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The Miki-type identity for the Apostol-Bernoulli numbers

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Abstract

We study analogues of the Miki, Matiyasevich, and Euler identities for the Apostol-Bernoulli numbers and obtain the analogues of the Miki and Euler identities for the Apostol-Genocchi numbers.

Keywords: Apostol-Bernoulli numbers; Apostol-Genocchi numbers; Miki identity; Matiyasevich identity; Euler identity

MSC: 05A19; 11B68

1. Introduction

The Apostol-Bernoulli numbers are defined in [2] as

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda) \frac{t^n}{n!}. \tag{1.1}$$

Note that at $\lambda = 1$ this generating function becomes

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

where B_n is the classical nth Bernoulli number. Moreover, $\mathcal{B}_0 = \mathcal{B}_0(\lambda) = 0$ while $B_0 = 1$ (see [9]). The Genocchi numbers are defined by the generating function

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!},$$

which are closely related to the classical Bernoulli numbers and the special values of the Euler polynomials. It is known that $G_n = 2(1-2^n)B_n$ and $G_n = nE_{n-1}(0)$, where $E_n(0)$ is a value of the Euler polynomials evaluated at 0 (sometimes are called the Euler numbers) [4, 10, 11]. Likewise the Apostol-Bernoulli numbers, the Apostol-Genocchi numbers are defined by their generating function as

$$\frac{2t}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{G}_n(\lambda)\frac{t^n}{n!} \tag{1.2}$$

with $\mathcal{G}_0 = \mathcal{G}_0(\lambda) = 0$.

Over the years, different identities were obtained for the Bernoulli numbers (for instance, see [3, 4, 6, 7, 10, 12, 16, 17]). The Euler identity for the Bernoulli numbers is given by (see [6, 15])

$$\sum_{k=2}^{n-2} \binom{n}{k} B_k B_{n-k} = -(n+1)B_n, \qquad (n \geq 4). \tag{1.3}$$

Its analogue for convolution of Bernoulli and Euler numbers was obtained in [10] using the p-adic integrals. The similar convolution was obtained for the generalized Apostol-Bernoulli polynomials in [13]. In 1978, Miki [15] found a special identity involving two different types of convolution between Bernoulli numbers:

$$\sum_{k=2}^{n-2} \frac{B_k}{k}\frac{B_{n-k}}{n-k} - \sum_{k=2}^{n-2}\binom{n}{k}\frac{B_k}{k}\frac{B_{n-k}}{n-k} = 2H_n\frac{B_n}{n}, \qquad (n \geq 4), \tag{1.4}$$

where $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is the nth harmonic number. Different kinds of proofs of this identity were represented in [1, 5, 8]. Gessel [8] generalized the Miki identity for the Bernoulli polynomials. Another generalization of the Miki identity for the Bernoulli and Euler polynomials was obtained in [16]. In 1997, Matiyasevich [1, 14] found an identity involving two types of convolution between Bernoulli numbers

$$(n+2)\sum_{k=2}^{n-2} B_k B_{n-k} - 2\sum_{k=2}^{n-2}\binom{n+2}{k} B_k B_{n-k} = n(n+1)B_n. \tag{1.5}$$

The analogues of the Euler, Miki and Matiyasevich identities for the Genocchi numbers were obtained in [1]. In this paper, we represent the analogues of these identities for the Apostol-Bernoulli and the Apostol-Genocchi numbers.

2. The analogues for the Apostol-Bernoulli numbers

In our work, we use the generating functions method to obtain new analogues of the known identities for the Apostol-Bernoulli numbers (see [1, 9]). It is easy to show that

$$\frac{1}{\lambda e^a - 1}\cdot\frac{1}{\mu e^b - 1} = \frac{1}{\lambda\mu e^{a+b} - 1}\left(1 + \frac{1}{\lambda e^a - 1} + \frac{1}{\mu e^b - 1}\right). \tag{2.1}$$

Let us take $a = xt$ and $b = x(1-t)$ and multiply both sides of the identity (2.1) by $t(1-t)x^2$.

$$\begin{aligned}
&\frac{tx}{\lambda e^{tx}-1}\frac{(1-t)x}{\mu e^{(1-t)x}-1} \\
&\quad= \frac{t(1-t)x^2}{\lambda\mu e^{tx+(1-t)x}}\left(1+\frac{1}{\lambda e^{tx}-1}+\frac{1}{\mu e^{(1-t)x}-1}\right) \\
&\quad= \frac{x}{\lambda\mu e^{x}-1}\left(t(1-t)x+(1-t)\frac{tx}{\lambda e^{tx}-1}+t\frac{(1-t)x}{\mu e^{(1-t)x}-1}\right). \qquad (2.2)
\end{aligned}$$

By using (1.1) and the Cauchy product, we get on the LH side of (2.2)

$$\begin{aligned}
\frac{tx}{\lambda e^{tx}-1}\frac{(1-t)x}{\mu e^{(1-t)x}-1} &= \left(\sum_{n=0}^{\infty}\mathcal{B}_n(\lambda)\frac{t^n x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{B}_n(\mu)\frac{(1-t)^n x^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(\lambda)t^k\mathcal{B}_{n-k}(\mu)(1-t)^{n-k}\right]\frac{x^n}{n!}, \qquad (2.3)
\end{aligned}$$

and on the RH side of (2.2) we obtain

$$\begin{aligned}
&\frac{x}{\lambda\mu e^{x}-1}\left(t(1-t)x+(1-t)\frac{tx}{\lambda e^{tx}-1}+t\frac{(1-t)x}{\mu e^{(1-t)x}-1}\right) \\
&= \sum_{n=0}^{\infty}\mathcal{B}_n(\lambda\mu)\frac{x^n}{n!}\cdot \\
&\qquad\cdot\left(t(1-t)x+(1-t)\sum_{n=0}^{\infty}\mathcal{B}_n(\lambda)\frac{t^n x^n}{n!}+t\sum_{n=0}^{\infty}\mathcal{B}_n(\mu)\frac{(1-t)^n x^n}{n!}\right) \\
&= t(1-t)\sum_{n=1}^{\infty}\mathcal{B}_{n-1}(\lambda\mu)n\frac{x^n}{n!} \\
&\qquad+(1-t)\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda)t^{n-k}\right]\frac{x^n}{n!} \\
&\qquad\qquad+t\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu)(1-t)^{n-k}\right]\frac{x^n}{n!}. \qquad (2.4)
\end{aligned}$$

By comparing the coefficients of $\frac{x^n}{n!}$ on left (2.3) and right (2.4) hand sides, we get

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu) \\
&\qquad= nt(1-t)\mathcal{B}_{n-1}(\lambda\mu)+(1-t)\sum_{k=0}^{n}\binom{n}{k}t^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda)
\end{aligned}$$

$$+\,t\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu). \tag{2.5}$$

It follows from (1.1) that $\mathcal{B}_n(1)=B_n$. It is well known that $B_0=1$, but from (1.1) we get $\mathcal{B}_0=0$. Therefore, we concentrate the members, containing the 0th index (the cases $k=0$ and $k=n$), out of the sums. The sum on the left hand side of (2.5) can be rewritten as

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu)\\
&=\sum_{k=1}^{n-1}\binom{n}{k}t^k(1-t)^{n-k}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu)\\
&\qquad\qquad+(1-t)^n\mathcal{B}_0(\lambda)\mathcal{B}_n(\mu)+t^n\mathcal{B}_n(\lambda)\mathcal{B}_0(\mu) \qquad (2.6)\\
&=\sum_{k=1}^{n-1}\binom{n}{k}t^k(1-t)^{n-k}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu)+(1-t)^n\delta_{1,\lambda}\mathcal{B}_n(\mu)+t^n\mathcal{B}_n(\lambda)\delta_{1,\mu},
\end{aligned}$$

where $\delta_{p,q}$ is the Kronecker symbol. On the right hand side of (2.5) we have that the first sum can be rewritten as

$$\begin{aligned}
&(1-t)\sum_{k=0}^{n}\binom{n}{k}t^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda)\\
&=(1-t)\sum_{k=1}^{n-1}\binom{n}{k}t^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda)\\
&\qquad\qquad+(1-t)t^n\mathcal{B}_0(\lambda\mu)\mathcal{B}_n(\lambda)+(1-t)\mathcal{B}_n(\lambda\mu)\mathcal{B}_0(\lambda) \qquad (2.7)\\
&=(1-t)\sum_{k=1}^{n-1}\binom{n}{k}t^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda)+(1-t)t^n\delta_{1,\lambda\mu}\mathcal{B}_n(\lambda)+(1-t)\mathcal{B}_n(\lambda\mu)\delta_{1,\lambda},
\end{aligned}$$

and the second sum can be rewritten as

$$\begin{aligned}
&t\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu)\\
&=t\sum_{k=1}^{n-1}\binom{n}{k}(1-t)^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu)\\
&\qquad\qquad+t(1-t)^n\mathcal{B}_0(\lambda\mu)\mathcal{B}_n(\mu)+t\mathcal{B}_n(\lambda\mu)\mathcal{B}_0(\mu) \qquad (2.8)\\
&=t\sum_{k=1}^{n-1}\binom{n}{k}(1-t)^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu)+t(1-t)^n\delta_{1,\lambda\mu}\mathcal{B}_n(\mu)+t\mathcal{B}_n(\lambda\mu)\delta_{1,\mu}.
\end{aligned}$$

By substituting the detailed expressions (2.6)–(2.8) back into (2.5), we get

$$\sum_{k=1}^{n-1}\binom{n}{k}t^k(1-t)^{n-k}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu)+(1-t)^n\delta_{1,\lambda}\mathcal{B}_n(\mu)+t^n\mathcal{B}_n(\lambda)\delta_{1,\mu}$$

$$
\begin{aligned}
&= nt(1-t)\mathcal{B}_{n-1}(\lambda\mu) + (1-t)\mathcal{B}_n(\lambda\mu)\delta_{1,\lambda} \\
&\quad + (1-t)\sum_{k=1}^{n-1}\binom{n}{k}t^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda) + (1-t)t^n\delta_{1,\lambda\mu}\mathcal{B}_n(\lambda) \\
&\quad + t\sum_{k=1}^{n-1}\binom{n}{k}(1-t)^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu) + t(1-t)^n\delta_{1,\lambda\mu}\mathcal{B}_n(\mu) \qquad (2.9) \\
&\quad + t\mathcal{B}_n(\lambda\mu)\delta_{1,\mu}.
\end{aligned}
$$

By dividing both sides of (2.9) by $t(1-t)$, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{n-1}\binom{n}{k}t^{k-1}(1-t)^{n-k-1}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu) + \frac{(1-t)^{n-1}}{t}\delta_{1,\lambda}\mathcal{B}_n(\mu) + \frac{t^{n-1}}{1-t}\mathcal{B}_n(\lambda)\delta_{1,\mu} \\
&= n\mathcal{B}_{n-1}(\lambda\mu) + \sum_{k=1}^{n-1}\binom{n}{k}t^{n-k-1}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda) + t^{n-1}\delta_{1,\lambda\mu}\mathcal{B}_n(\lambda) \qquad (2.10) \\
&+ \frac{1}{t}\mathcal{B}_n(\lambda\mu)\delta_{1,\lambda} + \sum_{k=1}^{n-1}\binom{n}{k}(1-t)^{n-k-1}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu) + (1-t)^{n-1}\delta_{1,\lambda\mu}\mathcal{B}_n(\mu) \\
&+ \frac{1}{1-t}\mathcal{B}_n(\lambda\mu)\delta_{1,\mu}.
\end{aligned}
$$

We rewrite the (2.10) as

$$
\begin{aligned}
&\sum_{k=1}^{n-1}\binom{n}{k}t^{k-1}(1-t)^{n-k-1}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu) \\
&\qquad = n\mathcal{B}_{n-1}(\lambda\mu) + \sum_{k=1}^{n-1}\binom{n}{k}t^{n-k-1}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda) \\
&\qquad\qquad + \sum_{k=1}^{n-1}\binom{n}{k}(1-t)^{n-k-1}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu) + A^{\delta}, \qquad (2.11)
\end{aligned}
$$

where

$$
\begin{aligned}
A^{\delta} &= \frac{1}{t}(\mathcal{B}_n(\lambda\mu) - (1-t)^{n-1}\mathcal{B}_n(\mu))\delta_{1,\lambda} + (t^{n-1}\mathcal{B}_n(\lambda) \\
&\quad + (1-t)^{n-1}\mathcal{B}_n(\mu))\delta_{1,\lambda\mu} + \frac{1}{1-t}(\mathcal{B}_n(\lambda\mu) - t^{n-1}\mathcal{B}_n(\lambda))\delta_{1,\mu}. \qquad (2.12)
\end{aligned}
$$

By integrating (2.11) between 0 and 1 with respect to t and using the formulae

$$
\int_0^1 t^p(1-t)^q dt = \frac{p!q!}{(p+q+1)!}, \qquad p,q \geq 0,
$$

$$\int_0^1 \frac{1-t^{p+1}-(1-t)^{p+1}}{t(1-t)}dt = 2\int_0^1 \frac{1-t^p}{1-t}dt = 2H_p, \qquad p \geq 1,$$

we obtain

$$\begin{aligned}
&\int_0^1 \sum_{k=1}^{n-1}\binom{n}{k} t^{k-1}(1-t)^{n-k-1}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu)dt \\
&\quad = \int_0^1 n\mathcal{B}_{n-1}(\lambda\mu)dt + \int_0^1 \sum_{k=1}^{n-1}\binom{n}{k} t^{n-k-1}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda)dt \\
&\quad + \int_0^1 \sum_{k=1}^{n-1}\binom{n}{k}(1-t)^{n-k-1}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu)dt + \int_0^1 A^\delta dt,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\sum_{k=1}^{n-1}\binom{n}{k}\frac{(k-1)!(n-k-1)!}{(n-1)!}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu) \\
&\qquad = n\mathcal{B}_{n-1}(\lambda\mu) + \sum_{k=1}^{n-1}\binom{n}{k}\mathcal{B}_k(\lambda\mu)\frac{\mathcal{B}_{n-k}(\lambda)}{n-k} \\
&\qquad\qquad + \sum_{k=1}^{n-1}\binom{n}{k}\mathcal{B}_k(\lambda\mu)\frac{\mathcal{B}_{n-k}(\mu)}{n-k} + \int_0^1 A^\delta dt. \qquad (2.13)
\end{aligned}$$

By dividing both sides of (2.13) by n and performing elementary transformations of the binomial coefficients of (2.13), we can state the following result.

Theorem 2.1. *For all $n \geq 2$,*

$$\begin{aligned}
&\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(\lambda)}{k}\frac{\mathcal{B}_{n-k}(\mu)}{n-k} \\
&= \mathcal{B}_{n-1}(\lambda\mu) + \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\lambda\mu)}{k}\frac{\mathcal{B}_{n-k}(\lambda)+\mathcal{B}_{n-k}(\mu)}{n-k} + \frac{1}{n}\int_0^1 A^\delta dt, \qquad (2.14)
\end{aligned}$$

where A^δ is given by (2.12).

We have to consider different possible cases for λ and μ values.

Example 2.2. Let $\lambda = 1$, $\mu = 1$. It follows from (2.12) that

$$A^\delta = \frac{1-(1-t)^{n-1}}{t}B_n + (t^{n-1}+(1-t)^{n-1})B_n + \frac{1-t^{n-1}}{1-t}B_n. \qquad (2.15)$$

Therefore, the integrating of (2.15) between 0 and 1 with respect to t gives

$$\int_0^1 A^\delta dt = \int_0^1 \frac{1-t^n-(1-t)^n}{t(1-t)} B_n dt + \int_0^1 t^{n-1} B_n dt + \int_0^1 (1-t)^{n-1} B_n dt$$
$$= 2H_n B_n. \tag{2.16}$$

By substituting (2.16) back into (2.14) and replacing all $\mathcal{B}$ by B consistently with the case condition, we get

$$\sum_{k=1}^{n-1} \frac{B_k}{k}\frac{B_{n-k}}{n-k} - 2\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{B_k}{k}\frac{B_{n-k}}{n-k} = B_{n-1} + 2H_n\frac{B_n}{n}.$$

Note that for even $n \geq 4$, all summands, containing odd-indexed Bernoulli numbers, equal zero. Thus, the sums must be limited from $k=2$ up to $n-2$ over even indexes only. Moreover, the term B_{n-1} on the RH side disappears from the same reason. Now we have

$$\sum_{k=2}^{n-2} \frac{B_k}{k}\frac{B_{n-k}}{n-k} - 2\sum_{k=2}^{n-2}\binom{n-1}{k-1}\frac{B_k}{k}\frac{B_{n-k}}{n-k} = 2H_n\frac{B_n}{n}.$$

In order to obtain the Miki identity (1.4), let us consider the sum

$$2\sum_{k=2}^{n-2}\binom{n-1}{k-1}\frac{B_k}{k}\frac{B_{n-k}}{n-k} = \frac{2}{n}\sum_{k=2}^{n-2}\frac{1}{n-k}\binom{n}{k}B_kB_{n-k}.$$

Finally, using $2\sum_{k=2}^{n-2}\frac{1}{n-k}\binom{n}{k}B_kB_{n-k} = n\sum_{k=2}^{n-2}\binom{n}{k}\frac{B_k}{k}\frac{B_{n-k}}{n-k}$ (see [1]), we obtain the known Miki identity (1.4) (see [1, 8, 15]).

Corollary 2.3. *Let $\mu \neq 1$. For all $n \geq 2$, the following identities are valid*

$$\sum_{k=1}^{n-1} \frac{B_k}{k}\frac{\mathcal{B}_{n-k}(\mu)}{n-k} - \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\mu)}{k}\frac{B_{n-k}+\mathcal{B}_{n-k}(\mu)}{n-k}$$
$$= \mathcal{B}_{n-1}(\mu) + H_{n-1}\frac{\mathcal{B}_n(\mu)}{n}, \tag{2.17}$$

$$\sum_{k=1}^{n-1} \frac{\mathcal{B}_k(\frac{1}{\mu})}{k}\frac{\mathcal{B}_{n-k}(\mu)}{n-k} - \frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}B_k\frac{\mathcal{B}_{n-k}(\frac{1}{\mu})+\mathcal{B}_{n-k}(\mu)}{n-k} = B_{n-1}. \tag{2.18}$$

Moreover, if $\lambda, \mu, \lambda\mu \neq 1$, then

$$\sum_{k=1}^{n-1} \frac{\mathcal{B}_k(\lambda)}{k}\frac{\mathcal{B}_{n-k}(\mu)}{n-k} - \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\lambda\mu)}{k}\frac{\mathcal{B}_{n-k}(\lambda)+\mathcal{B}_{n-k}(\mu)}{n-k}$$
$$= \mathcal{B}_{n-1}(\lambda\mu). \tag{2.19}$$

Proof. In the case $\lambda = 1$, $\mu \neq 1$, we have from (2.12) that $A^\delta = \frac{1-(1-t)^{n-1}}{t}\mathcal{B}_n(\mu)$. The integrating between 0 and 1 with respect to t gives

$$\int_0^1 A^\delta dt = \int_0^1 \frac{1-(1-t)^{n-1}}{t}\mathcal{B}_n(\mu)dt = H_{n-1}\mathcal{B}_n(\mu). \tag{2.20}$$

By substituting (2.20) into (2.14), we obtain

$$\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(\lambda)}{k}\frac{\mathcal{B}_{n-k}(\mu)}{n-k}$$
$$= \mathcal{B}_{n-1}(\lambda\mu) + \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\lambda\mu)}{k}\frac{\mathcal{B}_{n-k}(\lambda)+\mathcal{B}_{n-k}(\mu)}{n-k} + \frac{1}{n}H_{n-1}\mathcal{B}_n(\mu).$$

By taking into account that $\lambda = 1$ and $\mathcal{B}_p(1) = B_p$, we get the identity (2.17).

In order to prove (2.18), we suppose that $\lambda = \frac{1}{\mu} \neq 1$. Then, from (2.12), we obtain that $A^\delta = t^{n-1}\mathcal{B}_n(\frac{1}{\mu}) + (1-t)^{n-1}\mathcal{B}_n(\frac{1}{\mu})$. By integrating of A^δ between 0 and 1 with respect to t, we get

$$\int_0^1 A^\delta dt = \int_0^1 t^{n-1}\mathcal{B}_n(\frac{1}{\mu})dt + \int_0^1 (1-t)^{n-1}\mathcal{B}_n(\mu)dt = \frac{\mathcal{B}_n(\frac{1}{\mu})+\mathcal{B}_n(\mu)}{n}. \tag{2.21}$$

By substituting (2.21) into (2.14), we obtain

$$\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(\lambda)}{k}\frac{\mathcal{B}_{n-k}(\mu)}{n-k} = \mathcal{B}_{n-1}(\lambda\mu) + \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\lambda\mu)}{k}\frac{\mathcal{B}_{n-k}(\lambda)}{n-k}$$
$$+ \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\lambda\mu)}{k}\frac{\mathcal{B}_{n-k}(\mu)}{n-k} + \frac{\mathcal{B}_n(\lambda)+\mathcal{B}_n(\frac{1}{\lambda})}{n^2}.$$

By substituting $\lambda = \frac{1}{\mu}$ into the last equation and using the facts that $\mathcal{B}_p(1) = B_p$ and $B_0 = 1$, we obtain (2.18).

Equation (2.19) follows from the fact that $A^\delta = 0$ for $\lambda, \mu, \lambda\mu \neq 1$. □

By integrating both sides of (2.9) from 0 to 1 with respect to t and multiplying by $(n+1)(n+2)$, we obtain the following result, which is an analogue of the Matiyasevich identity (1.5).

Theorem 2.4. *For all $n \geq 2$,*

$$(n+2)\sum_{k=1}^{n-1}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu) - \sum_{k=1}^{n-1}\binom{n+2}{k}\mathcal{B}_k(\lambda\mu)\left(\mathcal{B}_{n-k}(\lambda)+\mathcal{B}_{n-k}(\mu)\right)$$
$$= \frac{n(n+1)(n+2)}{6}\mathcal{B}_{n-1}(\lambda\mu) \tag{2.22}$$
$$+ \frac{(n-1)(n+2)}{2}\left(\mathcal{B}_n(\mu)\delta_{1,\lambda} + \mathcal{B}_n(\lambda)\delta_{1,\mu}\right) + (\mathcal{B}_n(\lambda)+\mathcal{B}_n(\mu))\delta_{1,\lambda\mu}.$$

Example 2.5. Let $\lambda = 1$, $\mu = 1$. Then, by using the fact that $\mathcal{B}_p(1) = B_p$, we obtain

$$(n+2)\sum_{k=1}^{n-1} B_k B_{n-k} - 2\sum_{k=1}^{n-1}\binom{n+2}{k} B_k B_{n-k} = n(n+1)B_n + \frac{n(n+1)(n+2)}{6}B_{n-1}. \tag{2.23}$$

Finally, by assuming that n is even and $n \geq 4$, we get that all terms, containing odd indexed Bernoulli numbers, equal zero. Under this condition the $(n-1)$st Bernoulli number on the RH side disappears, and the summation limits are from 2 till $n-2$. Thus, we obtain (1.5) (see also [1]).

Corollary 2.6. *Let $\mu \neq 1$. Then, for all $n \geq 2$, the following identities are valid:*

$$(n+2)\sum_{k=1}^{n-1} B_k \mathcal{B}_{n-k}(\mu) - \sum_{k=1}^{n-1}\binom{n+2}{k}\mathcal{B}_k(\mu)\left(B_{n-k} + \mathcal{B}_{n-k}(\mu)\right) = \frac{n(n+1)(n+2)}{6}\mathcal{B}_{n-1}(\mu) + \frac{(n-1)(n+2)}{2}\mathcal{B}_n(\mu), \tag{2.24}$$

$$(n+2)\sum_{k=1}^{n-1} \mathcal{B}_k(\frac{1}{\mu})\mathcal{B}_{n-k}(\mu) - \sum_{k=1}^{n-1}\binom{n+2}{k} B_k \left(\mathcal{B}_{n-k}(\frac{1}{\mu}) + \mathcal{B}_{n-k}(\mu)\right) = \frac{n(n+1)(n+2)}{6}B_{n-1} + \mathcal{B}_n(\frac{1}{\mu}) + \mathcal{B}_n(\mu). \tag{2.25}$$

Moreover, if $\lambda, \mu, \lambda\mu \neq 1$, then

$$(n+2)\sum_{k=1}^{n-1} \mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu) - \sum_{k=1}^{n-1}\binom{n+2}{k}\mathcal{B}_k(\lambda\mu)\left(\mathcal{B}_{n-k}(\lambda) + \mathcal{B}_{n-k}(\mu)\right) = \frac{n(n+1)(n+2)}{6}\mathcal{B}_{n-1}(\mu). \tag{2.26}$$

Proof. By substituting $\lambda = 1$ into (2.22) and using the facts that $\mathcal{B}_p(1) = B_p$ and $\delta_{1,\mu} = \delta_{1,\lambda\mu} = 0$, we obtain (2.24). By substituting $\lambda = \frac{1}{\mu}$ into (2.22) and using the fact that $\delta_{1,\lambda} = \delta_{1,\mu} = 0$, we obtain (2.25). Equation (2.26) follows from (2.22) by using the fact that $\delta_{1,\lambda} = \delta_{1,\mu} = \delta_{1,\lambda\mu} = 0$. $\square$

By dividing (2.9) by t and substituting $t = 0$, we obtain the following analogue of the Euler identity (1.3).

Theorem 2.7 (The Euler identity analogue)**.** *For all $n \geq 2$,*

$$\sum_{k=1}^{n-1}\binom{n}{k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu) = n\mathcal{B}_1(\lambda)\mathcal{B}_{n-1}(\mu) - n\mathcal{B}_{n-1}(\lambda\mu) - n\mathcal{B}_{n-1}(\lambda\mu)\mathcal{B}_1(\lambda) - (n-1)\mathcal{B}_n(\mu)\delta_{1,\lambda} - \mathcal{B}_n(\lambda)\delta_{1,\mu} - \mathcal{B}_n(\mu)\delta_{1,\lambda\mu}. \tag{2.27}$$

Proof. By dividing (2.9) by t, we obtain

$$\begin{aligned}\sum_{k=1}^{n-1}\binom{n}{k}t^{k-1}(1-t)^{n-k}\mathcal{B}_k(\lambda)\mathcal{B}_{n-k}(\mu)+\frac{(1-t)^n}{t}\delta_{1,\lambda}\mathcal{B}_n(\mu)+t^{n-1}\mathcal{B}_n(\lambda)\delta_{1,\mu}\\ =n(1-t)\mathcal{B}_{n-1}(\lambda\mu)+(1-t)\sum_{k=1}^{n-1}\binom{n}{k}t^{n-k-1}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda)\\ +(1-t)t^{n-1}\delta_{1,\lambda\mu}\mathcal{B}_n(\lambda)+\frac{(1-t)}{t}\mathcal{B}_n(\lambda\mu)\delta_{1,\lambda}\\ +\sum_{k=1}^{n-1}\binom{n}{k}(1-t)^{n-k}\mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu)\\ +(1-t)^n\delta_{1,\lambda\mu}\mathcal{B}_n(\mu)+\mathcal{B}_n(\lambda\mu)\delta_{1,\mu}.\end{aligned} \tag{2.28}$$

Consider now the difference $\frac{(1-t)^n}{t}\delta_{1,\lambda}\mathcal{B}_n(\mu)-\frac{1-t}{t}\mathcal{B}_n(\lambda\mu)\delta_{1,\lambda}$. It is obviously that

$$\begin{aligned}\frac{(1-t)^n}{t}\delta_{1,\lambda}\mathcal{B}_n(\mu)-\frac{1-t}{t}\mathcal{B}_n(\lambda\mu)\delta_{1,\lambda}\\ =\frac{(1-t)^n}{t}\delta_{1,\lambda}\mathcal{B}_n(\mu)-\frac{1-t}{t}\mathcal{B}_n(\mu)\delta_{1,\lambda}\\ =\delta_{1,\lambda}\mathcal{B}_n(\mu)\frac{\sum_{j=0}^{n}\binom{n}{j}(-t)^j-1+t}{t}\\ =\delta_{1,\lambda}\mathcal{B}_n(\mu)\left(-\sum_{j=2}^{n}\binom{n}{j}(-t)^{j-1}-(n-1)\right).\end{aligned} \tag{2.29}$$

By substituting $t=0$ into (2.28) and using (2.29), we obtain (2.27). □

Example 2.8. Let $\lambda=1$, $\mu=1$. Then, by using the fact that $B_0=1$, we get

$$\sum_{k=0}^{n-1}\binom{n}{k}B_kB_{n-k}=-nB_n-nB_{n-1}.$$

Note that for $n\geq 4$, the odd Bernoulli numbers equal to zero and, thus, only one of the members on the right hand side will stay. Therefore, by assuming that $n\geq 4$ and n is even, we obtain the Euler identity (1.3) (see also [1, 6]).

Corollary 2.9. *For all $n\geq 2$ and $\mu\neq 1$, the following identities are valid:*

$$\sum_{k=1}^{n-1}\binom{n}{k}\mathcal{B}_k(\mu)\mathcal{B}_{n-k}(\mu)=-(n-1)\mathcal{B}_n(\mu)-n\mathcal{B}_{n-1}(\mu), \tag{2.30}$$

$$\sum_{k=0}^{n-1}\binom{n}{k}B_k\mathcal{B}_{n-k}(\frac{1}{\mu})=n\mathcal{B}_1(\mu)\mathcal{B}_{n-1}(\frac{1}{\mu})+nB_{n-1}\mathcal{B}_1(\frac{1}{\mu}). \tag{2.31}$$

Moreover, if $\lambda, \mu, \lambda\mu \neq 1$*, then*

$$\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{B}_k(\lambda\mu)\mathcal{B}_{n-k}(\mu) = n\mathcal{B}_1(\lambda)\mathcal{B}_{n-1}(\mu) - n(1+\mathcal{B}_1(\lambda))\mathcal{B}_{n-1}(\lambda\mu).$$

Identity (2.30) is obtained by substituting $\lambda = 1$ into (2.27), and Identity (2.31) is obtained in case $\lambda\mu = 1$. Note that here we use the fact that $\mathcal{B}_1(\mu) = \frac{1}{\mu-1}$ and, therefore, $\mathcal{B}_1(1/\mu) = -(\mathcal{B}_1(\mu)+1)$.

3. Identities for the Apostol-Genocchi numbers

Following the same technique we used in the previous section, we will obtain the analogues of the Miki and Euler identities for the Apostol-Genocchi numbers. It is easy to show that

$$\frac{1}{\lambda e^a + 1} \cdot \frac{1}{\mu e^b + 1} = \frac{1}{\lambda\mu e^{a+b} - 1}\left(1 - \frac{1}{\lambda e^a + 1} - \frac{1}{\mu e^b + 1}\right). \tag{3.1}$$

Let us take $a = xt$ and $b = (1-t)x$ and multiply both sides of the (3.1) by $4t(1-t)x^2$. We get

$$\begin{aligned}&\frac{2tx}{\lambda e^{tx}+1} \cdot \frac{2(1-t)x}{\mu e^{(1-t)x}+1}\\ &\qquad = 2 \cdot \frac{x}{\lambda\mu e^x - 1}\left(2t(1-t)x - (1-t)\frac{2tx}{\lambda e^{tx}+1} - t\frac{2(1-t)x}{\mu e^{(1-t)x}+1}\right),\end{aligned}$$

By using (1.1) and (1.2), we get

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} \mathcal{G}_n(\lambda)\frac{t^n x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \mathcal{G}_n(\mu)\frac{(1-t)^n x^n}{n!}\right)\\ &= 2\sum_{n=0}^{\infty} \mathcal{B}_n(\lambda\mu)\frac{x^n}{n!} \cdot\\ &\qquad \cdot \left(2t(1-t)x - (1-t)\sum_{n=0}^{\infty} \mathcal{G}_n(\lambda)\frac{t^n x^n}{n!} - t\sum_{n=0}^{\infty} \mathcal{G}_n(\mu)\frac{(1-t)^n x^n}{n!}\right).\end{aligned}$$

Therefore, by applying the Cauchy product and extracting the coefficients of $\frac{x^n}{n!}$, we obtain

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_k(\lambda)\mathcal{G}_{n-k}(\mu)t^k(1-t)^{n-k}\\ &\qquad = 4t(1-t)n\mathcal{B}_{n-1}(\lambda\mu) - 2(1-t)\sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k(\lambda\mu)\mathcal{G}_{n-k}(\lambda)t^{n-k}\end{aligned}$$

$$- 2t\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(\lambda\mu)\mathcal{G}_{n-k}(\mu)(1-t)^{n-k}. \qquad (3.2)$$

Now we divide (3.2) by $t(1-t)$ and then integrate with respect to t from 0 to 1. By using the facts that $\mathcal{B}_0 = 0$, $B_0 = 1$, and $\mathcal{G}_0 = G_0 = 0$, we obtain the following statement, that is an analogue of the Miki identity (1.4) for the Apostol-Genocchi numbers.

Theorem 3.1. *For all $n \geq 2$,*

$$\sum_{k=1}^{n-1}\frac{\mathcal{G}_k(\lambda)}{k}\frac{\mathcal{G}_{n-k}(\mu)}{n-k} + 2\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\lambda\mu)}{k}\frac{\mathcal{G}_{n-k}(\lambda)+\mathcal{G}_{n-k}(\mu)}{n-k}$$
$$= 4\mathcal{B}_{n-1}(\lambda\mu) - \frac{2}{n^2}\left(\mathcal{G}_n(\lambda)+\mathcal{G}_n(\mu)\right)\delta_{1,\lambda\mu}.$$

Example 3.2. Let $\lambda = \mu = 1$. Then

$$\sum_{k=1}^{n-1}\frac{G_k}{k}\frac{G_{n-k}}{n-k} + 4\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{B_k}{k}\frac{G_{n-k}}{n-k} = 4B_{n-1} - \frac{4G_n}{n^2}.$$

Let us suppose now that $n \geq 4$ and n is even. Then, the facts that both odd indexed Bernoulli and Genocchi numbers equal zero imply

$$\sum_{k=2}^{n-2}\frac{G_k}{k}\frac{G_{n-k}}{n-k} + 4\sum_{k=2}^{n-2}\binom{n-1}{k-1}\frac{B_k}{k}\frac{G_{n-k}}{n-k} = -\frac{4G_n}{n^2}.$$

Multiplying both sides of this equation by n and using $\frac{n}{k(n-k)} = \frac{1}{k} + \frac{1}{n-k}$ and $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$ yield

$$2\sum_{k=2}^{n-2}\frac{G_kG_{n-k}}{n-k} + 4\sum_{k=2}^{n-2}\binom{n}{k}\frac{B_kG_{n-k}}{n-k} = -\frac{4G_n}{n}.$$

By dividing both sides by 2 and replacing the indexes k by $n-k$ and vice versa, we obtain the following analogue of the Miki identity (1.4) for the Genocchi numbers

$$\sum_{k=2}^{n-2}\frac{G_kG_{n-k}}{k} + 2\sum_{k=2}^{n-2}\binom{n}{k}\frac{G_kB_{n-k}}{k} = -\frac{2G_n}{n}.$$

Note that this coincides with [1, Proposition 4.1] for the numbers B'_n, which are defined as $G_n = 2B'_n$.

Corollary 3.3. *Let $\mu \neq 1$. For $n \geq 2$,*

$$\sum_{k=1}^{n-1}\frac{G_k}{k}\frac{\mathcal{G}_{n-k}(\mu)}{n-k} + 2\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\mu)}{k}\frac{G_{n-k}+\mathcal{G}_{n-k}(\mu)}{n-k} = 4\mathcal{B}_{n-1}(\mu),$$

$$\sum_{k=1}^{n-1}\frac{\mathcal{G}_k(\frac{1}{\mu})}{k}\frac{\mathcal{G}_{n-k}(\mu)}{n-k}+2\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{B_k}{k}\frac{\mathcal{G}_{n-k}(\frac{1}{\mu})+\mathcal{G}_{n-k}(\mu)}{n-k}$$
$$=4B_{n-1}-\frac{2}{n^2}\left(\mathcal{G}_n(\frac{1}{\mu})+\mathcal{G}_n(\mu)\right).$$

Moreover, if $\lambda,\mu,\lambda\mu\neq 1$*, then*

$$\sum_{k=1}^{n-1}\frac{\mathcal{G}_k(\lambda)}{k}\frac{\mathcal{G}_{n-k}(\mu)}{n-k}+2\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{B}_k(\lambda\mu)}{k}\frac{\mathcal{G}_{n-k}(\lambda)+\mathcal{G}_{n-k}(\mu)}{n-k}=4\mathcal{B}_{n-1}(\lambda\mu).$$

In order to obtain the analogues of the Euler identity, we divide (3.2) by $t(1-t)$ and subsitute $t=0$.

Theorem 3.4. *For all* $n\geq 2$*,*

$$\sum_{k=1}^{n-1}\binom{n}{k}\mathcal{B}_k(\lambda\mu)\mathcal{G}_{n-k}(\mu)=n\mathcal{B}_{n-1}(\lambda\mu)(2-\mathcal{G}_1(\lambda))-\frac{n\mathcal{G}_1(\lambda)\mathcal{G}_{n-1}(\mu)}{2}-\mathcal{G}_n(\mu)\delta_{1,\lambda\mu}.$$

Example 3.5. Let $\lambda=\mu=1$. Then, since $G_1=1$, we obtain

$$\sum_{k=1}^{n-1}\binom{n}{k}B_kG_{n-k}=nB_{n-1}-\frac{n}{2}G_{n-1}-G_n.$$

By using the fact that all odd indexed Bernoulli and Genocchi numbers starting from $n=3$ disappear, we obtain for all even $n\geq 4$, $\sum_{k=2}^{n-2}\binom{n}{k}B_kG_{n-k}=-G_n$, where the summation is over even indexed numbers (see also [1]).

Here are some identities of the Euler type for the Apostol-Genocchi numbers following from Theorem 3.4.

Corollary 3.6. *Let* $\lambda\neq 1$*. For* $n\geq 2$*,*

$$\sum_{k=1}^{n-1}\binom{n}{k}\mathcal{B}_k(\lambda)\mathcal{G}_{n-k}(\lambda)=n\mathcal{B}_{n-1}(\lambda)-\frac{n\mathcal{G}_{n-1}(\lambda)}{2},$$
$$\sum_{k=1}^{n-1}\binom{n}{k}\mathcal{B}_k(\lambda)G_{n-k}=n\mathcal{B}_{n-1}(\lambda)(2-\mathcal{G}_1(\lambda))-\frac{n\mathcal{G}_1(\lambda)G_{n-1}}{2},$$
$$\sum_{k=0}^{n-2}\binom{n}{k}B_k\mathcal{G}_{n-k}(\frac{1}{\lambda})=-\frac{n\mathcal{G}_1(\lambda)\mathcal{G}_{n-1}(\frac{1}{\lambda})}{2}.$$

Moreover, if $\lambda,\mu,\lambda\mu\neq 1$*, then*

$$\sum_{k=1}^{n-1}\mathcal{B}_k(\lambda\mu)\mathcal{G}_{n-k}(\mu)=n\mathcal{B}_{n-1}(\lambda\mu)(2-\mathcal{G}_1(\lambda))-\frac{n\mathcal{G}_1(\lambda)\mathcal{G}_{n-1}(\mu)}{2}.$$

Here we used the facts that $B_0 = 1$ and $2 - \mathcal{G}_1(\lambda) = \mathcal{G}_1(\frac{1}{\lambda})$. Another series of the identities of the Miki and the Euler types for the Apostol-Genocchi numbers can be obtained in the same manner, when the following, easily proved, equation

$$\frac{1}{\lambda e^a - 1} \cdot \frac{1}{\mu e^b + 1} = \frac{1}{\lambda\mu e^{a+b} + 1}\left(1 + \frac{1}{\lambda e^a - 1} - \frac{1}{\mu e^b + 1}\right)$$

is taken as a basis for the generating function approach. The following result may be proved in the same way as Theorem 3.1. Let us take $a = xt$ and $b = (1-t)x$ and multiply both sides of the two last identities by $4t(1-t)x^2$. We get

$$\begin{aligned} &2 \cdot \frac{tx}{\lambda e^{tx} - 1} \cdot \frac{2(1-t)x}{\mu e^{(1-t)x} + 1} \\ &\quad = \frac{2x}{\lambda\mu e^x + 1}\left(2t(1-t)x + 2(1-t)\frac{tx}{\lambda e^{tx} - 1} - t\frac{2(1-t)x}{\mu e^{(1-t)x} + 1}\right). \end{aligned} \tag{3.3}$$

Again, we use (1.1) and (1.2) and apply the Cauchy product in order to extract the coefficients of $\frac{x^n}{n!}$ on both sides of (3.3). Thus, we obtain

$$\begin{aligned} &2\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(\lambda)\mathcal{G}_{n-k}(\mu)t^k(1-t)^{n-k} \\ &\quad = 2t(1-t)n\mathcal{G}_{n-1}(\lambda\mu) + 2(1-t)\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k(\lambda\mu)\mathcal{B}_{n-k}(\lambda)t^{n-k} \\ &\qquad - t\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k(\lambda\mu)\mathcal{G}_{n-k}(\mu)(1-t)^{n-k}. \end{aligned} \tag{3.4}$$

Now we divide both equations by $t(1-t)$ and then integrate with respect to t from 0 to 1. By using the facts that $\mathcal{B}_0 = 0$, $B_0 = 1$, and $\mathcal{G}_0 = G_0 = 0$, we obtain the following statement, that is another analogue of the Miki identity for the Apostol-Genocchi numbers.

Theorem 3.7. *For all $n \geq 2$,*

$$\begin{aligned} &\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(\lambda)}{k}\frac{\mathcal{G}_{n-k}(\mu)}{n-k} - \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{G}_k(\lambda\mu)}{k}\frac{\mathcal{B}_{n-k}(\lambda) - \frac{1}{2}\mathcal{G}_{n-k}(\mu)}{n-k} \\ &\qquad = \mathcal{G}_{n-1}(\lambda\mu) + \frac{\mathcal{G}_n(\mu)}{n}H_{n-1}\delta_{1,\lambda}. \end{aligned} \tag{3.5}$$

Example 3.8. Let $\lambda = \mu = 1$. Then, for all $n \geq 2$,

$$\sum_{k=1}^{n-1}\frac{B_k}{k}\frac{G_{n-k}}{n-k} - \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{G_k}{k}\frac{B_{n-k} - \frac{1}{2}G_{n-k}}{n-k} = G_{n-1} + \frac{G_n}{n}H_{n-1}.$$

It is known that the Genocchi and Bernoulli numbers are related as

$$G_n = 2(1-2^n)B_n$$

(see [1]). By substituting this identity into the difference $B_{n-k}-\frac{1}{2}G_{n-k}$ under the second summation, we obtain

$$\sum_{k=1}^{n-1}\frac{B_k}{k}\frac{G_{n-k}}{n-k}-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{G_k}{k}\frac{B_{n-k}-(1-2^{n-k})B_{n-k}}{n-k}=G_{n-1}+\frac{G_n}{n}H_{n-1}.$$

Note that for $n\geq 3$, the odd-indexed Bernoulli and Genocchi numbers disappear, therefore, let us assume now that n is even and $n\geq 4$. Thus, we have

$$\sum_{k=2}^{n-2}\frac{B_k}{k}\frac{G_{n-k}}{n-k}-\sum_{k=2}^{n-2}\binom{n-1}{k-1}\frac{G_k}{k}\frac{2^{n-k}B_{n-k}}{n-k}=\frac{G_n}{n}H_{n-1}.$$

Using the binomial identity $\binom{n-1}{k-1}=\binom{n-1}{n-k}$ leads to

$$\sum_{k=2}^{n-2}\frac{B_k}{k}\frac{G_{n-k}}{n-k}-\sum_{k=2}^{n-2}\binom{n-1}{n-k}\frac{G_k}{k}\frac{2^{n-k}B_{n-k}}{n-k}=\frac{G_n}{n}H_{n-1}.$$

We replace k by $n-k$ under the second summation. Finally, using the notation $G_n=2B_n'$, proposed in [1], and dividing both sides by 2 lead to the statement (4.2) of [1, Proposition 4.1]

$$\sum_{k=2}^{n-2}\frac{B_k}{k}\frac{B'_{n-k}}{n-k}-\sum_{k=2}^{n-2}\binom{n-1}{k}\frac{2^kB_k}{k}\frac{B'_{n-k}}{n-k}=\frac{B'_n}{n}H_{n-1}.$$

Corollary 3.9. *Let $\mu\neq 1$. For all $n\geq 2$,*

$$\sum_{k=1}^{n-1}\frac{B_k}{k}\frac{\mathcal{G}_{n-k}(\mu)}{n-k}-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{G}_k(\mu)}{k}\frac{B_{n-k}-\frac{1}{2}\mathcal{G}_{n-k}(\mu)}{n-k}=\mathcal{G}_{n-1}(\mu)+\frac{\mathcal{G}_n(\mu)}{n}H_{n-1}.$$

Due to the asymmetry of λ and μ in the (3.5), we get the following corollary of the Theorem 3.7.

Corollary 3.10. *Let $\lambda\neq 1$. For all $n\geq 2$,*

$$\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(\lambda)}{k}\frac{G_{n-k}}{n-k}-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{G}_k(\lambda)}{k}\frac{\mathcal{B}_{n-k}(\lambda)-\frac{1}{2}G_{n-k}}{n-k}=\mathcal{G}_{n-1}(\lambda),$$

$$\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(\lambda)}{k}\frac{\mathcal{G}_{n-k}(\frac{1}{\lambda})}{n-k}-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{G_k}{k}\frac{\mathcal{B}_{n-k}(\lambda)-\frac{1}{2}\mathcal{G}_{n-k}(\frac{1}{\lambda})}{n-k}=G_{n-1}. \tag{3.6}$$

Moreover, if $\lambda,\mu,\lambda\mu\neq 1$, then

$$\sum_{k=1}^{n-1}\frac{\mathcal{B}_k(\lambda)}{k}\frac{\mathcal{G}_{n-k}(\mu)}{n-k}-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\mathcal{G}_k(\lambda\mu)}{k}\frac{\mathcal{B}_{n-k}(\lambda)-\frac{1}{2}\mathcal{G}_{n-k}(\mu)}{n-k}=\mathcal{G}(\lambda\mu).$$

By dividing (3.2) and (3.4) by t and then substituting $t = 0$, we obtain the following analogue of the Euler identity.

Theorem 3.11. *For all $n \geq 2$,*

$$\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{G}_k(\lambda\mu)\mathcal{G}_{n-k}(\mu) = 2n\mathcal{G}_{n-1}(\lambda\mu) + 2(n-1)\mathcal{G}_n(\lambda\mu)\delta_{1,\lambda} + 2n\mathcal{B}_1(\lambda)\left(\mathcal{G}_{n-1}(\lambda\mu) - \mathcal{G}_{n-1}(\mu)\right). \quad (3.7)$$

Example 3.12. Let $\lambda = \mu = 1$. Then

$$\sum_{k=1}^{n-1} \binom{n}{k} G_k G_{n-k} = 2nG_{n-1} + 2(n-1)G_n.$$

By using the fact that all odd indexed Bernoulli and Genocchi numbers starting from $n = 3$ disappear, we obtain a more familiar form for all even $n \geq 4$, $\sum_{k=2}^{n-2} \binom{n}{k} G_k G_{n-k} = 2(n-1)G_n$, where the summation is over even indexed numbers (see also [1]).

Corollary 3.13. *Let $\lambda \neq 1$ and $n \geq 2$. Then the following identities are valid*

$$\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{G}_k(\lambda)\mathcal{G}_{n-k}(\lambda) = 2n\mathcal{G}_{n-1}(\lambda) + 2(n-1)\mathcal{G}_n(\lambda), \quad (3.8)$$

$$\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{G}_k(\lambda)G_{n-k} = 2n\mathcal{G}_{n-1}(\lambda) + 2n\mathcal{B}_{n-1}(\lambda)(\mathcal{G}_{n-1}(\lambda) - G_{n-1}), \quad (3.9)$$

$$\sum_{k=1}^{n-1} \binom{n}{k} G_k\mathcal{G}_{n-k}(\frac{1}{\lambda}) = 2nG_{n-1} + 2n\mathcal{B}_1(\frac{1}{\lambda})\left(\mathcal{G}_{n-1}(\frac{1}{\lambda}) - G_{n-1}\right). \quad (3.10)$$

Moreover, if $\lambda, \mu, \lambda\mu \neq 1$, then

$$\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{G}_k(\lambda\mu)\mathcal{G}_{n-k}(\mu) = 2n\mathcal{G}_{n-1}(\lambda\mu) + 2n\mathcal{B}_{n-1}(\lambda)(\mathcal{G}_{n-1}(\lambda\mu) - \mathcal{G}_{n-1}(\mu)). \quad (3.11)$$

Proof. Replacing λ and μ in (3.7), and substituting $\mu = 1$ lead to

$$\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{G}_k(\lambda)\mathcal{G}_{n-k}(\lambda) = 2n\mathcal{G}_{n-1}(\lambda) + 2(n-1)\mathcal{G}_n(\lambda) + 2n\left(-\frac{1}{2}\right)\left(\mathcal{G}_{n-1}(\lambda) - \mathcal{G}_{n-1}(\lambda)\right).$$

The last summand equals zero, and we obtain the identiy (3.8). By substituting $\mu = 1$ into (3.7) we obtain (3.9). Substituting $\mu = \frac{1}{\lambda}$ into (2.14) and using the fact that $1 + \mathcal{B}_1(\lambda) = -\mathcal{B}_1(\frac{1}{\lambda})$ lead to (3.10). The second summand on the RH of the (3.7) disappears since $\lambda \neq 1$, and we obtain (3.11). □

Remark 3.14. As it was mentioned above, the classical Bernoulli and Genocchi numbers are connected via the following relationship $G_n = 2(1-2^n)B_n$. It is easy to see that also the Apostol-Bernoulli and Apostol-Genocchi numbers satisfy $\mathcal{G}_n(\lambda) = -2\mathcal{B}_n(-\lambda)$. Moreover, the Apostol-Bernoulli numbers satisfy $\mathcal{B}_{2n}(\lambda) = \mathcal{B}_{2n}(\frac{1}{\lambda})$ and $\mathcal{B}_{2n+1}(\lambda) = -\mathcal{B}_{2n+1}(\frac{1}{\lambda})$ for $\lambda \neq 1$. In the same manner, the Apostol-Genocchi numbers satisfy $\mathcal{G}_{2n}(\lambda) = \mathcal{G}_{2n}(\frac{1}{\lambda})$ and $\mathcal{G}_{2n+1}(\lambda) = -\mathcal{G}_{2n+1}(\frac{1}{\lambda})$ for $n > 0$. These relationships allow to obtain new identities from those considered in the current paper.

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