U_j$ and $Y = \zeta$ to (3.3), we get $h_i^\ell(U_j, \zeta) = \delta_{ij}$. On the other hand, replacing $X$ by $U$ to (3.12)$_1$, we have $h_i^\ell(U_j, \zeta) = -(\alpha - 1)\delta_{ij}$. It follows that $\alpha = 0$. Since $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. \hfill \Box

**4. Recurrent and Lie recurrent structure tensors**

**Definition 4.1.** The structure tensor field $F$ of $M$ is said to be
(1) recurrent [11] if there exists a 1-form $\varpi$ on $M$ such that
\[(\nabla_X F)Y = \varpi(X)FY,\]

(2) Lie recurrent [11] if there exists a 1-form $\vartheta$ on $M$ such that
\[(\mathcal{L}_X F)Y = \vartheta(X)FY,\]

where $\mathcal{L}_X$ denotes the Lie derivative on $M$ with respect to $X$, that is,
\[(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \tag{4.1}\]

In case $\vartheta = 0$, i.e., $\mathcal{L}_X F = 0$, we say that $F$ is Lie parallel.

**Theorem 4.2.** There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric $\phi$-symmetric connection such that $\zeta$ is tangent to $M$ and the structure tensor field $F$ is recurrent.

**Proof.** Assume that $F$ is recurrent. From (3.18), we obtain
\[
\varpi(X)FY = \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X
- \sum_{i=1}^{r} h_i^e(X, Y)U_i - \sum_{a=r+1}^{n} h_a^e(X, Y)W_a
+ \{\alpha g(X, Y) + \beta \bar{g}(JX, Y) - \theta(X)\theta(Y)\} \zeta
- (\alpha - 1)\theta(Y)X - \beta \theta(Y)FX.
\]
Replacing $Y$ by $\xi_j$ to this and using the fact that $F\xi_j = -V_j$, we get
\[
\varpi(X)V_j = \sum_{k=1}^{r} h_k^e(X, \xi_j)U_k + \sum_{b=r+1}^{n} h_b^e(X, \xi_j)W_b - \beta u_j(X)\zeta.
\]
Taking the scalar product with $U_j$, we get $\varpi = 0$. It follows that $F$ is parallel with respect to $\nabla$. By Corollary 3.2, we have our theorem. $\square$

**Theorem 4.3.** Let $M$ be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a non-metric $\phi$-symmetric connection such that $\zeta$ is tangent to $M$ and $F$ is Lie recurrent. Then we have the following results:

1. $F$ is Lie parallel,
2. the function $\alpha$ satisfies $\alpha = 0$,
3. $\tau_{ij}$ and $\rho_{ia}$ satisfy $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,
\[
\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)g(A_{N_k}V_j, N_i) - \beta \theta(X)\delta_{ij}.
\]
Proof. (1) Using (2.13), (3.2), (3.18), (4.1) and the fact that $\theta \circ F = 0$, we get
\[ \vartheta(X)FY = -\nabla FY X + F\nabla Y X \]  
(4.2)

Taking the scalar product with $\xi_j$ and then, $Y$ by $V_j$ to (4.2), respectively, we have
\[ -\vartheta(X)V_j = \nabla V_j X + F\nabla \xi_j X \]  
(4.3)

Taking the scalar product with $U_i$ to (4.3) and $N_i$ to (4.4) respectively, we get
\[ -\delta_{ij} \vartheta(X) = g(\nabla \xi_j X, U_i) - \bar{g}(\nabla \xi_j X, N_i), \]
\[ \delta_{ij} \vartheta(X) = g(\nabla V_j X, U_i) - \bar{g}(\nabla \xi_j X, N_i). \]

Comparing these two equations, we get $\vartheta = 0$. Thus $F$ is Lie parallel.

(2) Taking the scalar product with $\zeta$ to (4.4), we get $g(\nabla \xi_j X, \zeta) = \alpha u_j(X)$. Taking $X = U_i$ to this result and using (3.15), we obtain $\alpha = 0$.

(3) Taking the scalar product with $N_i$ to (4.3) such that $X = W_a$ and using (3.4), (3.6)4, (3.8) and (3.17), we get $h_a^i(U_i, V_j) = \rho_{ia}(\xi_j)$. On the other hand, taking the scalar product with $W_a$ to (4.4) such that $X = U_i$ and using (3.15), we have $h_a^i(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h_a^i(U_i, V_j) = 0$.

Taking the scalar product with $U_i$ to (4.3) such that $X = W_a$ and using (3.4), (3.6)2, 4, (3.8) and (3.17), we get $\epsilon_a \rho_{ia}(V_j) = \lambda_{aj}(U_i)$. On the other hand, taking the scalar product with $W_a$ to (4.3) such that $X = U_i$ and using (3.1)2 and (3.15), we get $\epsilon_a \rho_{ia}(V_j) = -\lambda_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = \lambda_{aj}(U_i)$.

Taking the scalar product with $V_i$ to (4.3) such that $X = W_a$ and using (3.4), (3.6)2, (3.14)4 and (3.17), we obtain $\lambda_{ai}(V_j) = -\lambda_{aj}(V_i)$. On the other hand, taking the scalar product with $W_a$ to (4.3) such that $X = V_i$ and using (3.6)2 and (3.16), we have $\lambda_{ai}(V_j) = \lambda_{aj}(V_i)$. Thus we obtain $\lambda_{ai}(V_j) = 0$. 

\[ \lambda_{ai}(V_j) = \lambda_{aj}(V_i). \]
Taking the scalar product with $W_a$ to (4.3) such that $X = \xi_i$ and using (2.8), (3.3), (3.6) and (3.7), we get $h^i_\ell (V_j, W_a) = \lambda_{ai} (\xi_j)$. On the other hand, taking the scalar product with $V_i$ to (4.4) such that $X = W_a$ and using (3.3) and (3.7), we get $h^i_\ell (V_j, W_a) = -\lambda_{ai} (\xi_j)$. Thus $\lambda_{ai} (\xi_j) = 0$ and $h^i_\ell (V_j, W_a) = 0$.

Summarizing the above results, we obtain

$$
\rho_{ia} (\xi_j) = 0, \quad \rho_{ia} (V_j) = 0, \quad \lambda_{ai} (U_j) = 0, \quad \lambda_{ai} (V_j) = 0, \quad \lambda_{ai} (\xi_j) = 0. \tag{4.5}
$$

$\rho^a_i (U_i, V_j) = h^i_\ell (U_i, W_a) = 0, \quad h^i_\ell (V_j, W_a) = h^a_i (V_j, V_i) = 0.$

Taking the scalar product with $N_i$ to (4.2) and using (3.14), we have

$$
- \bar{g} (\nabla_{FY} X, N_i) + g (\nabla_Y X, U_i) - \beta \theta (Y) v_i (X) \tag{4.6}
$$

$$
+ \sum_{k=1}^r \epsilon_k (Y) \bar{g} (A_{N_k} X, N_i) + \sum_{a=r+1}^n \epsilon_a w_a (Y) \rho_{ia} (X) = 0.
$$

Replacing $X$ by $V_j$ to (4.6) and using (3.7), (3.16) and (4.5), we have

$$
h^i_\ell (FX, U_i) + \tau_{ij} (X) + \beta \theta (X) \delta_{ij} = \sum_{k=1}^r \epsilon_k (X) \bar{g} (A_{N_k} V_j, N_i). \tag{4.7}
$$

Replacing $X$ by $\xi_j$ to (4.6) and using (2.8), (3.7) and (4.5), we have

$$
h^i_\ell (X, U_i) = \sum_{k=1}^r \epsilon_k (X) \bar{g} (A_{N_k} \xi_j, N_i) + \tau_{ij} (FX). \tag{4.8}
$$

Taking $X = U_k$ to this equation and using (3.14), we have

$$
h^i_\ell (U_k, V_j) = \bar{g} (A_{N_k} \xi_j, N_i). \tag{4.9}
$$

Taking $X = U_i$ to (4.2) and using (2.13), (3.3), (3.4) and (3.15), we get

$$
\sum_{k=1}^r \epsilon_k (Y) A_{N_k} U_i + \sum_{a=r+1}^n \epsilon_a (Y) A_{E_a} U_i - A_{N_i} Y \tag{4.10}
$$

$$
- F (A_{N_i} FY) - \sum_{j=1}^r \tau_{ij} (FY) U_j - \sum_{a=r+1}^n \rho_{ia} (FY) W_a = 0.
$$

Taking the scalar product with $V_j$ to (4.10) and using (3.8), (3.9), (3.14), (4.5) and (4.9), we get

$$
h^i_\ell (X, U_i) = - \sum_{k=1}^r \epsilon_k (X) \bar{g} (A_{N_k} \xi_j, N_i) - \tau_{ij} (FX).
$$

Comparing this equation with (4.8), we obtain

$$
\tau_{ij} (FX) + \sum_{k=1}^r \epsilon_k (X) \bar{g} (A_{N_k} \xi_j, N_i) = 0.
$$
Replacing $X$ by $U_h$ to this equation, we have $\bar{g}(A_{N_k} \xi_j, N_i) = 0$. Therefore,
\[ \tau_{ij}(FX) = 0, \quad h^\ell_j(FX, U_i) = 0. \] (4.11)
Taking $X = FY$ to (4.11)2, we get $h^\ell_j(FX, U_i) = 0$. Thus (4.7) is reduced to
\[ \tau_{ij}(X) = \sum_{k=1}^r u_k(X)\bar{g}(A_{N_k} V_j, N_i) - \beta(X)\delta_{ij}. \]

Taking the scalar product with $U_j$ to (4.10) such that $Y = W_a$ and using (3.4), (3.8), (3.9) and (3.14)2, we have
\[ h_i^a(W_a, U_j) = \epsilon_a h_i^a(U_i, U_j) = \epsilon_a h_i^a(U_j, U_i) = h_i^a(U_j, W_a). \] (4.12)

Taking the scalar product with $W_a$ to (4.10), we have
\[ \epsilon_a \rho_{ia}(FY) = -h_i^a(Y, W_a) \]
\[ + \sum_{k=1}^r u_k(Y)h_k^a(U_i, W_a) + \sum_{b=r+1}^n \epsilon_b w_b(Y)h_b^a(U_i, W_a). \]

Taking the scalar product with $U_i$ to (4.2) and then, taking $X = W_a$ and using (3.4), (3.6)4, (3.8), (3.9), (3.14)2, (3.17) and (4.12), we obtain
\[ \epsilon_a \rho_{ia}(FY) = h_i^a(Y, W_a) \]
\[ - \sum_{k=1}^r u_k(Y)h_k^a(U_i, W_a) - \sum_{b=r+1}^n \epsilon_b w_b(Y)h_b^a(U_i, W_a). \]
Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$. \hfill \Box

5. Indefinite generalized Sasakian space forms

**Definition 5.1.** An indefinite trans-Sasakian manifold $\tilde{M}$ is said to be a **indefinite generalized Sasakian space form** and denote it by $\tilde{M}(f_1, f_2, f_3)$ if there exist three smooth functions $f_1$, $f_2$ and $f_3$ on $\tilde{M}$ such that
\[ \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = f_1\{\bar{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \bar{g}(\tilde{X}, \tilde{Z})\tilde{Y}\} \]
\[ + f_2\{\bar{g}(\tilde{X}, J\tilde{Z})J\tilde{Y} - \bar{g}(\tilde{Y}, J\tilde{Z})J\tilde{X} + 2\bar{g}(\tilde{X}, J\tilde{Y})J\tilde{Z}\} \]
\[ + f_3\{\theta(\tilde{X})\theta(\tilde{Z})\tilde{Y} - \theta(\tilde{Y})\theta(\tilde{Z})\tilde{X} \]
\[ + \bar{g}(\tilde{X}, \tilde{Z})\theta(\tilde{Y})\zeta - \bar{g}(\tilde{Y}, \tilde{Z})\theta(\tilde{X})\zeta\}, \]
where $\tilde{R}$ is the curvature tensor of the Levi-Civita connection $\tilde{\nabla}$.

The notion of generalized Sasakian space form was introduced by Alegre et al. [3], while the indefinite generalized Sasakian space forms were introduced by Jin [8]. Sasakian space form, Kenmotsu space form and cosymplectic space form are important kinds of generalized Sasakian space forms such that
respectively, where \( c \) is a constant J-sectional curvature of each space forms.

Denote by \( \bar{R} \) the curvature tensors of the non-metric \( \phi \)-symmetric connection \( \bar{\nabla} \) on \( M \). By directed calculations from (1.2), (1.5) and (2.1), we see that

\[
\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \bar{R}(\bar{X}, \bar{Y})\bar{Z} + (\bar{\nabla}_X \theta)(\bar{Z})J\bar{Y} - (\bar{\nabla}_Y \theta)(\bar{Z})J\bar{X} - \theta(\bar{Z})\{\alpha(\theta(Y)X - \theta(X)Y) + \beta(\theta(Y)JX - \theta(X)JY)
+ 2\beta g(X, JY)\zeta}\}
\]

Denote by \( R \) and \( R^* \) the curvature tensors of the induced linear connections \( \nabla \) and \( \nabla^* \) on \( M \) and \( S(TM) \) respectively. Using the Gauss-Weingarten formulae, we obtain Gauss-Codazzi equations for \( M \) and \( S(TM) \) respectively:

\[
\bar{R}(X, Y)Z = R(X, Y)Z
+ \sum_{i=1}^{r}\{h_i^\ell(X, Z)A_{N_i}Y - h_i^\ell(Y, Z)A_{N_i}X\}
+ \sum_{a=r+1}^{n}\{h_a^s(X, Z)A_{E_a}Y - h_a^s(Y, Z)A_{E_a}X\}
+ \sum_{i=1}^{r}\{((\nabla_X h_i^\ell)(Y, Z) - (\nabla_Y h_i^\ell)(X, Z)
+ \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y, Z) - \tau_{ji}(Y)h_j^\ell(X, Z)]\}
+ \sum_{a=r+1}^{n}\{[\lambda_{ai}(X)h_a^s(Y, Z) - \lambda_{ai}(Y)h_a^s(X, Z)]
- \theta(X)h_a^s(FY, Z) + \theta(Y)h_a^s(FX, Z)\}N_i
+ \sum_{a=r+1}^{n}\{((\nabla_X h_a^s)(Y, Z) - (\nabla_Y h_a^s)(X, Z)
+ \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y, Z) - \rho_{ia}(Y)h_i^\ell(X, Z)]\}
+ \sum_{b=r+1}^{n}\{[\sigma_{ba}(X)h_b^s(Y, Z) - \sigma_{ba}(Y)h_b^s(X, Z)]
- \theta(X)h_b^s(FY, Z) + \theta(Y)h_b^s(FX, Z)\}E_a,
\]

\[
R(X, Y)PZ = R^*(X, Y)PZ
+ \sum_{i=1}^{r}\{h_i^s(X, PZ)A_{\xi_i}Y - h_i^s(Y, PZ)A_{\xi_i}X\}
\]

\[
\sum_{i=1}^{r} \left( (\nabla_X h^*_i)(Y, PZ) - (\nabla_Y h^*_i)(X, PZ) \right)
+ \sum_{k=1}^{r} \left[ \tau_{ik}(Y)h^*_k(X, PZ) - \tau_{ik}(X)h^*_k(Y, PZ) \right]
- \theta(X)h^*_i(FY, PZ) + \theta(Y)h^*_i(FX, PZ) \right) \xi_i,
\]

Taking the scalar product with \( \xi_i \) and \( N_i \) to (5.2) by turns and then, substituting (5.3) and (5.1) and using (3.6)
we get

\[
(\nabla_X h^*_i)(Y, Z) - (\nabla_Y h^*_i)(X, Z)
+ \sum_{j=1}^{r} \left\{ \tau_{ij}(X)h^*_j(Y, Z) - \tau_{ij}(Y)h^*_j(X, Z) \right\}
+ \sum_{a=r+1}^{n} \left\{ \lambda_{ai}(X)h^*_a(Y, Z) - \lambda_{ai}(Y)h^*_a(X, Z) \right\}
- \theta(X)h^*_i(FY, Z) + \theta(Y)h^*_i(FX, Z)
- (\nabla_X \theta)(Z)u_i(Y) + (\nabla_Y \theta)(Z)u_i(X)
+ \beta \theta(Z)\{ \theta(Y)u_i(X) - \theta(X)u_i(Y) \}
= f_2\{u_i(Y)\bar{g}(X, JZ) - u_i(X)\bar{g}(Y, JZ) + 2u_i(Z)\bar{g}(X, JY)\},
\]

\[
(\nabla_X h^*_i)(Y, PZ) - (\nabla_Y h^*_i)(X, PZ)
+ \sum_{j=1}^{r} \left\{ \tau_{ij}(X)h^*_j(Y, PZ) - \tau_{ij}(Y)h^*_j(X, PZ) \right\}
- \sum_{a=r+1}^{n} \left\{ \epsilon_a\{ \rho_{ia}(X)h^*_a(Y, PZ) - \rho_{ia}(Y)h^*_a(X, PZ) \right\}
+ \sum_{j=1}^{r} \left\{ h^*_j(X, PZ)\eta_i(A_{N_j}Y) - h^*_j(Y, PZ)\eta_i(A_{N_j}X) \right\}
- \theta(X)h^*_i(FY, PZ) + \theta(Y)h^*_i(FX, PZ)
- (\nabla_X \theta)(PZ)v_i(Y) + (\nabla_Y \theta)(PZ)v_i(X)
+ \alpha \theta(PZ)\{ \theta(Y)\eta_i(X) - \theta(X)\eta_i(Y) \}
+ \beta \theta(PZ)\{ \theta(Y)v_i(X) - \theta(X)v_i(Y) \}
= f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y) \}
+ f_2\{v_i(Y)\bar{g}(X, JPZ) - v_i(X)\bar{g}(Y, JPZ) + 2v_i(PZ)\bar{g}(X, JY) \}
+ f_3\{ \theta(X)\eta_i(Y) - \theta(Y)\eta_i(X) \} \theta(PZ).
\]

**Theorem 5.2.** Let \( M \) be a generic lightlike submanifold of an indefinite generalized Sasakian space form \( \bar{M}(f_1, f_2, f_3) \) with a non-metric \( \phi \)-symmetric connection such that \( \zeta \) is tangent to \( M \). Then \( \alpha, \beta, f_1, f_2 \) and \( f_3 \) satisfy
(1) \( \alpha \) is a constant on \( M \),

(2) \( \alpha \beta = 0 \), and

(3) \( f_1 - f_2 = \alpha^2 - \beta^2 \) and \( f_1 - f_3 = \alpha^2 - \beta^2 - \zeta \beta \).

**Proof.** Applying \( \bar{\nabla}_X \) to \( \theta(U_i) = 0 \) and \( \theta(V_i) = 0 \) by turns and using (2.4), (3.15), (3.16) and the facts that \( F \zeta = 0 \) and \( \zeta \) belongs to \( S(TM) \), we get

\[
(\bar{\nabla}_X \theta)(U_i) = \alpha \eta_i(X) + \beta v_i(X), \quad (\bar{\nabla}_X \theta)(V_i) = \beta u_i(X).
\]

Applying \( \nabla_X \) to (3.14): \( h^\ell_j(Y, U_i) = h^*_i(Y, V_j) \) and using (2.1), (2.12), (3.7), (3.9), (3.11), (3.12), (3.14)\(_{1,2,4} \), (3.15) and (3.16), we obtain

\[
(\nabla_X h^\ell_j)(Y, U_i) = (\nabla_X h^*_i)(Y, V_j)
\]

\[
- \sum_{k=1}^{r} \{ \tau_{kj}(X)h^\ell_k(Y, U_i) + \tau_{ik}(X)h^*_k(Y, V_j) \}
\]

\[
- \sum_{a=r+1}^{n} \{ \lambda_{aj}(X)h^*_a(U_i) + \epsilon_a \rho_{ia}(X)h^*_a(Y, V_j) \}
\]

\[
+ \sum_{k=1}^{r} \{ h^*_i(Y, U_k)h^\ell_k(X, \xi_j) + h^*_i(X, U_k)h^\ell_k(Y, \xi_j) \}
\]

\[
- g(A^*_i X, F(A^*_{N_k} Y)) - g(A^*_j Y, F(A^*_i X))
\]

\[
- \sum_{k=1}^{r} h^\ell_j(X, V_k)\eta_k(A^*_{N_k} Y)
\]

\[
- \beta(\alpha - 1)\{ u_j(Y)v_i(X) - u_j(X)v_i(Y) \}
\]

\[
- \alpha(\alpha - 1)u_j(Y)\eta_i(X) - \beta^2 u_j(X)\eta_i(Y).
\]

Substituting this equation into the modification equation, which is change \( i \) into \( j \) and \( Z \) into \( U_i \) from (5.5), and using (3.6)\(_3 \) and (3.14)\(_3 \), we have

\[
(\nabla_X h^*_i)(Y, V_j) - (\nabla_Y h^*_i)(X, V_j)
\]

\[
- \sum_{k=1}^{r} \{ \tau_{ik}(X)h^*_k(Y, V_j) - \tau_{ik}(Y)h^*_k(X, V_j) \}
\]

\[
- \sum_{a=r+1}^{n} \{ \rho_{ia}(X)h^*_a(Y, V_j) - \rho_{ia}(Y)h^*_a(X, V_j) \}
\]

\[
+ \sum_{k=1}^{r} \{ h^*_k(X, V_j)\eta_i(A^*_{N_k} Y) - h^*_k(Y, V_j)\eta_i(A^*_{N_k} X) \}
\]

\[
- \theta(X)h^*_i(FY, V_j) + \theta(Y)h^*_i(FX, V_j)
\]

\[
- \beta(2\alpha - 1)\{ u_j(Y)v_i(X) - u_j(X)v_i(Y) \}.
\]
\[- (\alpha^2 - \beta^2)\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\}
= f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\bar{g}(X,JY)\}.
\]

Comparing this equation with (5.6) such that $PZ = V_j$, we obtain
\[
\{f_1 - f_2 - \alpha^2 + \beta^2\}\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\}
= 2\alpha\beta\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\}.
\]

Taking $Y = U_j$, $X = \xi_i$ and $Y = U_j$, $X = V_i$ to this by turns, we have
\[
f_1 - f_2 = \alpha^2 - \beta^2 - \zeta\beta, \quad U_j\alpha = 0.
\]

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$ and using (2.3) and the fact: $\theta \circ J = 0$, we get
\[
(\bar{\nabla}_X\theta)(\zeta) = 0. \quad (5.8)
\]

Applying $\bar{\nabla}_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (1.1) and (2.5), we have
\[
(\bar{\nabla}_X\eta)(Y) = -g(A_{N_i}X, Y) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y) - \theta(Y)v_i(X). \quad (5.9)
\]

Applying $\nabla_X$ to $h_i^*(Y, \zeta) = -(\alpha - 1)v_i(Y) + \beta\eta_i(Y)$ and using (3.9), (3.10), (3.20), (5.9) and the fact that $\alpha\beta = 0$, we get
\[
(\nabla_X h_i^*)(Y, \zeta) = -(X\alpha)v_i(Y) + (X\beta)\eta_i(Y) + (\alpha - 1)\{g(A_{N_i}X, FY) + g(A_{N_i}Y, FX)
\]
\[
- \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) - \sum_{\alpha=r+1}^{n} \epsilon_{\alpha}w_{\alpha}(Y)\rho_{\alpha i}(X)
\]
\[
- \sum_{j=1}^{r} u_j(Y)\eta_i(A_{N_j}X) - (\alpha - 1)\theta(Y)\eta_i(X)\}
\]
\[
- \beta\{g(A_{N_i}X, Y) + g(A_{N_i}Y, X) - \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y)
\]
\[
- \beta\theta(X)\eta_i(Y)\}.
\]

Substituting this and (3.13)$_2$ into (5.6) with $PZ = \zeta$ and using (5.8), we get
\[
\{X\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(X)\}\eta_i(Y)
\]
\[
= \{Y\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(Y)\}\eta_i(X)
\]
\[
= (X\alpha)v_i(Y) - (Y\alpha)v_i(X).
\]

Taking $X = \zeta$, $Y = \xi_i$ and $X = U_j$, $Y = V_i$ to this by turns, we have
\[
f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \quad U_j\alpha = 0.
\]
Applying $\nabla_Y$ to (3.11)\_2 and using (3.10) and (3.19), we get
\[
(\nabla_X h^\ell_i)(Y, \zeta) = -(X\alpha)u_i(Y) \\
+ (\alpha - 1)\left(\sum_{j=1}^{r} u_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\lambda_{ai}(X)
+ h^\ell_i(X, FY) + h^\ell_i(Y, FX)\right) \\
- \beta\{h^\ell_i(Y, X) + \theta(Y)u_i(X) - \theta(X)u_i(Y)\}.
\]
Substituting this into (5.5) such that $Z = \zeta$ and using (3.3) and (5.8), we have
\[
(X\alpha)u_i(Y) = (Y\alpha)u_i(X).
\]
Taking $Y = U_i$ to this result and using the fact that $U_i\alpha = 0$, we have $X\alpha = 0$. Therefore $\alpha$ is a constant. This completes the proof of the theorem. \(\square\)

**Theorem 5.3.** Let $M$ be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ with a non-metric $\phi$-symmetric connection such that $\zeta$ is tangent to $M$. If $F$ is Lie recurrent, then
\[
\alpha = 0, \quad f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = -\zeta\beta.
\]

**Proof.** By Theorem 4.2, we shown that $\alpha = 0$ and we have (4.11)\_2. Applying $\nabla_X$ to (4.11)\_2: $h^\ell_i(Y, U_j) = 0$ and using (3.11)\_2, (3.15) and (4.11)\_2, we have
\[
(\nabla_X h^\ell_i)(Y, U_j) = -h^\ell_i(Y, F(A_{N_j} X)) - \sum_{a=r+1}^{n} \rho_{ja}(X)h^\ell_i(Y, W_a) \\
+ \beta u_i(Y)v_j(X).
\]
Substituting this into (5.5) with $Z = U_j$ and using (5.7)\_1, we obtain
\[
h^\ell_i(X, F(A_{N_j} Y)) - h^\ell_i(Y, F(A_{N_j} X)) \\
+ \sum_{a=r+1}^{n} \{\rho_{ja}(Y)h^\ell_i(X, W_a) - \rho_{ja}(X)h^\ell_i(Y, W_a)\} \\
+ \sum_{a=r+1}^{n} \{\lambda_{ai}(X)h^a_{N_j}(Y, U_j) - \lambda_{ai}(Y)h^a_{N_j}(X, U_j)\} \\
= f_2(\xi_j(X)\eta_j(X) - \eta_{ij}(X)\eta_j(Y) + 2\delta_{ij}\tilde{g}(X, JY)).
\]
Taking $Y = U_i$ and $X = \xi_j$ to this and using (3.3) and (4.5)\_1,3,5, we have
\[
3f_2 = h^\ell_i(\xi_j, F(A_{N_j} X)) + \sum_{a=r+1}^{n} \rho_{ja}(U_i)h^\ell_i(\xi_j, W_a).
\] (5.10)

In general, replacing $X$ by $\xi_j$ to (3.7) and using (3.3) and (3.6)\_7, we get
\[
h^\ell_i(X, \xi_j) = g(A_{\xi_i}^* \xi_j, X). \quad \text{From this and (3.6)\_1, we obtain } A_{\xi_i}^* \xi_j = -A_{\xi_j}^* \xi_i. \quad \text{Thus}
$A^*_i \xi_j$ are skew-symmetric with respect to $i$ and $j$. On the other hand, in case $M$ is Lie recurrent, taking $Y = U_j$ to (4.10), we have $A_{N_i}U_j = A_{N_j}U_i$. Thus $A_{N_i}U_j$ are symmetric with respect to $i$ and $j$. Therefore, we get

$$h^\ell_i(\xi_j, F(A_{N_j}U_i)) = g(A^*_i, \xi_j, F(A_{N_j}U_i)) = 0.$$  

Also, by using (3.4), (3.6)$_2$, (3.14)$_4$ and (4.5)$_4$, we have

$$h^\ell_i(\xi_j, W_a) = \epsilon_a h^*_a(\xi_j, V_i) = \epsilon_a h^*_a(V_i, \xi_j) = -\lambda_ja(V_i) = 0.$$  

Thus we get $f_2 = 0$ by (5.10). Therefore, $f_1 = -\beta^2$ and $f_3 = -\zeta \beta$.

\textbf{Theorem 5.4.} Let $M$ be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a non-metric $\phi$-symmetric connection such that $\zeta$ is tangent to $M$. If $U_i$s or $V_i$s are parallel with respect to $\nabla$, then $\bar{M}(f_1, f_2, f_3)$ is a flat manifold with an indefinite cosymplectic structure;

$$\alpha = \beta = 0, \quad f_1 = f_2 = f_3 = 0.$$  

Proof. (1) If $U_i$s are parallel with respect to $\nabla$, then we have (3.22). As $\alpha = 0$, we get $f_1 = f_2 = f_3$ by Theorem 5.2. Applying $\nabla_Y$ to (3.22)$_5$, we obtain

$$(\nabla_X h^*_i)(Y, U_j) = 0.$$  

Substituting this equation and (3.22) into (5.6) with $PZ = U_j$, we have

$$f_1\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)\} + f_2\{v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$  

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we get $f_1 + f_2 = 0$. Thus we see that $f_1 = f_2 = f_3 = 0$ and $\bar{M}$ is flat.

(2) If $V_i$s are parallel with respect to $\nabla$, then we have (3.23) and $\alpha = 0$. As $\alpha = 0$, we get $f_1 = f_2 = f_3$ by Theorem 5.2. From (3.14)$_1$ and (3.23)$_5$, we have

$$h^*_i(Y, V_j) = 0.$$  

Applying $\nabla_X$ to this equation and using the fact that $\nabla_X V_j = 0$, we have

$$(\nabla_X h^*_i)(Y, V_j) = 0.$$  

Substituting these two equations into (5.6) such that $PZ = V_j$, we obtain

$$\sum_{a=r+1}^{n} \epsilon_a \{\rho_{ia}(Y)h^*_a(X, V_j) - \rho_{ia}(X)h^*_a(Y, V_j)\}$$  

$$+ \sum_{k=1}^{r} \{h^\ell_k(X, V_j)\eta_i(A_{N_k}Y) - h^\ell_k(Y, V_j)\eta_i(A_{N_k}X)\}$$  

$$= f_1\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} + 2f_2\delta_{ij}g(X, JY).$$  

Taking $X = \xi_i$ and $Y = U_j$ to this equation and using (3.3), (3.23)$_{3,4,5}$ and the fact that $h^*_a(U_j, V_j) = \epsilon_a h^*_a(U_j, W_a) = 0$ due to (3.3)$_1$, (3.14)$_4$ and (3.23)$_5$, we obtain $f_1 + 2f_2 = 0$. It follows that $f_1 = f_2 = f_3 = 0$ and $\bar{M}$ is flat. \qed
Definition 5.5. An \( r \)-lightlike submanifold \( M \) is called totally umbilical [6] if there exist smooth functions \( A_i \) and \( B_a \) on a neighborhood \( U \) such that
\[
\begin{align*}
    h_i^+(X, Y) &= A_i g(X, Y), \\
    h_a^-(X, Y) &= B_a g(X, Y).
\end{align*}
\] (5.11)

In case \( A_i = B_a = 0 \), we say that \( M \) is totally geodesic.

Theorem 5.6. Let \( M \) be a generic lightlike submanifold of an indefinite generalized Sasakian space form \( \bar{M}(f_1, f_2, f_3) \) with a non-metric \( \phi \)-symmetric connection such that \( \zeta \) is tangent to \( M \). If \( M \) is totally umbilical, then \( \bar{M}(f_1, f_2, f_3) \) is an indefinite Sasakian space form such that
\[
\begin{align*}
    \alpha &= 1, \quad \beta = 0; \quad f_1 = \frac{2}{3}; \quad f_2 = f_3 = -\frac{1}{3}.
\end{align*}
\]

Proof. Taking \( Y = \zeta \) to (5.11)\(_{1,2} \) by turns and using (3.12)\(_{1,2} \), we have
\[
\begin{align*}
    A_i \theta(X) &= -(\alpha - 1)u_i(X), \\
    B_a \theta(X) &= -(\alpha - 1)w_a(X),
\end{align*}
\]
respectively. Taking \( X = \zeta \) and \( X = U_i \) to the first equation by turns, we have \( A_i = 0 \) and \( \alpha = 1 \) respectively. Taking \( X = \zeta \) to the second equation, we have \( B_a = 0 \). As \( A_i = B_a = 0 \), \( M \) is totally geodesic. As \( \alpha = 1 \) and \( \beta = 0 \), \( M \) is an indefinite Sasakian manifold and \( f_1 - 1 = f_2 = f_3 \) by Theorem 5.2.

Taking \( Z = U_j \) to (5.5) and using (5.7)\(_1 \) and \( h_\ell^i = h_a^i = 0 \), we get
\[
(f_2 + 1)\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y)\} + 2\delta_{ij} f_2 \bar{g}(X, JY) = 0.
\]
Taking \( X = \xi_j \) and \( Y = U_i \), we have \( f_2 = -\frac{1}{3} \). Thus \( f_1 = \frac{2}{3} \) and \( f_3 = -\frac{1}{3} \). \( \square \)

Definition 5.7. (1) A screen distribution \( S(TM) \) is said to be totally umbilical [6] in \( M \) if there exist smooth functions \( \gamma_i \) on a neighborhood \( U \) such that
\[
h_i^+(X, PY) = \gamma_i g(X, PY).
\]
In case \( \gamma_i = 0 \), we say that \( S(TM) \) is totally geodesic in \( M \).

(2) An \( r \)-lightlike submanifold \( M \) is said to be screen conformal [8] if there exist non-vanishing smooth functions \( \varphi_i \) on \( U \) such that
\[
h_i^+(X, PY) = \varphi_i h_\ell^i(X, PY).
\] (5.12)

Theorem 5.8. Let \( M \) be a generic lightlike submanifold of an indefinite generalized Sasakian space form \( \bar{M}(f_1, f_2, f_3) \) with a non-metric \( \phi \)-symmetric connection such that \( \zeta \) is tangent to \( M \). If \( S(TM) \) is totally umbilical or \( M \) is screen conformal, then \( \bar{M}(f_1, f_2, f_3) \) is an indefinite Sasakian space form;
\[
\begin{align*}
    \alpha &= 1, \quad \beta = 0; \quad f_1 = 0, \quad f_2 = f_3 = -1.
\end{align*}
\]
Proof. (1) If $S(TM)$ is totally umbilical, then (3.13)$_2$ is reduced to
\[ \gamma_i \theta(X) = - (\alpha - 1) v_i(X) + \beta \eta_i(X). \]
Replacing $X$ by $V_i$, $\xi_i$ and $\zeta$ respectively, we have $\alpha = 1$, $\beta = 0$ and $\gamma_i = 0$. As $\gamma_i = 0$, $S(TM)$ is totally geodesic, and $h^s_i(X, U_k) = 0$ and $h^f_j(X, U_k) = 0$. As $\alpha = 1$ and $\beta = 0$, $M$ is an indefinite Sasakian manifold and $f_1 - 1 = f_2 = f_3$ by Theorem 5.1. Taking $PZ = U_k$ to (5.6) with $h^*_i = 0$, we get
\[ f_1 \{ v_k(Y) \eta_i(X) - v_k(X) \eta_i(Y) \} + \{ v_i(Y) \eta_k(X) - v_i(X) \eta_k(Y) \} = 0. \]
Taking $X = \xi_i$ and $Y = V_k$, we have $f_1 = 0$. Thus $f_2 = f_3 = -1$.

(2) If $M$ is screen conformal, then, from (3.12)$_2$, (3.13)$_2$ and (5.12), we have
\[ (\alpha - 1) \{ v_i(X) - \beta \eta_i(X) = \varphi_i (\alpha - 1) u_i(X) \}. \]
Taking $X = V_i$ and $X = \xi_i$ to this equation by turns, we have $\alpha = 1$ and $\beta = 0$. As $\alpha = 1$ and $\beta = 0$, $M$ is an indefinite Sasakian manifold and $f_1 - 1 = f_2 = f_3$ by Theorem 5.1.

Denote by $\mu_i$ the $r$-th vector fields on $S(TM)$ such that $\mu_i = U_i - \varphi_i V_i$. Then $J \mu_i = N_i - \varphi_i \xi_i$. Using (3.14)$_1, 2, 3, 4$ and (5.12), we get
\[ h^f_j(X, \mu_i) = 0, \quad h^s_a(X, \mu_i) = 0. \quad (5.13) \]
Applying $\nabla_Y$ to (5.12), we have
\[ (\nabla_X h^*_i)(Y, PZ) = (X \varphi_i) h^f_i(Y, PZ) + \varphi_i (\nabla_X h^f_i)(Y, PZ). \]
Substituting this equation and (5.12) into (5.6) and using (5.5), we have
\[
\sum_{j=1}^{r} \{(X \varphi_i) \delta_{ij} - \varphi_i \tau_{ji}(X) - \varphi_j \tau_{ij}(X) - \eta_i(A_{N_j} X) \} h^f_j(Y, PZ)
\]
\[
- \sum_{j=1}^{r} \{(Y \varphi_i) \delta_{ij} - \varphi_i \tau_{ji}(Y) - \varphi_j \tau_{ij}(Y) - \eta_i(A_{N_j} Y) \} h^f_j(X, PZ)
\]
\[
- \sum_{a=r+1}^{n} \{ \epsilon_a \rho_{ia}(X) + \varphi_i \lambda_{ai}(X) \} h^s_a(Y, PZ)
\]
\[
\quad + \sum_{a=r+1}^{n} \{ \epsilon_a \rho_{ia}(Y) + \varphi_i \lambda_{ai}(Y) \} h^s_a(X, PZ)
\]
\[
- (\nabla_X \theta)(PZ) \{ v_i(Y) - \varphi u_i(Y) \} + (\nabla_Y \theta)(PZ) \{ v_i(X) - \varphi u_i(X) \}
\]
\[
- \alpha \{ \theta(X) \eta_i(Y) - \theta(Y) \eta_i(X) \} \theta(PZ)
\]
\[
= f_1 \{ g(Y, PZ) \eta_i(X) - g(X, PZ) \eta_i(Y) \}
\]
\[
\quad + f_2 \{ [v_i(Y) - \varphi_i u_i(Y)] \tilde{g}(X, JPZ) - [v_i(X) - \varphi_i u_i(X)] \tilde{g}(Y, JPZ) \}. \]
Replacing $PZ$ by $\mu_j$ to this and using (5.7) and (5.13), we obtain

$$
\begin{align*}
&+ 2[v_i(PZ) - \varphi_i u_i(PZ)]\tilde{g}(X, JY) \\
+ f_3\{\theta(X)\eta_h(Y) - \theta(Y)\eta_h(X)\}\theta(PZ).
\end{align*}
$$

Taking $X = \xi_i$ and $Y = V_j$, we get $f_1 = 0$. Thus $f_2 = f_3 = -1$.

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**References**


