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The $h(x)$-Lucas quaternion polynomials

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Abstract

In this paper, we study $h(x)$-Lucas quaternion polynomials considering several properties involving these polynomials and we present the exponential generating functions and the Poisson generating functions of the $h(x)$-Lucas quaternion polynomials. Also, by using Binet's formula we give the Cassini's identity, Catalan's identity and d'Ocagne's identity of the $h(x)$-Lucas quaternion polynomials.

Keywords: Lucas polynomials, recurrences, quaternion.

MSC: 11B39, 11B37, 11R52

1. Introduction

The Lucas sequence, $\{L_n\}$, is defined by the recurrence relation, for $n > 1$

$$L_{n+1} = L_n + L_{n-1}$$

where $L_0 = 2$, $L_1 = 1$.

In [13], Nalli and Haukkanen introduced the $h(x)$-Lucas polynomials.

Definition 1.1 ([13]). Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Lucas polynomials $\{L_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x), n \geq 1, \tag{1.1}$$

with initial conditions $L_{h,0}(x) = 2$, $L_{h,1}(x) = h(x)$.

The quaternions are such numbers which extend the complex numbers. They are members of noncommutative algebra. A quaternion p is defined in the form

$$p = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2 and a_3 are real numbers and i, j, k are standart orthonormal basis in $\mathbb{R}^3$ which satisfy the quaternion multiplication rules as

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i \quad ki = -ik = j.$$

The conjugate of the quaternion p is denoted by $\overline{p}$ and $\overline{p} = a_0 - a_1 i - a_2 j - a_3 k$.

We start by recalling some basic results concerning quaternion algebra $\mathbb{H}$, it is well known that the algebra $\mathbb{H} = \{a = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 | a_i \in \mathbb{R}, i = \{0, 1, 2, 3\}\}$ of real quaternions define a $four-$ dimensional vector space over $\mathbb{R}$ having basis $e_0 \cong 1$, $e_1 \cong i$, $e_2 \cong j$ and $e_3 \cong k$ which satisfies the following multiplication rules.

$$\begin{aligned} {e_s}^2 = -1, s \in \{1, 2, 3\}, \quad e_1 e_2 = -e_2 e_1 = e_3, \quad & e_2 e_3 = -e_3 e_2 = e_1, \\ & e_3 e_1 = -e_1 e_3 = e_2. \end{aligned} \tag{1.2}$$

In [8], Horodam defined the nth Lucas quaternions as follows.

Definition 1.2 ([8])**.** The Lucas quaternion numbers that are given for the nth classic Lucas L_n number are defined by the following recurrence relations:

$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}$$

where $n = 0, \mp 1, \mp 2, \ldots$.

The Lucas quaternions have been studied in several papers (see, for example [1, 2, 7, 10, 15]). Recently, in [2], Ari considered the $h(x)$-Lucas quaternion polynomials, he derived the Binet formula and generating function of $h(x)$-Lucas quaternion polynomial sequence.

In this paper, we study $h(x)$-Lucas quaternion polynomials considering several properties involving these polynomials and we present the exponential generating functions and the Poisson generating functions of the $h(x)$-Lucas quaternion polynomials. Also, by using Binet's formula we give the Cassini's identity, the Catalan's identity and the d'Ocagne's identity of the $h(x)$-Lucas quaternion polynomials.

2. The $h(x)$-Lucas quaternion polynomials and some properties

Let e_i $(i = 0, 1, 2, 3)$ be a basis of $\mathbb{H}$, which satisfy the multiplication rules (1.2). Let $h(x)$ be a polynomial with real coefficients. In [2], Ari introduced the $h(x)$-Lucas quaternion polynomials as follows:

Definition 2.1 ([2])**.** Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Lucas quaternion polynomials $\{T_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$T_{h,n}(x) = \sum_{s=0}^{3} L_{h,n+s}(x)e_s \tag{2.1}$$

where $L_{h,n}(x)$ is the nth $h(x)$-Lucas polynomial.

The conjugate of $T_{h,n}(x)$ is given by

$$\overline{T_{h,n}(x)} = L_{h,n}(x)e_0 - L_{h,n+1}(x)e_1 - L_{h,n+2}(x)e_2 - L_{h,n+3}(x)e_3.$$

For $n = 0$,

$$\begin{aligned} T_{h,0}(x) &= \sum_{s=0}^{3} L_{h,s}(x)e_s \\ &= L_{h,0}(x)e_0 + L_{h,1}(x)e_1 + L_{h,2}(x)e_2 + L_{h,3}(x)e_3 \\ &= 2e_0 + h(x)e_1 + (h^2(x) + 2)e_2 + (h^3(x) + 3h(x))e_3. \end{aligned}$$

For $n = 1$,

$$\begin{aligned} T_{h,1}(x) &= \sum_{s=0}^{3} L_{h,s+1}(x)e_s \\ &= L_{h,1}(x)e_0 + L_{h,2}(x)e_1 + L_{h,3}(x)e_2 + L_{h,4}(x)e_3 \\ &= h(x)e_0 + (h^2(x) + 2)e_1 + (h^3(x) + 3h(x))e_2 \\ &\quad + (h^4(x) + 4h^2(x) + 2)e_3. \end{aligned}$$

From the recurrence relation (2.1), using the recurrence relation (1.1) and some properties of summation formulas, we obtain that

$$\begin{aligned} T_{h,n+1}(x) &= \sum_{s=0}^{3} L_{h,s+1+n}(x)e_s \\ &= \sum_{s=0}^{3} \Big(h(x)L_{h,s+n}(x) + L_{h,s+n-1}(x)\Big)e_s \\ &= h(x)\sum_{s=0}^{3} L_{h,s+n}(x)e_s + \sum_{s=0}^{3} L_{h,s+n-1}(x)e_s \\ &= h(x)T_{h,n}(x) + T_{h,n-1}(x) \end{aligned}$$

and so

$$T_{h,n+1}(x) = h(x)T_{h,n}(x) + T_{h,n-1}(x).$$

In [13], authors studied some combinatorial properties of $h(x)$-Fibonacci and $h(x)$-Lucas polynomials and present properties of these polynomials. They obtained the following Binet's formula for $L_{h,n}(x)$

$$L_{h,n}(x) = \alpha^n(x) + \beta^n(x) \tag{2.2}$$

where

$$\alpha(x) = \frac{h(x) + \sqrt{h^2(x)+4}}{2}, \qquad \beta(x) = \frac{h(x) - \sqrt{h^2(x)+4}}{2} \tag{2.3}$$

are roots of the characteristic equation $y^2 - h(x)y - 1 = 0$ of the recurrence relation (1.1).

Ari in [2] calculated the Binet-style formula for $T_{h,n}(x)$,

$$T_{h,n}(x) = \alpha^*(x)\alpha^n(x) + \beta^*(x)\beta^n(x) \tag{2.4}$$

where $\alpha(x)$ and $\beta(x)$ as in (2.3) and $\alpha^*(x) = \sum_{s=0}^{3} \alpha^s(x)e_s$, $\beta^*(x) = \sum_{s=0}^{3} \beta^s(x)e_s$. The following basic identities are needed for our purpose in proving.

$$\alpha(x) + \beta(x) = h(x), \quad \alpha(x)\beta(x) = -1, \quad \alpha(x) - \beta(x) = \sqrt{h^2(x)+4} \tag{2.5}$$

and

$$\frac{\alpha(x)}{\beta(x)} = -\alpha^2(x), \qquad \frac{\beta(x)}{\alpha(x)} = -\beta^2(x).$$

Also,

$$1 + h(x)\alpha(x) = \alpha^2(x), \qquad 1 + h(x)\beta(x) = \beta^2(x), \tag{2.6}$$

and

$$1 + \alpha^2(x) = \alpha(x)\sqrt{h^2(x)+4}, \qquad 1 + \beta^2(x) = -\beta(x)\sqrt{h^2(x)+4}. \tag{2.7}$$

The following Lemma, related with the $h(x)$-Lucas polynomials and it will be useful in the proof of one property of the $h(x)$-Lucas quaternion polynomials in the next Theorem.

Lemma 2.2. *For $n \geq 0$,*

$$L^2{}_{h,n}(x) + L^2{}_{h,n+1}(x) = L_{h,2n}(x) + L_{h,2n+2}(x).$$

Proof. Using (2.2) and (2.5), we get

$$\begin{aligned}
L^2{}_{h,n}(x) + L^2{}_{h,n+1}(x) &= (\alpha^n(x) + \beta^n(x))^2 + (\alpha^{n+1}(x) + \beta^{n+1}(x))^2 \\
&= \alpha^{2n}(x) + 2\alpha^n(x)\beta^n(x) + \beta^{2n}(x) \\
&\quad + \alpha^{2n+2}(x) + 2\alpha^{n+1}(x)\beta^{n+1}(x) + \beta^{2n+2}(x) \\
&= \alpha^{2n}(x) + \beta^{2n}(x) + \alpha^{2n+2}(x) + \beta^{2n+2}(x) \\
&= L_{h,2n}(x) + L_{h,2n+2}(x).
\end{aligned}$$

So the proof is complete. □

Theorem 2.3. *For $n \geq 0$, the following statements hold:*

(i) $(T_{h,n}(x))^2 + (T_{h,n+1}(x))^2 = (\alpha^{2*}(x)\alpha^{2n+1}(x) - \beta^{2*}(x)\beta^{2n+1}(x))(\alpha(x) - \beta(x))$.

(ii) $\frac{(T_{h,n}(x))^2 + (T_{h,n+1}(x))^2}{(\alpha(x)-\beta(x))} = (\alpha^{2*}(x)\alpha^{2n+1}(x) - \beta^{2*}(x)\beta^{2n+1}(x))$.

(iii) $\overline{T_{h,n}(x)} + T_{h,n}(x) = 2L_{h,n}(x)e_0$.

(iv) $(T_{h,n}(x))^2 = 2L_{h,n}(x)e_0T_{h,n}(x) - T_{h,n}(x)\overline{T_{h,n}(x)} = T_{h,n}(x)(2L_{h,n}(x)e_0 - \overline{T_{h,n}(x)})$.

(v) $T_{h,n}(x)\overline{T_{h,n}(x)} = ((h(x))^2 + 2)(L_{h,2n+4}(x) + L_{h,2n+2}(x))$.

(vi) $T_{h,1}(x) - \alpha(x)T_{h,0}(x) = -\beta^*(x)\sqrt{h^2(x) + 4}$.
In particular $\frac{T_{h,1}(x)-\alpha(x)T_{h,0}(x)}{\alpha(x)-\beta(x)} = -\beta^*(x)$.

(vii) $T_{h,1}(x) - \beta(x)T_{h,0}(x) = \alpha^*(x)\sqrt{h^2(x) + 4}$.
In particular $\frac{T_{h,1}(x)-\beta(x)T_{h,0}(x)}{\alpha(x)-\beta(x)} = \alpha^*(x)$.

Proof. *(i)* From (2.4), (2.5) and (2.7), we obtain

$$
\begin{aligned}
&(T_{h,n}(x))^2 + (T_{h,n+1}(x))^2 \\
&= (\alpha^*(x)\alpha^n(x) + \beta^*(x)\beta^n(x))^2 + (\alpha^*(x)\alpha^{n+1}(x) + \beta^*(x)\beta^{n+1}(x))^2 \\
&= \alpha^{2*}(x)\alpha^{2n}(x)(1 + \alpha^2(x)) + \beta^{2*}(x)\beta^{2n}(x)(1 + \beta^2(x)) \\
&= \alpha^{2*}(x)\alpha^{2n+1}(x)\sqrt{h^2(x) + 4} - \beta^{2*}(x)\beta^{2n+1}(x)\sqrt{h^2(x) + 4} \\
&= (\alpha^{2*}(x)\alpha^{2n+1}(x) - \beta^{2*}(x)\beta^{2n+1}(x))(\alpha(x) - \beta(x)).
\end{aligned}
$$

(ii) The proof of *(ii)* follows immediately from *(i)*.

(iii) Using the definition of $\overline{T_{h,n}(x)}$ and some computations, we have

$$
\begin{aligned}
\overline{T_{h,n}(x)} &= L_{h,n}(x)e_0 - L_{h,n+1}(x)e_1 - L_{h,n+2}(x)e_2 - L_{h,n+3}(x)e_3 \\
&= 2L_{h,n}(x)e_0 - \sum_{s=0}^{3} L_{h,n+s}(x)e_s \\
&= 2L_{h,n}(x)e_0 - T_{h,n}(x),
\end{aligned}
$$

and the result follows.

(iv) By *(iii)*, *(iv)* holds.

(v) Using Definition 2.1, the definition of $\overline{T_{h,n}(x)}$, Lemma 2.2 and (1.1) we obtain

$$
\begin{aligned}
T_{h,n}(x)\overline{T_{h,n}(x)} &= \sum_{s=0}^{3} L_{h,n+s}(x)e_s\overline{T_{h,n}(x)} \\
&= \big(L_{h,n}(x)e_0 + L_{h,n+1}(x)e_1 + L_{h,n+2}(x)e_2 + L_{h,n+3}(x)e_3\big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \big(L_{h,n}(x)e_0 - L_{h,n+1}(x)e_1 - L_{h,n+2}(x)e_2 - L_{h,n+3}(x)e_3\big)\\
&= L^2{}_{h,n}(x) + L^2{}_{h,n+1}(x) + L^2{}_{h,n+2}(x) + L^2{}_{h,n+3}(x)\\
&= L_{h,2n}(x) + L_{h,2n+2}(x) + L_{h,2n+4}(x) + L_{h,2n+6}(x)\\
&= L_{h,2n}(x) + L_{h,2n+2}(x) + L_{h,2n+4}(x) + h(x)L_{h,2n+5}(x)\\
&\quad + L_{h,2n+4}(x)\\
&= 2L_{h,2n+2}(x) + h^2(x)L_{h,2n+2}(x) + 2L_{h,2n+4}(x)\\
&\quad + h^2(x)L_{h,2n+4}(x)\\
&= \big(2 + (h(x))^2\big)\big(L_{h,2n+2}(x) + L_{h,2n+4}(x)\big).
\end{aligned}
$$

(vi) Since

$$L_{h,s+1}(x) - \beta(x)L_{h,s}(x) = \alpha^s(x)\big(\alpha(x) - \beta(x)\big)$$

and

$$L_{h,s+1}(x) - \alpha(x)L_{h,s}(x) = \beta^s(x)\big(\alpha(x) - \beta(x)\big),$$

using the definition of $\beta^*(x)$, Definition2.1 and Eq.(2.5), we have

$$
\begin{aligned}
&T_{h,1}(x) - \alpha(x)T_{h,0}(x)\\
&= L_{h,1}(x)e_0 + L_{h,2}(x)e_1 + L_{h,3}(x)e_2 + L_{h,4}(x)e_3\\
&\quad - \alpha(x)\big(L_{h,0}(x)e_0 + L_{h,1}(x)e_1 + L_{h,2}(x)e_2 + L_{h,3}(x)e_3\big)\\
&= \big(L_{h,1}(x) - \alpha(x)L_{h,0}(x)\big)e_0 + \big(L_{h,2}(x) - \alpha(x)L_{h,1}(x)\big)e_1\\
&\quad + \big(L_{h,3}(x) - \alpha(x)L_{h,2}(x)\big)e_2 + \big(L_{h,4}(x) - \alpha(x)L_{h,3}(x)\big)e_3\\
&= -\beta^0(x)(\alpha(x) - \beta(x))e_0 - \beta^1(x)(\alpha(x) - \beta(x))e_1\\
&\quad - \beta^2(x)(\alpha(x) - \beta(x))e_2 - \beta^3(x)(\alpha(x) - \beta(x))e_3\\
&= -\sqrt{h^2(x) + 4}(e_0 + \beta^1(x)e_1 + \beta^2(x)e_2 + \beta^3(x)e_3)\\
&= -\sqrt{h^2(x) + 4}\sum_{s=0}^{3} \beta^s(x)e_s\\
&= -\sqrt{h^2(x) + 4}\beta^*(x).
\end{aligned}
$$

which completes the first part of the proof of (vi). The proof of the remaining part can be obtained from previous result.

(vii) The proof is similar to part (vi) and thus, omitted. □

Theorem 2.4. *For* $n \geq 0$, $\sum_{k=0}^{n} \binom{n}{k}(h(x))^k T_{h,k}(x) = T_{h,2n}(x)$.

Proof. Using (2.4) and (2.6), we obtain

$$\sum_{k=0}^{n} \binom{n}{k}(h(x))^k T_{h,k}(x) = \sum_{k=0}^{n} \binom{n}{k}(h(x))^k[\alpha^*(x)\alpha^k(x) + \beta^*(x)\beta^k(x)]$$

$$
\begin{aligned}
&= \alpha^*(x)\sum_{k=0}^{n}\binom{n}{k}(h(x))^k\alpha^k(x)\\
&\quad + \beta^*(x)\sum_{k=0}^{n}\binom{n}{k}(h(x))^k\beta^k(x)\\
&= \alpha^*(x)(1+h(x)\alpha(x))^n + \beta^*(x)(1+h(x)\beta(x))^n\\
&= \alpha^*(x)\alpha^{2n}(x) + \beta^*(x)\beta^{2n}(x)\\
&= T_{h,2n}(x).
\end{aligned}
$$

□

Theorem 2.5. *The sum of the first m terms of the sequence $\{T_{h,m}(x)\}_{m=0}^{\infty}$ is given by*

$$\sum_{k=0}^{m} T_{h,k}(x) = \frac{T_{h,0}(x) - T_{h,m}(x) - T_{h,m+1}(x) - \alpha^*(x)\beta(x) - \beta^*(x)\alpha(x)}{(1-\alpha(x))(1-\beta(x))}.$$

Proof. From (2.4), (2.5) and some calculations, we get

$$
\begin{aligned}
\sum_{k=0}^{m} T_{h,k}(x) &= \sum_{k=0}^{m}(\alpha^*(x)\alpha^k(x) + \beta^*(x)\beta^k(x))\\
&= \alpha^*(x)\sum_{k=0}^{m}\alpha^k(x) + \beta^*(x)\sum_{k=0}^{m}\beta^k(x)\\
&= \alpha^*(x)\Big(\frac{1-\alpha^{m+1}(x)}{1-\alpha(x)}\Big) + \beta^*(x)\Big(\frac{1-\beta^{m+1}(x)}{1-\beta(x)}\Big)\\
&= \frac{\alpha^*(x) - \alpha^*(x)\beta(x) - \alpha^*(x)\alpha^{m+1}(x) + \alpha^*(x)\alpha^m(x)\alpha(x)\beta(x)}{(1-\beta(x))(1-\alpha(x))}\\
&\quad + \frac{\beta^*(x) - \beta^*(x)\alpha(x) - \beta^*(x)\beta^{m+1}(x) + \beta^*(x)\alpha(x)\beta(x)\beta^m(x)}{(1-\beta(x))(1-\alpha(x))}\\
&= \frac{T_{h,0}(x) - T_{h,m}(x) - T_{h,m+1}(x) - \alpha^*(x)\beta(x) - \beta^*(x)\alpha(x)}{(1-\alpha(x))(1-\beta(x))}.
\end{aligned}
$$

So the proof is complete. □

3. Exponential generating functions for the $h(x)$-Lucas quaternion polynomials

In this section, we give the exponential generating functions for the sequence of the $h(x)$-Lucas quaternion polynomials. The exponential generating function of a sequence $\{b_k\}_{k=0}^{\infty}$ is given by

$$EG(b_k, l) = \sum_{k=0}^{\infty} b_k \frac{l^k}{k!}.$$

Theorem 3.1. *The exponential generating function for the $h(x)$-Lucas quaternion polynomials are*

$$\sum_{k=0}^{\infty} \frac{T_{h,k}(x)}{k!} l^k = \alpha^*(x) e^{\alpha(x)l} + \beta^*(x) e^{\beta(x)l}. \tag{3.1}$$

Proof. From the Binet-style formula for the $h(x)$-Lucas quaternion polynomials, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{T_{h,k}(x)}{k!} l^k &= \sum_{k=0}^{\infty} \Big(\alpha^*(x)\alpha^k(x) + \beta^*(x)\beta^k(x) \Big) \frac{l^k}{k!} \\
&= \alpha^*(x) \sum_{k=0}^{\infty} \frac{(\alpha(x)l)^k}{k!} + \beta^*(x) \sum_{k=0}^{\infty} \frac{(\beta(x)l)^k}{k!} \\
&= \alpha^*(x) e^{\alpha(x)l} + \beta^*(x) e^{\beta(x)l}.
\end{aligned}$$ □

4. Poisson generating functions for the $h(x)$-Lucas quaternion polynomials

In this section, we present Poisson generating functions for the sequence of the $h(x)$-Lucas quaternion polynomials.

Lemma 4.1. *The Poisson generating functions for the $h(x)$-Lucas quaternion polynomials are*

$$\sum_{k=0}^{\infty} \frac{T_{h,k}(x)}{k!} l^k e^{-l} = \frac{\alpha^*(x) e^{\alpha(x)l} + \beta^*(x) e^{\beta(x)l}}{e^l}. \tag{4.1}$$

Proof. Since $PG(b_n, x) = e^{-l} EG(b_n, x)$, we have the result by Theorem 3.1. □

5. Catalan's, Cassini's and d'Ocagne's identity for the $h(x)$-Lucas quaternion polynomials

In this section, we compute Catalan's identity, Cassini's identity and d'Ocagne's identity for the $h(x)$-Lucas quaternion polynomials, we start with Catalan's identity.

Theorem 5.1. *For $n \geq m \geq 1$, Catalan identity for the $h(x)$-Lucas quaternion polynomials is*

$$\begin{aligned}
T_{h,n+m}(x) T_{h,n-m}(x) - T^2{}_{h,n}(x) &= (-1)^{n-m} (\alpha^m(x) - \beta^m(x)) \\
&\quad \times 1 \Big(\alpha^*(x)\beta^*(x)\alpha^m(x) - \beta^*(x)\alpha^*(x)\beta^m(x) \Big).
\end{aligned}$$

Proof. Using (2.4) and (2.5), we obtain

$$\begin{aligned}
&T_{h,n+m}(x)T_{h,n-m}(x) - T^2{}_{h,n}(x)\\
&= \Big(\alpha^*(x)\alpha^{n+m}(x) + \beta^*(x)\beta^{n+m}(x)\Big)\Big(\alpha^*(x)\alpha^{n-m}(x) + \beta^*(x)\beta^{n-m}(x)\Big)\\
&\quad - \Big(\alpha^*(x)\alpha^n(x) + \beta^*(x)\beta^n(x)\Big)^2\\
&= \alpha^*(x)\beta^*(x)\alpha^{n+m}(x)\beta^{n-m}(x) + \beta^*(x)\alpha^*(x)\beta^{n+m}(x)\alpha^{n-m}(x)\\
&\quad - \alpha^*(x)\beta^*(x)\alpha^n(x)\beta^n(x) - \beta^*(x)\alpha^*(x)\alpha^n(x)\beta^n(x)\\
&= \alpha^*(x)\beta^*(x)\big(\alpha(x)\beta(x)\big)^n\big(\frac{\alpha^m(x)}{\beta^m(x)} - 1\big)\\
&\quad + \beta^*(x)\alpha^*(x)\big(\alpha(x)\beta(x)\big)^n\big(\frac{\beta^m(x)}{\alpha^m(x)} - 1\big)\\
&= \alpha^*(x)\beta^*(x)(-1)^n\alpha^m(x)\big(\frac{\alpha^m(x) - \beta^m(x)}{(\alpha(x)\beta(x))^m}\big)\\
&\quad + \beta^*(x)\alpha^*(x)(-1)^n\beta^m(x)\big(\frac{\beta^m(x) - \alpha^m(x)}{(\alpha(x)\beta(x))^m}\big)\\
&= (-1)^{n-m}(\alpha^m(x) - \beta^m(x))\Big(\alpha^*(x)\beta^*(x)\alpha^m(x) - \beta^*(x)\alpha^*(x)\beta^m(x)\Big).
\end{aligned}$$

So Theorem 5.1 is proved. □

Theorem 5.2. *For any natural number n, Cassini identity for the $h(x)$-Lucas quaternion polynomials is*

$$\begin{aligned}
T_{h,n+1}(x)T_{h,n-1}(x) - T^2{}_{h,n}(x) &= (-1)^{n-1}(\alpha(x) - \beta(x))\\
&\quad \times \Big(\alpha^*(x)\beta^*(x)\alpha(x) - \beta^*(x)\alpha^*(x)\beta(x)\Big).
\end{aligned}$$

Proof. Taking $m = 1$ in Catalan's identity, the proof is completed. □

Theorem 5.3 (d'Ocagne's identity)**.** *Suppose that n is a nonnegative integer number and m any natural number. If $m > n$, then*

$$\begin{aligned}
&T_{h,m}(x)T_{h,n+1}(x) - T_{h,m+1}(x)T_{h,n}(x)\\
&= (-1)^n(\alpha(x) - \beta(x))\Big(\beta^*(x)\alpha^*(x)\beta^{m-n}(x) - \alpha^*(x)\beta^*(x)\alpha^{m-n}(x)\Big).
\end{aligned}$$

Proof. From (2.4) and (2.5), we obtain

$$\begin{aligned}
&T_{h,m}(x)T_{h,n+1}(x) - T_{h,m+1}(x)T_{h,n}(x)\\
&= \Big(\alpha^*(x)\alpha^m(x) + \beta^*(x)\beta^m(x)\Big)\Big(\alpha^*(x)\alpha^{n+1}(x) + \beta^*(x)\beta^{n+1}(x)\Big)\\
&\quad - \Big(\alpha^*(x)\alpha^{m+1}(x) + \beta^*(x)\beta^{m+1}(x)\Big)\Big(\alpha^*(x)\alpha^n(x) + \beta^*(x)\beta^n(x)\Big)\\
&= \alpha^*(x)\beta^*(x)\alpha^m(x)\beta^n(x)\Big(\beta(x) - \alpha(x)\Big) + \beta^*(x)\alpha^*(x)\beta^m(x)\alpha^n(x)
\end{aligned}$$

$$\times \Big(\alpha(x) - \beta(x)\Big)$$
$$= \alpha^*(x)\beta^*(x)\alpha^{m-n}(x)\Big(\alpha(x)\beta(x)\Big)^n\Big(\beta(x) - \alpha(x)\Big) + \beta^*(x)\alpha^*(x)\beta^{m-n}(x)$$
$$\times \Big(\alpha(x)\beta(x)\Big)^n\Big(\alpha(x) - \beta(x)\Big)$$
$$= (-1)^n(\alpha(x) - \beta(x))\Big(\beta^*(x)\alpha^*(x)\beta^{m-n}(x) - \alpha^*(x)\beta^*(x)\alpha^{m-n}(x)\Big).$$

So, the proof is complete. □

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