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Generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection

Dae Ho Jin[a], **Jae Won Lee**[b*]

[a]Department of Mathematics Education
Dongguk University, Gyeongju 38066, Republic of Korea
jindh@dongguk.ac.kr

[b]Department of Mathematics Education and RINS
Gyeongsang National University, Jinju 52828, Republic of Korea
leejaew@gnu.ac.kr

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Abstract

Jin [13] introduced the notion of non-metric ϕ-symmetric connection on semi-Riemannian manifolds and studied lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection [12]. We study further the geometry of this subject. In this paper, we study generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection.

Keywords: non-metric ϕ-symmetric connection, generic lightlike submanifold, indefinite trans-Sasakian structure

MSC: 53C25, 53C40, 53C50

1. Introduction

The notion of non-metric ϕ-symmetric connection on indefinite almost contact manifolds or indefinite almost complex manifolds was introduced by Jin [12, 13]. Here we quote Jin's definition in itself as follows:

*Corresponding author

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *non-metric ϕ-symmetric connection* if it and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\,\theta(\bar{Y})\phi(\bar{X}, \bar{Z}) - \theta(\bar{Z})\phi(\bar{X}, \bar{Y}), \tag{1.1}$$

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}, \tag{1.2}$$

where ϕ and J are tensor fields of types $(0, 2)$ and $(1, 1)$ respectively, and θ is an 1-form associated with a smooth vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Throughout this paper, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

In case $\phi = \bar{g}$ in (1.1), the above non-metric ϕ-symmetric connection reduces to so-called the quarter-symmetric non-metric connection. Quarter-symmetric non-metric connection was intorduced by S. Golad [7], and then, studied by many authors [2, 4, 19, 20]. In case $\phi = \bar{g}$ in (1.1) and $J = I$ in (1.2), the above non-metric ϕ-symmetric connection reduces to so-called the semi-symmetric non-metric connection. Semi-symmetric non-metric connection was intorduced by Ageshe and Chafle [1] and later studied by many geometers.

The notion of generic lightlike submanifolds on indefinite almost contact manifolds or indefinite almost complext manifolds was introduced by Jin-Lee [14] and later, studied by Duggal-Jin [6], Jin [9, 10] and Jin-Lee [16] and several geometers. We cite Jin-Lee's definition in itself as follows:

A lightlike submanifold M of an indefinite almost contact manifold $\bar{M}$ is said to be *generic* if there exists a screen distribution $S(TM)$ on M such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.3}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ on $\bar{M}$, *i.e.*, $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The geometry of generic lightlike submanifolds is an extension of that of lightlike hypersurfaces and half lightlike submanifolds of codimension 2. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

The notion of trans-Sasakian manifold, of type (α, β), was introduced by Oubina [18]. If $\bar{M}$ is a semi-Riemannian manifold with a trans-Sasakian structure of type (α, β), then $\bar{M}$ is called an *indefinite trans-Sasakian manifold of type* (α, β). Indefinite Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of indefinite trans-Sasakian manifolds such that

$$\alpha = 1,\ \ \beta = 0; \quad \alpha = 0,\ \ \beta = 1; \quad \alpha = \beta = 0, \quad \text{respectively.}$$

In this paper, we study generic lightlike submanifolds M of an indefinite trans-Sasakian manifold $\bar{M} = (\bar{M}, J, \zeta, \theta, \bar{g})$ with a non-metric ϕ-symmetric connection, in which the tensor field J in (1.2) is identical with the indefinite almost contact structure tensor field J of $\bar{M}$, the tensor field ϕ in (1.1) is identical with the fundamental 2-form associated with J, that is,

$$\phi(\bar{X}, \bar{Y}) = \bar{g}(J\bar{X}, \bar{Y}), \tag{1.4}$$

and the 1-form θ, defined by (1.1) and (1.2), is identical with the structure 1-form θ of the indefinite almost contact metric structure $(J,\,\zeta,\,\theta,\,\bar{g})$ of $\bar{M}$.

Remark 1.1. Denote $\widetilde{\nabla}$ by the unique Levi-Civita connection of $(\bar{M},\bar{g})$ with respect to the metric $\bar{g}$. It is known [13] that *a linear connection $\bar{\nabla}$ on $\bar{M}$ is non-metric ϕ-symmetric connection if and only if it satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})J\bar{X}. \tag{1.5}$$

For the rest of this paper, by the *non-metric ϕ-symmetric connection* we shall mean the *non-metric ϕ-symmetric connection defined by* (1.5).

2. Non-metric ϕ-symmetric connections

An odd-dimensional semi-Riemannian manifold $(\bar{M},\,\bar{g})$ is called an *indefinite trans-Sasakian manifold* if there exist (1) a structure set $\{J,\,\zeta,\,\theta,\,\bar{g}\}$, where J is a $(1,1)$-type tensor field, ζ is a vector field and θ is a 1-form such that

$$\begin{aligned} J^2\bar{X} &= -\bar{X} + \theta(\bar{X})\zeta, \quad \theta(\zeta)=1, \quad \theta(\bar{X}) = \epsilon\,\bar{g}(\bar{X},\zeta), \\ \theta\circ J &= 0, \qquad \bar{g}(J\bar{X},\,J\bar{Y}) = \bar{g}(\bar{X},\,\bar{Y}) - \epsilon\,\theta(\bar{X})\theta(\bar{Y}), \end{aligned} \tag{2.1}$$

(2) two smooth functions α and β, and a Levi-Civita connection $\widetilde{\nabla}$ such that

$$(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X},\bar{Y})\zeta - \epsilon\,\theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X},\bar{Y})\zeta - \epsilon\,\theta(\bar{Y})J\bar{X}\},$$

where ϵ denotes $\epsilon=1$ or -1 according as ζ is spacelike or timelike respectively. $\{J,\,\zeta,\,\theta,\,\bar{g}\}$ is called an *indefinite trans-Sasakian structure of type* (α,β).

In the entire discussion of this article, we shall assume that the vector field ζ is a spacelike one, *i.e.*, $\epsilon=1$, without loss of generality.

Let $\bar{\nabla}$ be a non-metric ϕ-symmetric connection on $(\bar{M},\bar{g})$. Using (1.5) and the fact that $\theta\circ J=0$, the equation in the item (2) is reduced to

$$\begin{aligned} (\bar{\nabla}_{\bar{X}}J)\bar{Y} \;=\;& \alpha\{\bar{g}(\bar{X},\bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} \\ &+\; \beta\{\bar{g}(J\bar{X},\bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\} + \theta(\bar{Y})\{\bar{X} - \theta(\bar{X})\zeta\}. \end{aligned} \tag{2.2}$$

Replacing $\bar{Y}$ by ζ to (2.2) and using $J\zeta=0$ and $\theta(\bar{\nabla}_{\bar{X}}\zeta)=0$, we obtain

$$\bar{\nabla}_{\bar{X}}\zeta \;= -(\alpha-1)J\bar{X} + \beta\{\bar{X} - \theta(\bar{X})\zeta\}. \tag{2.3}$$

Let (M,g) be an m-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $(\bar{M},\bar{g})$ of dimension $(m+n)$. Then the radical distribution $Rad(TM) = TM\cap TM^{\perp}$ on M is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1\le r\le \min\{m,\,n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$

in TM and $TM^{\perp}$ respectively, which are called the *screen distribution* and the *co-screen distribution* of M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots \quad \in \{1, \ldots, r\}, \qquad a, b, c, \ldots \quad \in \{r+1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1, \cdots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \cdots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

We say that a lightlike submanifold $M = (M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

(1) *r-lightlike submanifold* if $1 \leq r < \min\{m, n\}$;

(2) *co-isotropic submanifold* if $1 \leq r = n < m$;

(3) *isotropic submanifold* if $1 \leq r = m < n$;

(4) *totally lightlike submanifold* if $1 \leq r = m = n$.

The above three classes (2)$\sim$(4) are particular cases of the class (1) as follows:

$$S(TM^{\perp}) = \{0\}, \qquad S(TM) = \{0\}, \qquad S(TM) = S(TM^{\perp}) = \{0\}$$

respectively. The geometry of r-lightlike submanifolds is more general than that of the other three types. For this reason, we consider only r-lightlike submanifolds M, with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \cdots, \xi_r, N_1, \cdots, N_r, F_{r+1}, \cdots, F_m, E_{r+1}, \cdots, E_n\},$$

where $\{F_{r+1}, \cdots, F_m\}$ and $\{E_{r+1}, \cdots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

In the sequel, we shall assume that ζ is tangent to M. Călin [5] proved that *if ζ is tangent to M, then it belongs to $S(TM)$* which we assumed in this paper. Let P

be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{a=r+1}^{n} h_a^s(X,Y)E_a, \tag{2.4}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \tag{2.5}$$

$$\bar{\nabla}_X E_a = -A_{E_a}X + \sum_{i=1}^{r} \lambda_{ai}(X)N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X)E_b; \tag{2.6}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X,PY)\xi_i, \tag{2.7}$$

$$\nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.8}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are called the *shape operators*, and τ_{ij}, ρ_{ia}, λ_{ai} and σ_{ab} are 1-forms.

Let M be a generic lightlike submanifold of $\bar{M}$. From (1.3) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM on M is decomposed as follows:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{2.9}$$

Consider local null vector fields U_i and V_i for each i, local non-null unit vector fields W_a for each a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{2.10}$$

$$u_i(X) = g(X,V_i), \quad v_i(X) = g(X,U_i), \quad w_a(X) = \epsilon_a g(X,W_a). \tag{2.11}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{2.12}$$

Applying J to (2.12) and using $(2.1)_1$ and (2.10), we have

$$F^2X = -X + \theta(X)\zeta + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{2.13}$$

In the following, we say that F is the *structure tensor field* on M.

3. Structure equations

Let $\bar{M}$ be an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection $\bar{\nabla}$. In the following, we shall assume that ζ is tangent to M. Călin [5] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which we assumed in this paper. Using (1.1), (1.2), (1.4), (2.4) and (2.12), we see that

$$\begin{aligned}
(\nabla_X g)(Y,Z) &= \sum_{i=1}^{r}\{h_i^\ell(X,Y)\eta_i(Z) + h_i^\ell(X,Z)\eta_i(Y)\} && (3.1)\\
&\quad - \theta(Y)\phi(X,Z) - \theta(Z)\phi(X,Y), &&\\
T(X,Y) &= \theta(Y)FX - \theta(X)FY, && (3.2)\\
h_i^\ell(X,Y) - h_i^\ell(Y,X) &= \theta(Y)u_i(X) - \theta(X)u_i(Y), && (3.3)\\
h_a^s(X,Y) - h_a^s(Y,X) &= \theta(Y)w_a(X) - \theta(X)w_a(Y), && (3.4)\\
\phi(X,\xi_i) = u_i(X), \quad &\phi(X,N_i) = v_i(X), \quad \phi(X,E_a) = w_a(X), && (3.5)\\
\phi(X,V_i) = 0, \quad &\phi(X,U_i) = -\eta_i(X), \quad \phi(X,W_a) = 0, &&
\end{aligned}$$

for all i and a, where η_i's are 1-forms such that $\eta_i(X) = \bar{g}(X, N_i)$.

From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. Applying $\bar{\nabla}_X$ to $g(\xi_i,\xi_j) = 0$, $\bar{g}(\xi_i, E_a) = 0$, $\bar{g}(N_i, N_j) = 0$, $\bar{g}(N_i, E_a) = 0$ and $\bar{g}(E_a, E_b) = \epsilon\delta_{ab}$ by turns and using (1.1) and (2.4) ∼ (2.6), we obtain

$$\begin{aligned}
&h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = -\epsilon_a\lambda_{ai}(X),\\
&\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \qquad \eta_i(A_{E_a}X) = \epsilon_a\rho_{ia}(X), \qquad (3.6)\\
&\epsilon_b\sigma_{ab} + \epsilon_a\sigma_{ba} = 0; \quad h_i^\ell(X,\xi_i) = 0, \quad h_i^\ell(\xi_j,\xi_k) = 0, \quad A^*_{\xi_i}\xi_i = 0.
\end{aligned}$$

Definition 3.1. We say that a lightlike submanifold M of $\bar{M}$ is

(1) *irrotational*[17] if $\bar{\nabla}_X\xi_i \in \Gamma(TM)$ for all $i \in \{1, \cdots, r\}$,

(2) *solenoidal* [15] if A_{W_a} and A_{N_i} are $S(TM)$-valued for all α and i.

From (2.4) and $(3.1)_2$, the item (1) is equivalent to

$$h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = \lambda_{ai}(X) = 0.$$

By using $(3.1)_4$, the item (2) is equivalent to

$$\eta_j(A_{N_i}X) = 0, \qquad \rho_{ia}(X) = \eta_i(A_{E_a}X) = 0.$$

The local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X,Y) + \theta(Y)u_i(X) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{3.7}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X,Y) + \theta(Y)w_a(X) - \sum_{k=1}^{r} \lambda_{ak}(X)\eta_k(Y), \tag{3.8}$$

$$h_i^*(X,PY) = g(A_{N_i}X,PY) + \theta(PY)v_i(X). \tag{3.9}$$

Replacing Y by ζ to (2.4) and using (2.3), (2.12), (3.7) and (3.8), we have

$$\nabla_X\zeta = -(\alpha-1)FX + \beta(X - \theta(X)\zeta), \tag{3.10}$$

$$\theta(A^*_{\xi_i}X) = -\alpha u_i(X), \qquad h_i^\ell(X,\zeta) = -(\alpha-1)u_i(X), \tag{3.11}$$

$$\theta(A_{E_a}X) = -\{\epsilon_a(\alpha-1)+1\}w_a(X), \qquad h_a^s(X,\zeta) = -(\alpha-1)w_a(X). \tag{3.12}$$

Applying $\bar\nabla_X$ to $\bar g(\zeta,N_i)$ and using (2.3), (2.5) and (3.9), we have

$$\theta(A_{N_i}X) = -\alpha v_i(X) + \beta\eta_i(X), \qquad h_i^*(X,\zeta) = -(\alpha-1)v_i(X) + \beta\eta_i(X). \tag{3.13}$$

Applying $\bar\nabla_X$ to $(2.10)_{1,2,3}$ and (2.12) by turns and using (2.2), (2.4) $\sim$ (2.8), (2.10) $\sim$ (2.12) and (3.7) $\sim$ (3.9), we have

$$\begin{aligned} &h_j^\ell(X,U_i) = h_i^*(X,V_j), \qquad \epsilon_a h_i^*(X,W_a) = h_a^s(X,U_i),\\ &h_j^\ell(X,V_i) = h_i^\ell(X,V_j), \qquad \epsilon_a h_i^\ell(X,W_a) = h_a^s(X,V_i),\\ &\epsilon_b h_b^s(X,W_a) = \epsilon_a h_a^s(X,W_b), \end{aligned} \tag{3.14}$$

$$\begin{aligned} \nabla_X U_i \;=\; & F(A_{N_i}X) + \sum_{j=1}^{r}\tau_{ij}(X)U_j + \sum_{a=r+1}^{n}\rho_{ia}(X)W_a \\ & - \{\alpha\eta_i(X) + \beta v_i(X)\}\zeta, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \nabla_X V_i \;=\; & F(A^*_{\xi_i}X) - \sum_{j=1}^{r}\tau_{ji}(X)V_j + \sum_{j=1}^{r}h_j^\ell(X,\xi_i)U_j \\ & - \sum_{a=r+1}^{n}\epsilon_a\lambda_{ai}(X)W_a - \beta u_i(X)\zeta, \end{aligned} \tag{3.16}$$

$$\nabla_X W_a \;=\; F(A_{E_a}X) + \sum_{i=1}^{r}\lambda_{ai}(X)U_i + \sum_{b=r+1}^{n}\sigma_{ab}(X)W_b \tag{3.17}$$

$$
\begin{aligned}
&\quad - \beta w_a(X)\zeta, \\
(\nabla_X F)(Y) &= \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X \qquad (3.18) \\
&\quad - \sum_{i=1}^{r} h_i^\ell(X,Y) U_i - \sum_{a=r+1}^{n} h_a^s(X,Y) W_a \\
&\quad + \{\alpha g(X,Y) + \beta \bar{g}(JX,Y) - \theta(X)\theta(Y)\}\zeta \\
&\quad - (\alpha - 1)\theta(Y)X - \beta\theta(Y)FX, \\
(\nabla_X u_i)(Y) &= -\sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) - \sum_{a=r+1}^{n} w_a(Y)\lambda_{ai}(X) \qquad (3.19) \\
&\quad - h_i^\ell(X, FY) - \beta\theta(Y) u_i(X), \\
(\nabla_X v_i)(Y) &= \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) \qquad (3.20) \\
&\quad + \sum_{j=r+1}^{r} u_j(Y)\eta_i(A_{N_j} X) - g(A_{N_i} X, FY) \\
&\quad - (\alpha - 1)\theta(Y)\eta_i(X) - \beta\theta(Y) v_i(X).
\end{aligned}
$$

Theorem 3.2. *There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection such that ζ is tangent to M and F satisfies the following equation:*

$$
(\nabla_X F)Y = (\nabla_Y F)X, \qquad \forall\, X,\, Y \in \Gamma(TM).
$$

Proof. Assume that $(\nabla_X F)Y - (\nabla_Y F)X = 0$. From (3.18) we obtain

$$
\begin{aligned}
&\sum_{i=1}^{r} \{u_i(Y) A_{N_i} X - u_i(X) A_{N_i} Y\} \qquad (3.21) \\
&+ \sum_{a=r+1}^{n} \{w_a(Y) A_{E_a} X - w_a(X) A_{E_a} Y\} - 2\beta \bar{g}(X, JY)\zeta \\
&+ \{\theta(X) u_i(Y) - \theta(Y) u_i(X)\} U_i + \{\theta(X) w_a(Y) - \theta(Y) w_a(X)\} W_a \\
&+ (\alpha - 1)\{\theta(X)Y - \theta(Y)X\} + \beta\{\theta(X)FY - \theta(Y)FX\} = 0.
\end{aligned}
$$

Taking the scalar product with ζ and using $(3.12)_1$ and $(3.13)_1$, we have

$$
\begin{aligned}
&\alpha \sum_{i=1}^{r} \{u_i(Y) v_i(X) - u_i(X) v_i(Y)\} \\
&= \beta \sum_{i=1}^{r} \{u_i(Y)\eta_i(X) - u_i(X)\eta_i(Y)\} - 2\beta \bar{g}(X, JY).
\end{aligned}
$$

Taking $X = V_j$, $Y = U_j$ and $X = \xi_j$, $Y = U_j$ to this equation by turns, we obtain $\alpha = 0$ and $\beta = 0$, respectively. Taking $X = \xi_i$ to (3.21), we have

$$\theta(X)\xi_i + \sum_{j=1}^{r} u_j(X)A_{N_j}\xi_i + \sum_{a=r+1}^{n} w_a(X)A_{E_a}\xi_i = 0.$$

Taking $X = U_k$ and $X = W_b$ to this equation, we have

$$A_{N_k}\xi_i = 0, \qquad A_{E_b}\xi_i = 0.$$

Therefore, we get $\theta(X)\xi_i = 0$. It follows that $\theta(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction to $\theta(\zeta) = 1$. Thus we have our theorem. □

Corollary 3.3. *There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection such that ζ is tangent to M and F is parallel with respect to the connection ∇.*

Theorem 3.4. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a non-metric ϕ-symmetric connection such that ζ is tangent to M. If U_is or V_is are parallel with respect to ∇, then $\alpha = \beta = 0$, i.e., $\bar{M}$ is an indefinite cosymplectic manifold. Furthermore, if U_i is parellel, M is solenoidal and $\tau_{ij} = 0$, if V_i is parallel, M is irrotational and $\tau_{ij} = 0$.*

Proof. (1) If U_i is parallel with respect to ∇, then, taking the scalar product with ζ, V_j, W_a, U_j and N_j to (3.15) such that $\nabla_X U_i = 0$ respectively, we get

$$\alpha = \beta = 0, \quad \tau_{ij} = 0, \quad \rho_{ia} = 0, \quad \eta_j(A_{N_i}X) = 0, \quad h_i^*(X, U_j) = 0. \tag{3.22}$$

As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. As $\rho_{ia} = 0$ and $\eta_j(A_{N_i}X) = 0$, M is solenoidal.

(2) If V_i is parallel with respect to ∇, then, taking the scalar product with ζ, U_j, V_j, W_a and N_j to (3.16) with $\nabla_X V_i = 0$ respectively, we get

$$\beta = 0, \quad \tau_{ji} = 0, \quad h_j^\ell(X, \xi_i) = 0, \quad \lambda_{ai} = 0, \quad h_i^\ell(X, U_j) = 0. \tag{3.23}$$

As $h_j^\ell(X, \xi_i) = 0$ and $\lambda_{ai} = 0$, M is irrotational.

As $h_i^\ell(X, U_j) = 0$, we get $h_i^\ell(\zeta, U_j) = 0$. Taking $X = U_j$ and $Y = \zeta$ to (3.3), we get $h_i^\ell(U_j, \zeta) = \delta_{ij}$. On the other hand, replacing X by U to $(3.12)_1$, we have $h_i^\ell(U_j, \zeta) = -(\alpha - 1)\delta_{ij}$. It follows that $\alpha = 0$. Since $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. □

4. Recurrent and Lie recurrent structure tensors

Definition 4.1. The structure tensor field F of M is said to be

(1) *recurrent* [11] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY,$$

(2) *Lie recurrent* [11] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \tag{4.1}$$

In case $\vartheta = 0$, *i.e.*, $\mathcal{L}_X F = 0$, we say that F is *Lie parallel.*

Theorem 4.2. *There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection such that ζ is tangent to M and the structure tensor field F is recurrent.*

Proof. Assume that F is recurrent. From (3.18), we obtain

$$\begin{aligned}\varpi(X)FY \;=\; & \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X \\ & - \sum_{i=1}^{r} h_i^\ell(X,Y)U_i - \sum_{a=r+1}^{n} h_a^s(X,Y)W_a \\ & + \{\alpha g(X,Y) + \beta\bar{g}(JX,Y) - \theta(X)\theta(Y)\}\zeta \\ & - (\alpha - 1)\theta(Y)X - \beta\theta(Y)FX.\end{aligned}$$

Replacing Y by ξ_j to this and using the fact that $F\xi_j = -V_j$, we get

$$\varpi(X)V_j = \sum_{k=1}^{r} h_k^\ell(X,\xi_j)U_k + \sum_{b=r+1}^{n} h_b^s(X,\xi_j)W_b - \beta u_j(X)\zeta.$$

Taking the scalar product with U_j, we get $\varpi = 0$. It follows that F is parallel with respect to ∇. By Corollary 3.2, we have our theorem. $\square$

Theorem 4.3. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a non-metric ϕ-symmetric connection such that ζ is tangent to M and F is Lie recurrent. Then we have the following results:*

(1) *F is Lie parallel,*

(2) *the function α satisfies $\alpha = 0$,*

(3) *τ_{ij} and ρ_{ia} satisfy $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,*

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)g(A_{N_k}V_j, N_i) - \beta\theta(X)\delta_{ij}.$$

Proof. (1) Using (2.13), (3.2), (3.18), (4.1) and the fact that $\theta \circ F = 0$, we get

$$\begin{aligned}\vartheta(X)FY &= -\nabla_{FY}X + F\nabla_Y X \\ &+ \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X \\ &- \sum_{i=1}^{r}\{h_i^{\ell}(X,Y) - \theta(Y)u_i(X)\}U_i \\ &- \sum_{a=r+1}^{n}\{h_a^{s}(X,Y) - \theta(Y)w_a(X)\}W_a \\ &+ \alpha\{g(X,Y)\zeta - \theta(Y)X\} - \beta\theta(Y)FX.\end{aligned} \tag{4.2}$$

Replacing Y by ξ_j and then, Y by V_j to (4.2), respectively, we have

$$\begin{aligned}-\vartheta(X)V_j &= \nabla_{V_j}X + F\nabla_{\xi_j}X \\ &- \sum_{i=1}^{r} h_i^{\ell}(X,\xi_j)U_i - \sum_{a=r+1}^{n} h_a^{s}(X,\xi_j)W_a,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\vartheta(X)\xi_j &= -\nabla_{\xi_j}X + F\nabla_{V_j}X + \alpha u_j(X)\zeta \\ &- \sum_{i=1}^{r} h_i^{\ell}(X,V_j)U_i - \sum_{a=r+1}^{n} h_a^{s}(X,V_j)W_a.\end{aligned} \tag{4.4}$$

Taking the scalar product with U_i to (4.3) and N_i to (4.4) respectively, we get

$$\begin{aligned}-\delta_{ij}\vartheta(X) &= g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i), \\ \delta_{ij}\vartheta(X) &= g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i).\end{aligned}$$

Comparing these two equations, we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with ζ to (4.4), we get $g(\nabla_{\xi_j}X, \zeta) = \alpha u_j(X)$. Taking $X = U_i$ to this result and using (3.15), we obtain $\alpha = 0$.

(3) Taking the scalar product with N_i to (4.3) such that $X = W_a$ and using (3.4), $(3.6)_4$, (3.8) and (3.17), we get $h_a^s(U_i, V_j) = \rho_{ia}(\xi_j)$. On the other hand, taking the scalar product with W_a to (4.4) such that $X = U_i$ and using (3.15), we have $h_a^s(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h_a^s(U_i, V_j) = 0$.

Taking the scalar product with U_i to (4.3) such that $X = W_a$ and using (3.4), $(3.6)_{2,4}$, (3.8) and (3.17), we get $\epsilon_a\rho_{ia}(V_j) = \lambda_{aj}(U_i)$. On the other hand, taking the scalar product with W_a to (4.3) such that $X = U_i$ and using $(3.1)_2$ and (3.15), we get $\epsilon_a\rho_{ia}(V_j) = -\lambda_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = \lambda_{aj}(U_i) = 0$.

Taking the scalar product with V_i to (4.3) such that $X = W_a$ and using (3.4), $(3.6)_2$, $(3.14)_4$ and (3.17), we obtain $\lambda_{ai}(V_j) = -\lambda_{aj}(V_i)$. On the other hand, taking the scalar product with W_a to (4.3) such that $X = V_i$ and using $(3.6)_2$ and (3.16), we have $\lambda_{ai}(V_j) = \lambda_{aj}(V_i)$. Thus we obtain $\lambda_{ai}(V_j) = 0$.

Taking the scalar product with W_a to (4.3) such that $X = \xi_i$ and using (2.8), (3.3), $(3.6)_2$ and (3.7), we get $h_i^\ell(V_j, W_a) = \lambda_{ai}(\xi_j)$. On the other hand, taking the scalar product with V_i to (4.4) such that $X = W_a$ and using (3.3) and (3.17), we get $h_i^\ell(V_j, W_a) = -\lambda_{ai}(\xi_j)$. Thus $\lambda_{ai}(\xi_j) = 0$ and $h_i^\ell(V_j, W_a) = 0$.

Summarizing the above results, we obtain

$$\begin{aligned} &\rho_{ia}(\xi_j) = 0, \quad \rho_{ia}(V_j) = 0, \quad \lambda_{ai}(U_j) = 0, \quad \lambda_{ai}(V_j) = 0, \quad \lambda_{ai}(\xi_j) = 0, \\ &h_a^s(U_i, V_j) = h_j^\ell(U_i, W_a) = 0, \qquad h_i^\ell(V_j, W_a) = h_a^s(V_j, V_i) = 0. \end{aligned} \tag{4.5}$$

Taking the scalar product with N_i to (4.2) and using $(3.1)_4$, we have

$$\begin{aligned} &-\bar{g}(\nabla_{FY}X, N_i) + g(\nabla_Y X, U_i) - \beta\theta(Y)v_i(X) \\ &+ \sum_{k=1}^{r} u_k(Y)\bar{g}(A_{N_k}X, N_i) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) = 0. \end{aligned} \tag{4.6}$$

Replacing X by V_j to (4.6) and using (3.7), (3.16) and $(4.5)_2$, we have

$$h_j^\ell(FX, U_i) + \tau_{ij}(X) + \beta\theta(X)\delta_{ij} = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i). \tag{4.7}$$

Replacing X by ξ_j to (4.6) and using (2.8), (3.7) and $(4.5)_1$, we have

$$h_j^\ell(X, U_i) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) + \tau_{ij}(FX). \tag{4.8}$$

Taking $X = U_k$ to this equation and using $(3.14)_1$, we have

$$h_i^*(U_k, V_j) = \bar{g}(A_{N_k}\xi_j, N_i). \tag{4.9}$$

Taking $X = U_i$ to (4.2) and using (2.13), (3.3), (3.4) and (3.15), we get

$$\begin{aligned} &\sum_{k=1}^{r} u_k(Y)A_{N_k}U_i + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}U_i - A_{N_i}Y \\ &- F(A_{N_i}FY) - \sum_{j=1}^{r} \tau_{ij}(FY)U_j - \sum_{a=r+1}^{n} \rho_{ia}(FY)W_a = 0. \end{aligned} \tag{4.10}$$

Taking the scalar product with V_j to (4.10) and using (3.8), (3.9), $(3.14)_1$, $(4.5)_6$ and (4.9), we get

$$h_j^\ell(X, U_i) = -\sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) - \tau_{ij}(FX).$$

Comparing this equation with (4.8), we obtain

$$\tau_{ij}(FX) + \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) = 0.$$

Replacing X by U_h to this equation, we have $\bar{g}(A_{N_k}\xi_j, N_i) = 0$. Therefore,

$$\tau_{ij}(FX) = 0, \qquad h_j^\ell(X, U_i) = 0. \tag{4.11}$$

Taking $X = FY$ to $(4.11)_2$, we get $h_j^\ell(FX, U_i) = 0$. Thus (4.7) is reduced to

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i) - \beta\theta(X)\delta_{ij}.$$

Taking the scalar product with U_j to (4.10) such that $Y = W_a$ and using (3.4), (3.8), (3.9) and $(3.14)_2$, we have

$$h_i^*(W_a, U_j) = \epsilon_a h_a^s(U_i, U_j) = \epsilon_a h_a^s(U_j, U_i) = h_i^*(U_j, W_a). \tag{4.12}$$

Taking the scalar product with W_a to (4.10), we have

$$\begin{aligned}\epsilon_a\rho_{ia}(FY) &= -h_i^*(Y, W_a)\\ &\quad + \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) + \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h_b^s(U_i, W_a).\end{aligned}$$

Taking the scalar product with U_i to (4.2) and then, taking $X = W_a$ and using (3.4), $(3.6)_4$, (3.8), (3.9), $(3.14)_2$, (3.17) and (4.12), we obtain

$$\begin{aligned}\epsilon_a\rho_{ia}(FY) &= h_i^*(Y, W_a)\\ &\quad - \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) - \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h_b^s(U_i, W_a).\end{aligned}$$

Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$. □

5. Indefinite generalized Sasakian space forms

Definition 5.1. An indefinite trans-Sasakian manifold $\bar{M}$ is said to be a *indefinite generalized Sasakian space form* and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned}\widetilde{R}(\bar{X}, \bar{Y})\bar{Z} \;=\; & f_1\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}\}\\ & + f_2\{\bar{g}(\bar{X}, J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y}, J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\}\\ & + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X}\\ & \qquad + \bar{g}(\bar{X}, \bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y}, \bar{Z})\theta(\bar{X})\zeta\},\end{aligned} \tag{5.1}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\bar{\nabla}$.

The notion of generalized Sasakian space form was introduced by Alegre *et. al.* [3], while the indefinite generalized Sasakian space forms were introduced by Jin [8]. Sasakian space form, Kenmotsu space form and cosymplectic space form are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

Denote by $\bar{R}$ the curvature tensors of the non-metric ϕ-symmetric connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (1.2), (1.5) and (2.1), we see that

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(\bar{X},\bar{Y})\bar{Z} &+ (\bar{\nabla}_{\bar{X}}\theta)(\bar{Z})J\bar{Y} - (\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z})J\bar{X} \\ &- \theta(\bar{Z})\{\alpha[\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}] + \beta[\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}] \\ &+ 2\beta\bar{g}(X, JY)\zeta\}.\end{aligned} \tag{5.2}$$

Denote by R and R^* the curvature tensors of the induced linear connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulae, we obtain Gauss-Codazzi equations for M and $S(TM)$ respectively:

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z & \\ &+ \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\ &+ \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\ + \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) &- (\nabla_Y h_i^\ell)(X,Z) \\ &+ \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\ &+ \sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)] \\ &- \theta(X)h_i^\ell(FY,Z) + \theta(Y)h_i^\ell(FX,Z)\}N_i \\ + \sum_{a=r+1}^{n}\{(\nabla_X h_a^s)(Y,Z) &- (\nabla_Y h_a^s)(X,Z) \\ &+ \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_i^\ell(X,Z)] \\ &+ \sum_{b=r+1}^{n}[\sigma_{ba}(X)h_b^s(Y,Z) - \sigma_{ba}(Y)h_b^s(X,Z)] \\ &- \theta(X)h_a^s(FY,Z) + \theta(Y)h_a^s(FX,Z)\}E_a,\end{aligned} \tag{5.3}$$

$$\begin{aligned}R(X,Y)PZ = R^*(X,Y)PZ & \\ &+ \sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y - h_i^*(Y,PZ)A_{\xi_i}X\}\end{aligned} \tag{5.4}$$

$$+ \sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y, PZ) - (\nabla_Y h_i^*)(X, PZ)$$

$$+ \sum_{k=1}^{r}[\tau_{ik}(Y)h_k^*(X, PZ) - \tau_{ik}(X)h_k^*(Y, PZ)]$$
$$- \theta(X)h_i^*(FY, PZ) + \theta(Y)h_i^*(FX, PZ)\}\xi_i,$$

Taking the scalar product with ξ_i and N_i to (5.2) by turns and then, substituting (5.3) and (5.1) and using $(3.6)_4$ and (5.4), we get

$$\begin{aligned}
&(\nabla_X h_i^\ell)(Y, Z) - (\nabla_Y h_i^\ell)(X, Z) \\
&+ \sum_{j=1}^{r}\{\tau_{ji}(X)h_j^\ell(Y, Z) - \tau_{ji}(Y)h_j^\ell(X, Z)\} \\
&+ \sum_{a=r+1}^{n}\{\lambda_{ai}(X)h_a^s(Y, Z) - \lambda_{ai}(Y)h_a^s(X, Z)\} \\
&- \theta(X)h_i^\ell(FY, Z) + \theta(Y)h_i^\ell(FX, Z) \\
&- (\bar{\nabla}_X\theta)(Z)u_i(Y) + (\bar{\nabla}_Y\theta)(Z)u_i(X) \\
&+ \beta\theta(Z)\{\theta(Y)u_i(X) - \theta(X)u_i(Y)\} \\
&= f_2\{u_i(Y)\bar{g}(X, JZ) - u_i(X)\bar{g}(Y, JZ) + 2u_i(Z)\bar{g}(X, JY)\},
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
&(\nabla_X h_i^*)(Y,\, PZ) - (\nabla_Y h_i^*)(X,\, PZ) \\
&- \sum_{j=1}^{r}\{\tau_{ij}(X)h_j^*(Y,\, PZ) - \tau_{ij}(Y)h_j^*(X,\, PZ)\} \\
&- \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h_a^s(Y,\, PZ) - \rho_{ia}(Y)h_a^s(X,\, PZ)\} \\
&+ \sum_{j=1}^{r}\{h_j^\ell(X,\, PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y,\, PZ)\eta_i(A_{N_j}X)\} \\
&- \theta(X)h_i^*(FY, PZ) + \theta(Y)h_i^*(FX, PZ) \\
&- (\bar{\nabla}_X\theta)(PZ)v_i(Y) + (\bar{\nabla}_Y\theta)(PZ)v_i(X) \\
&+ \alpha\theta(PZ)\{\theta(Y)\eta_i(X) - \theta(X)\eta_i(Y)\} \\
&+ \beta\theta(PZ)\{\theta(Y)v_i(X) - \theta(X)v_i(Y)\} \\
&= f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \\
&+ f_2\{v_i(Y)\bar{g}(X, JPZ) - v_i(X)\bar{g}(Y, JPZ) + 2v_i(PZ)\bar{g}(X, JY)\} \\
&+ f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ).
\end{aligned} \tag{5.6}$$

Theorem 5.2. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a non-metric ϕ-symmetric connection such that ζ is tangent to M. Then α, β, f_1, f_2 and f_3 satisfy*

(1) α *is a constant on* M*,*

(2) $\alpha\beta = 0$*, and*

(3) $f_1 - f_2 = \alpha^2 - \beta^2$ *and* $f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta$*.*

Proof. Applying $\bar{\nabla}_X$ to $\theta(U_i) = 0$ and $\theta(V_i) = 0$ by turns and using (2.4), (3.15), (3.16) and the facts that $F\zeta = 0$ and ζ belongs to $S(TM)$, we get

$$(\bar{\nabla}_X\theta)(U_i) = \alpha\eta_i(X) + \beta v_i(X), \qquad (\bar{\nabla}_X\theta)(V_i) = \beta u_i(X). \tag{5.7}$$

Applying ∇_X to $(3.14)_1$: $h^\ell_j(Y, U_i) = h^*_i(Y, V_j)$ and using (2.1), (2.12), (3.7), (3.9), (3.11), (3.12), $(3.14)_{1,2,4}$, (3.15) and (3.16), we obtain

$$\begin{aligned}
(\nabla_X h^\ell_j)(Y, U_i) &= (\nabla_X h^*_i)(Y, V_j) \\
&- \sum_{k=1}^{r}\{\tau_{kj}(X)h^\ell_k(Y, U_i) + \tau_{ik}(X)h^*_k(Y, V_j)\} \\
&- \sum_{a=r+1}^{n}\{\lambda_{aj}(X)h^s_a(Y, U_i) + \epsilon_a\rho_{ia}(X)h^s_a(Y, V_j)\} \\
&+ \sum_{k=1}^{r}\{h^*_i(Y, U_k)h^\ell_k(X, \xi_j) + h^*_i(X, U_k)h^\ell_k(Y, \xi_j)\} \\
&- g(A^*_{\xi_j}X, F(A_{N_i}Y)) - g(A^*_{\xi_j}Y, F(A_{N_i}X)) \\
&- \sum_{k=1}^{r} h^\ell_j(X, V_k)\eta_k(A_{N_i}Y) \\
&- \beta(\alpha - 1)\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\} \\
&- \alpha(\alpha - 1)u_j(Y)\eta_i(X) - \beta^2 u_j(X)\eta_i(Y).
\end{aligned}$$

Substituting this equation into the modification equation, which is change i into j and Z into U_i from (5.5), and using $(3.6)_3$ and $(3.14)_3$, we have

$$\begin{aligned}
&(\nabla_X h^*_i)(Y, V_j) - (\nabla_Y h^*_i)(X, V_j) \\
&- \sum_{k=1}^{r}\{\tau_{ik}(X)h^*_k(Y, V_j) - \tau_{ik}(Y)h^*_k(X, V_j)\} \\
&- \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h^s_a(Y, V_j) - \rho_{ia}(Y)h^s_a(X, V_j)\} \\
&+ \sum_{k=1}^{r}\{h^\ell_k(X, V_j)\eta_i(A_{N_k}Y) - h^\ell_k(Y, V_j)\eta_i(A_{N_k}X)\} \\
&- \theta(X)h^*_i(FY, V_j) + \theta(Y)h^*_i(FX, V_j) \\
&- \beta(2\alpha - 1)\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\}
\end{aligned}$$

$$
\begin{aligned}
&- (\alpha^2-\beta^2)\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)\}\\
&= \ f_2\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)+2\delta_{ij}\bar{g}(X,JY)\}.
\end{aligned}
$$

Comparing this equation with (5.6) such that $PZ=V_j$, we obtain

$$
\begin{aligned}
&\{f_1-f_2-\alpha^2+\beta^2\}\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)\}\\
&= \ 2\alpha\beta\{u_j(Y)v_i(X)-u(_jX)v_i(Y)\}.
\end{aligned}
$$

Taking $Y=U_j$, $X=\xi_i$ and $Y=U_j$, $X=V_i$ to this by turns, we have

$$f_1-f_2=\alpha^2-\beta^2, \qquad\qquad \alpha\beta=0.$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta)=1$ and using (2.3) and the fact: $\theta\circ J=0$, we get

$$(\bar{\nabla}_X\theta)(\zeta)=0. \tag{5.8}$$

Applying $\bar{\nabla}_X$ to $\eta_i(Y)=\bar{g}(Y,N_i)$ and using (1.1) and (2.5), we have

$$(\nabla_X\eta)(Y)=-g(A_{N_i}X,Y)+\sum_{j=1}^{r}\tau_{ij}(X)\eta_j(Y)-\theta(Y)v_i(X). \tag{5.9}$$

Applying ∇_X to $h_i^*(Y,\zeta)=-(\alpha-1)v_i(Y)+\beta\eta_i(Y)$ and using (3.9), (3.10), (3.20), (5.9) and the fact that $\alpha\beta=0$, we get

$$
\begin{aligned}
(\nabla_X h_i^*)(Y,\zeta) \ &= -(X\alpha)v_i(Y)+(X\beta)\eta_i(Y)\\
&+(\alpha-1)\{g(A_{N_i}X,FY)+g(A_{N_i}Y,FX)\\
&\qquad -\sum_{j=1}^{r}v_j(Y)\tau_{ij}(X)-\sum_{a=r+1}^{n}\epsilon_a w_a(Y)\rho_{ia}(X)\\
&\qquad -\sum_{j=1}^{r}u_j(Y)\eta_i(A_{N_j}X)-(\alpha-1)\theta(Y)\eta_i(X)\}\\
&-\beta\{g(A_{N_i}X,Y)+g(A_{N_i}Y,X)-\sum_{j=1}^{r}\tau_{ij}(X)\eta_j(Y)\\
&\qquad -\beta\theta(X)\eta_i(Y)\}.
\end{aligned}
$$

Substituting this and $(3.13)_2$ into (5.6) with $PZ=\zeta$ and using (5.8), we get

$$
\begin{aligned}
&\{X\beta+(f_1-f_3-\alpha^2+\beta^2)\theta(X)\}\eta_i(Y)\\
&-\{Y\beta+(f_1-f_3-\alpha^2+\beta^2)\theta(Y)\}\eta_i(X)\\
&= \ (X\alpha)v_i(Y)-(Y\alpha)v_i(X).
\end{aligned}
$$

Taking $X=\zeta$, $Y=\xi_i$ and $X=U_j$, $Y=V_i$ to this by turns, we have

$$f_1-f_3=\alpha^2-\beta^2-\zeta\beta, \qquad\qquad U_j\alpha=0.$$

Applying ∇_Y to $(3.11)_2$ and using (3.10) and (3.19), we get

$$\begin{aligned}(\nabla_X h_i^\ell)(Y,\zeta) = &-(X\alpha)u_i(Y)\\ &+(\alpha-1)\{\sum_{j=1}^{r} u_j(Y)\tau_{ij}(X)+\sum_{a=r+1}^{n}\epsilon_a w_a(Y)\lambda_{ai}(X)\\ &\qquad +h_i^\ell(X,FY)+h_i^\ell(Y,FX)\}\\ &-\beta\{h_i^\ell(Y,X)+\theta(Y)u_i(X)-\theta(X)u_i(Y)\}.\end{aligned}$$

Substituting this into (5.5) such that $Z=\zeta$ and using (3.3) and (5.8), we have

$$(X\alpha)u_i(Y)=(Y\alpha)u_i(X).$$

Taking $Y=U_i$ to this result and using the fact that $U_i\alpha=0$, we have $X\alpha=0$. Therefore α is a constant. This completes the proof of the theorem. □

Theorem 5.3. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with a non-metric ϕ-symmetric connection such that ζ is tangent to M. If F is Lie recurrent, then*

$$\alpha=0,\qquad f_1=-\beta^2,\qquad f_2=0,\qquad f_3=-\zeta\beta.$$

Proof. By Theorem 4.2, we shown that $\alpha=0$ and we have $(4.11)_2$. Applying ∇_X to $(4.11)_2$: $h_i^\ell(Y,U_j)=0$ and using $(3.11)_2$, (3.15) and $(4.11)_2$, we have

$$\begin{aligned}(\nabla_X h_i^\ell)(Y,U_j) = &-h_i^\ell(Y,F(A_{N_j}X))-\sum_{a=r+1}^{n}\rho_{ja}(X)h_i^\ell(Y,W_a)\\ &+\beta u_i(Y)v_j(X).\end{aligned}$$

Substituting this into (5.5) with $Z=U_j$ and using $(5.7)_1$, we obtain

$$\begin{aligned}&h_i^\ell(X,F(A_{N_j}Y))-h_i^\ell(Y,F(A_{N_j}X))\\ &+\sum_{a=r+1}^{n}\{\rho_{ja}(Y)h_i^\ell(X,W_a)-\rho_{ja}(X)h_i^\ell(Y,W_a)\}\\ &+\sum_{a=r+1}^{n}\{\lambda_{ai}(X)h_a^s(Y,U_j)-\lambda_{ai}(Y)h_a^s(X,U_j)\}\\ &= f_2\{u_i(Y)\eta_j(X)-u_i(X)\eta_j(Y)+2\delta_{ij}\bar{g}(X,JY)\}.\end{aligned}$$

Taking $Y=U_i$ and $X=\xi_j$ to this and using (3.3) and $(4.5)_{1,3,5}$, we have

$$3f_2=h_i^\ell(\xi_j,F(A_{N_j}U_i))+\sum_{a=r+1}^{n}\rho_{ja}(U_i)h_i^\ell(\xi_j,W_a). \tag{5.10}$$

In general, replacing X by ξ_j to (3.7) and using (3.3) and $(3.6)_7$, we get $h_i^\ell(X,\xi_j)=g(A^*_{\xi_i}\xi_j,X)$. From this and $(3.6)_1$, we obtain $A^*_{\xi_i}\xi_j=-A^*_{\xi_j}\xi_i$. Thus

$A^*_{\xi_i}\xi_j$ are skew-symmetric with respect to i and j. On the other hand, in case M is Lie recurrent, taking $Y = U_j$ to (4.10), we have $A_{N_i}U_j = A_{N_j}U_i$. Thus $A_{N_i}U_j$ are symmetric with respect to i and j. Therefore, we get

$$h^\ell_i(\xi_j,\, F(A_{N_j}U_i)) = g(A^*_{\xi_i}\xi_j, F(A_{N_j}U_i)) = 0.$$

Also, by using (3.4), $(3.6)_2$, $(3.14)_4$ and $(4.5)_4$, we have

$$h^\ell_i(\xi_j, W_a) = \epsilon_a h^s_a(\xi_j, V_i) = \epsilon_a h^s_a(V_i, \xi_j) = -\lambda_{ja}(V_i) = 0.$$

Thus we get $f_2 = 0$ by (5.10). Therefore, $f_1 = -\beta^2$ and $f_3 = -\zeta\beta$. □

Theorem 5.4. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a non-metric ϕ-symmetric connection such that ζ is tangent to M. If U_is or V_is are parallel with respect to ∇, then $\bar{M}(f_1, f_2, f_3)$ is a flat manifold with an indefinite cosymplectic structure;*

$$\alpha = \beta = 0, \qquad f_1 = f_2 = f_3 = 0.$$

Proof. (1) If U_is are parallel with respect to ∇, then we have (3.22). As $\alpha = 0$, we get $f_1 = f_2 = f_3$ by Theorem 5.2. Applying ∇_Y to $(3.22)_5$, we obtain

$$(\nabla_X h^*_i)(Y, U_j) = 0.$$

Substituting this equation and (3.22) into (5.6) with $PZ = U_j$, we have

$$f_1\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)\} + f_2\{v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we get $f_1 + f_2 = 0$. Thus we see that $f_1 = f_2 = f_3 = 0$ and $\bar{M}$ is flat.

(2) If V_is are parallel with respect to ∇, then we have (3.23) and $\alpha = 0$. As $\alpha = 0$, we get $f_1 = f_2 = f_3$ by Theorem 5.2. From $(3.14)_1$ and $(3.23)_5$, we have

$$h^*_i(Y, V_j) = 0.$$

Applying ∇_X to this equation and using the fact that $\nabla_X V_j = 0$, we have

$$(\nabla_X h^*_i)(Y, V_j) = 0.$$

Substituting these two equations into (5.6) such that $PZ = V_j$, we obtain

$$\begin{aligned}
&\sum_{a=r+1}^{n} \epsilon_a\{\rho_{ia}(Y)h^s_a(X,\, V_j) - \rho_{ia}(X)h^s_a(Y,\, V_j)\} \\
&+ \sum_{k=1}^{r} \{h^\ell_k(X,\, V_j)\eta_i(A_{N_k}Y) - h^\ell_k(Y,\, V_j)\eta_i(A_{N_k}X)\} \\
&= f_1\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} + 2f_2\delta_{ij}\bar{g}(X, JY).
\end{aligned}$$

Taking $X = \xi_i$ and $Y = U_j$ to this equation and using (3.3), $(3.23)_{3,4,5}$ and the fact that $h^s_a(U_j, V_j) = \epsilon_a h^\ell_i(U_j, W_a) = 0$ due to (3.3), $(3.14)_4$ and $(3.23)_5$, we obtain $f_1 + 2f_2 = 0$. It follows that $f_1 = f_2 = f_3 = 0$ and $\bar{M}$ is flat. □

Definition 5.5. An r-lightlike submanifold M is called *totally umbilical* [6] if there exist smooth functions $\mathcal{A}_i$ and $\mathcal{B}_a$ on a neighborhood $\mathcal{U}$ such that

$$h_i^\ell(X,Y) = \mathcal{A}_i g(X,Y), \qquad h_a^s(X,Y) = \mathcal{B}_a g(X,Y). \tag{5.11}$$

In case $\mathcal{A}_i = \mathcal{B}_a = 0$, we say that M is *totally geodesic.*

Theorem 5.6. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a non-metric ϕ-symmetric connection such that ζ is tangent to M. If M is totally umbilical, then $\bar{M}(f_1, f_2, f_3)$ is an indefinite Sasakian space form such that*

$$\alpha = 1, \quad \beta = 0; \qquad f_1 = \frac{2}{3}, \quad f_2 = f_3 = -\frac{1}{3}.$$

Proof. Taking $Y = \zeta$ to $(5.11)_{1,\,2}$ by turns and using $(3.12)_{1,\,2}$, we have

$$\mathcal{A}_i\theta(X) = -(\alpha-1)u_i(X), \qquad \mathcal{B}_a\theta(X) = -(\alpha-1)w_a(X),$$

respectively. Taking $X = \zeta$ and $X = U_i$ to the first equation by turns, we have $\mathcal{A}_i = 0$ and $\alpha = 1$ respectively. Taking $X = \zeta$ to the second equation, we have $\mathcal{B}_a = 0$. As $\mathcal{A}_i = \mathcal{B}_a = 0$, M is totally geodesic. As $\alpha = 1$ and $\beta = 0$, $\bar{M}$ is an indefinite Sasakian manifold and $f_1 - 1 = f_2 = f_3$ by Theorem 5.2.

Taking $Z = U_j$ to (5.5) and using $(5.7)_1$ and $h_i^\ell = h_a^s = 0$, we get

$$(f_2+1)\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y)\} + 2\delta_{ij} f_2 \bar{g}(X, JY) = 0.$$

Taking $X = \xi_j$ and $Y = U_i$, we have $f_2 = -\frac{1}{3}$. Thus $f_1 = \frac{2}{3}$ and $f_3 = -\frac{1}{3}$. $\square$

Definition 5.7. (1) A screen distribution $S(TM)$ is said to be *totally umbilical* [6] in M if there exist smooth functions γ_i on a neighborhood $\mathcal{U}$ such that

$$h_i^*(X, PY) = \gamma_i g(X, PY).$$

In case $\gamma_i = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(2) An r-lightlike submanifold M is said to be *screen conformal* [8] if there exist non-vanishing smooth functions φ_i on $\mathcal{U}$ such that

$$h_i^*(X, PY) = \varphi_i h_i^\ell(X, PY). \tag{5.12}$$

Theorem 5.8. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a non-metric ϕ-symmetric connection such that ζ is tangent to M. If $S(TM)$ is totally umbilical or M is screen conformal, then $\bar{M}(f_1, f_2, f_3)$ is an indefinite Sasakian space form;*

$$\alpha = 1, \quad \beta = 0; \qquad f_1 = 0, \quad f_2 = f_3 = -1.$$

Proof. (1) If $S(TM)$ is totally umbilical, then $(3.13)_2$ is reduced to

$$\gamma_i\theta(X) = -(\alpha - 1)v_i(X) + \beta\eta_i(X).$$

Replacing X by V_i, ξ_i and ζ respectively, we have $\alpha = 1$, $\beta = 0$ and $\gamma_i = 0$. As $\gamma_i = 0$, $S(TM)$ is totally geodesic, and $h^s_a(X, U_k) = 0$ and $h^\ell_j(X, U_k) = 0$. As $\alpha = 1$ and $\beta = 0$, $\bar{M}$ is an indefinite Sasakian manifold and $f_1 - 1 = f_2 = f_3$ by Theorem 5.1. Taking $PZ = U_k$ to (5.6) with $h^*_i = 0$, we get

$$f_1[\{v_k(Y)\eta_i(X) - v_k(X)\eta_i(Y)\} + \{v_i(Y)\eta_k(X) - v_i(X)\eta_k(Y)\}] = 0.$$

Taking $X = \xi_i$ and $Y = V_k$, we have $f_1 = 0$. Thus $f_2 = f_3 = -1$.

(2) If M is screen conformal, then, from $(3.12)_2$, $(3.13)_2$ and (5.12), we have

$$(\alpha - 1)\{v_i(X) - \beta\eta_i(X) = \varphi_i(\alpha - 1)u_i(X)\}.$$

Taking $X = V_i$ and $X = \xi_i$ to this equation by turns, we have $\alpha = 1$ and $beta = 0$. As $\alpha = 1$ and $\beta = 0$, $\bar{M}$ is an indefinite Sasakian manifold and $f_1 - 1 = f_2 = f_3$ by Theorem 5.1.

Denote by μ_i the r-th vector fields on $S(TM)$ such that $\mu_i = U_i - \varphi_i V_i$. Then $J\mu_i = N_i - \varphi_i\xi_i$. Using $(3.14)_{1,2,3,4}$ and (5.12), we get

$$h^\ell_j(X, \mu_i) = 0, \qquad h^s_a(X, \mu_i) = 0. \tag{5.13}$$

Applying ∇_Y to (5.12), we have

$$(\nabla_X h^*_i)(Y, PZ) = (X\varphi_i)h^\ell_i(Y, PZ) + \varphi_i(\nabla_X h^\ell_i)(Y, PZ).$$

Substituting this equation and (5.12) into (5.6) and using (5.5), we have

$$\begin{aligned}
&\sum_{j=1}^{r}\{(X\varphi_i)\delta_{ij} - \varphi_i\tau_{ji}(X) - \varphi_j\tau_{ij}(X) - \eta_i(A_{N_j}X)\}h^\ell_j(Y, PZ)\\
&- \sum_{j=1}^{r}\{(Y\varphi_i)\delta_{ij} - \varphi_i\tau_{ji}(Y) - \varphi_j\tau_{ij}(Y) - \eta_i(A_{N_j}Y)\}h^\ell_j(X, PZ)\\
&- \sum_{a=r+1}^{n}\{\epsilon_a\rho_{ia}(X) + \varphi_i\lambda_{ai}(X)\}h^s_a(Y, PZ)\\
&+ \sum_{a=r+1}^{n}\{\epsilon_a\rho_{ia}(Y) + \varphi_i\lambda_{ai}(Y)\}h^s_a(X, PZ)\\
&- (\bar{\nabla}_X\theta)(PZ)\{v_i(Y) - \varphi u_i(Y)\} + (\bar{\nabla}_Y\theta)(PZ)\{v_i(X) - \varphi u_i(X)\}\\
&- \alpha\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ)\\
&= f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\}\\
&+ f_2\{[v_i(Y) - \varphi_i u_i(Y)]\bar{g}(X, JPZ) - [v_i(X) - \varphi_i u_i(X)]\bar{g}(Y, JPZ)
\end{aligned}$$

$$+2[v_i(PZ) - \varphi_i u_i(PZ)]\bar{g}(X, JY)\}$$
$$+ f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ).$$

Replacing PZ by μ_j to this and using (5.7) and (5.13), we obtain

$$f_1\{[v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)] - \varphi_j[u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)]\}$$
$$+ f_1\{[v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)] - \varphi_i[u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y)]\}$$
$$- 2f_2(\varphi_j + \varphi_i)\delta_{ij}\bar{g}(X, JY) = 0.$$

Taking $X = \xi_i$ and $Y = V_j$, we get $f_1 = 0$. Thus $f_2 = f_3 = -1$. □

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