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On a variant of the Lucas' square pyramid problem

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Abstract

In this paper we consider the problem of finding integers k such that the sum of k consecutive cubes starting at n^3 is a perfect square. We give an upper bound of k in terms of n and then, list all possible k when $1 < n \leq 300$.

Keywords: Diophantine equation, Lucas' square pyramid problem, sum of squares, sum of cubes

MSC: 11A99, 11D09, 11D25

1. Introduction

The problem of finding integers k such that the sum of k consecutive squares is a square has been initiated by Lucas [3], who formulated the problem as follows: when does a square pyramid of cannonballs contain a number of cannonballs which is a perfect square? This is equivalent to solving the diophantine equation

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = y^2. \tag{1.1}$$

It was not until 1918 that a complete solution to Lucas' problem was given by Watson [5]. He showed that the diophantine equation (1.1) has only two solutions, namely $(k, y) = (1, 1)$ and $(24, 70)$. It is natural to ask whether this phenomenon keeps occurring when the initial square is shifted. This is in fact equivalent to solving the following diophantine equation

$$n^2 + (n+1)^2 + \cdots + (n+k-1)^2 = y^2. \tag{1.2}$$

This problem has been considered by many authors from different points of view. For instance, Beeckmans [1] determined all values $1 \leq k \leq 1000$ for which equation (1.2) has solutions (n, y). Using the theory of elliptic curves Bremner, Stroeker and Tzanakis [2] found all solutions k and y to equation (1.2) when $1 \leq n \leq 100$. Stroeker [4] considered the question of when does a sum of k consecutive cubes starting at n^3 equal a perfect square. He [4], considered the case where k is a fixed integer. In this paper we take $n > 1$ a fixed integer and consider the question of when does a sum k consecutive cubes starting at n^3 equal a perfect square. We will give in theorem 1 an upper bound of k in term of n, and then use this upper bound to do some computations to list all possible k when $1 \leq n \leq 300$. Our method uses only elementary techniques.

2. The sum of k consecutive cubes being a square

Stroeker [4] considered the question of when does a sum of k consecutive cubes starting at n^3 equal a perfect square. He [4] considered the case where k is a fixed integer. This is equivalent to solving the following diophantine equation:

$$n^3 + (n+1)^3 + \cdots + (n+k-1)^3 = y^2. \tag{2.1}$$

The problem is interesting only when $n > 1$. In fact, when $n = 1$, because of the well known equality $1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$ equation (2.1) is always true for any value of the integer k. Stroeker [4] solved equation (2.1) for $2 \leq k \leq 50$ and $k = 98$. We prove the following.

Theorem 2.1. *If $n > 1$ is a fixed integer, there are only finitely many k such that the sum of k consecutive cubes starting at n^3 is a perfect square. Moreover, $k \leq \lfloor \frac{n^2}{\sqrt{2}} - n + 1 \rfloor$.*

Proof. The equality

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 = \left(\frac{(n-1)n}{2}\right)^2$$

leads

$$n^3 + (n+1)^3 + \cdots + (n+k-1)^3 = \left(\frac{(n+k)(n+k-1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2.$$

Hence equation (2.1) gives

$$\left(\frac{(n+k)(n+k-1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2 = y^2.$$

It is well known that the positive solutions of the last equation are given by

$$\begin{cases} \frac{(n+k)(n+k-1)}{2} = \alpha(a^2+b^2), \\ \frac{n(n-1)}{2} = \alpha(a^2-b^2) & \alpha \in \mathcal{N} \\ y = \alpha(2ab) \end{cases} \tag{2.2}$$

or

$$\begin{cases} \frac{(n+k)(n+k-1)}{2} = \alpha(a^2+b^2) \\ \frac{n(n-1)}{2} = \alpha(2ab) & \alpha \in \mathcal{N} \\ y = \alpha(a^2-b^2) \end{cases} \tag{2.3}$$

where $a, b \in \mathbb{N}$, $\gcd(a,b) = 1$, $a > b$, $a \neq b \pmod 2$. The first equation in system (2.2) gives that

$$(n+k-1)^2 < 2\alpha(a^2+b^2). \tag{2.4}$$

The second equation in system (2.2) gives

$$\frac{n^2}{2} > \frac{n(n-1)}{2} = \alpha(a^2-b^2) \geq \alpha(a+b).$$

Hence

$$\left(\frac{n^2}{2}\right)^2 > (\alpha(a+b))^2 \geq \alpha(a^2+b^2). \tag{2.5}$$

Inequality (2.4) combined with inequality (2.5) yield

$$(n+k-1)^2 < 2\alpha(a^2+b^2) \leq 2\left(\frac{n^2}{2}\right)^2.$$

Whereupon

$$n+k-1 < \frac{n^2}{\sqrt{2}},$$

hence,

$$k \leq \frac{n^2}{\sqrt{2}} - n + 1.$$

The second equation in system (2.3) implies that

$$\frac{n(n-1)}{2} = 2\alpha(ab),$$

hence

$$\frac{n^2}{4} > \alpha ab.$$

This last inequality combined with the first equation in system (2.3) yield

$$2\left(\frac{n^2}{4}\right)^2 > 2\alpha^2a^2b^2 > \alpha(a^2+b^2) > \left(\frac{n+k-1}{2}\right)^2.$$

Whereupon

$$k \leq \frac{n^2}{\sqrt{2}} - n + 1. \qquad \square$$

3. Some computations

Based upon Theorem 2.1, we wrote a program in MAPLE and found the solutions to equation (2.1) for $1 < n \leq 300$. The solutions are listed in the following table.

n	k	y^2
4	1	64
9	1	729
	17	104329
14	12	97344
	21	345744
16	1	4096
21	128	121528576
23	3	41616
25	1	15625
	5	99225
	15	518400
	98	56205009
28	8	254016
33	33	4322241
36	1	46656
49	1	117649
	291	3319833924
64	1	262144
	42	26904969
	48	34574400
69	32	19998784
78	105	268304400
81	1	531441
	28	24147396
	69	114383025
	644	68869504900
88	203	1765764441
96	5	4708900
97	98	336098889
100	1	1000000
105	64	171714816

Remark 3.1. Let $C_n = |\{(k, y) \text{ solution to equation (2.1) }\}|$. We see from theorem 1, that for every n, C_n is finite, and from the table above, that for $1 \leq n \leq 300$, $C_n \leq 7$. It would be interesting to see if there exists a constant C such that $C_n \leq C$ for every n.

111	39	87609600
118	5	8643600
	60	200505600
120	17	35808256
	722	125308212121
121	1	1771561
	1205	771665618025
133	32	106007616
144	1	2985984
	13	43956900
	21	77053284
	77	484968484
	82	540423009
	175	2466612225
	246	5647973409
153	18	76055841
	305	10817040025
165	287	10205848576
168	243	6902120241
169	1	4826809
	2022	5755695204609
176	45	353816100
	195	4473603225
189	423	34640654400
196	1	7529536
216	98	1875669481
	784	248961081600
217	63	976437504
	242	10499076225
	434	44214734529
221	936	446630236416
225	1	11390625
	35	498628900
	280	15560067600
	3143	32148582480784
232	87	1854594225
	175	6108204025
256	1	16777216
	169	7052640400
	336	29537234496
	1190	1090405850625
265	54	1349019441
	2209	9356875327801

289	1	24137569
	4616	144648440352144
295	76	2830240000
298	560	133210400400

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