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Convolution of second order linear recursive sequences I.

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Abstract

In this paper, we deal with convolutions of second order linear recursive sequences and give some special convolutions for Fibonacci-, Pell-, Jacobsthal- and Mersenne-sequences and their associated sequences.

Keywords: convolution, Fibonacci, generating function

MSC: 11B37, 11B39

1. Introduction

Let A, B be given real numbers with $AB \neq 0$. A second order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ is defined by the recursion

$$G_n = AG_{n-1} + BG_{n-2} \quad (n \geq 2),$$

where the initial terms G_0, G_1 are fixed real numbers with $|G_0| + |G_1| \neq 0$. For brevity, we use the following notation $G_n(G_0, G_1, A, B)$, too. The polynomial

$$p(x) = x^2 - Ax - B \tag{1.1}$$

is said to be the characteristic polynomial of the sequence $\{G_n\}_{n=0}^{\infty}$. If $D = A^2 + 4B \neq 0$ then the Binet formula of $\{G_n\}_{n=0}^{\infty}$ is

$$G_n = \frac{G_1 - \beta G_0}{\alpha - \beta}\alpha^n - \frac{G_1 - \alpha G_0}{\alpha - \beta}\beta^n,$$

where α, β are distinct roots of the characteristic polynomial. If $G_0 = 0$ and $G_1 = 1$ then $\{G_n\}_{n=0}^{\infty}$ is known as R-sequence $\{R_n\}_{n=0}^{\infty}$ with it's Binet formula

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{1.2}$$

If $G_0 = 2$ and $G_1 = A$ then the sequence is known as associated-R, or R-Lucas sequence $\{V_n\}_{n=0}^{\infty}$ with it's Binet formula

$$V_n = \alpha^n + \beta^n. \tag{1.3}$$

In the following sections, we will use the generating function and partial-fraction decomposition for the proofs. The generating function of $\{G_n\}_{n=0}^{\infty}$ (which can easily be verified by the well known methods) is

$$g(x) = \frac{G_0 + (G_1 - AG_0)x}{1 - Ax - Bx^2}. \tag{1.4}$$

The following table contains some special, well-known sequences with their initial terms, characteristic polynomial and generating function, where P-Lucas, J-Lucas and M-Lucas sequences are the associated sequences of Pell, Jacobsthal and Mersenne sequences, respectively.

Name	$G_n(G_0, G_1, A, B)$	Characteristic polynomial	Gen. function
Fibonacci	$F_n(0,1,1,1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{x}{1-x-x^2}$
Pell	$P_n(0,1,2,1$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{x}{1-2x-x^2}$
Jacobsthal	$J_n(0,1,1,2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{x}{1-x-2x^2}$
Mersenne	$M_n(0,1,3,-2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{x}{1-3x+2x^2}$
Lucas	$L_n(2,1,1,1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{2-x}{1-x-x^2}$
P-Lucas	$p_n(2,2,2,1)$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{2-2x}{1-2x-x^2}$
J-Lucas	$j_n(2,1,1,2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{2-x}{1-x-2x^2}$
M-Lucas	$m_n(2,3,3,-2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{2-3x}{1-3x+2x^2}$

Table 1: Named sequences

For further generating functions for second order linear recursive sequences see the paper of Mező [3].

We consider the sequence $\{c(n)\}_{n=0}^{\infty}$ given by the convolution of two different second order linear recursive sequences $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$:

$$c(n) = \sum_{k=0}^{n} G_k H_{n-k}.$$

Griffiths and Bramham [1] investigated the convolution of Lucas- and Jacobsthal-numbers and got the result:

$$c(n) = j_{n+1} - L_{n+1},$$

which can be found in the OEIS [2] with the following id: A264038.

In this paper, we deal with convolution of two different sequences, where all of the roots are distinct and the sequences are R-sequences or R-Lucas sequences. The convolution of sequences with themselves was investigated by Zhang W., Zhang Z., He P., Feng H. and many others. In [5], Feng and Zhang Z. generalized the previous results, i.e. they evaluated the following summation:

$$\sum_{a_1+a_2+\cdots+a_k=n} W_{ma_1} W_{ma_2} \cdots W_{ma_k}.$$

For example, the convolution of Fibonacci numbers with themselves was given as a corollary in [4] by Zhang W.:

$$\sum_{a+b=n} F_a F_b = \frac{1}{5}\left[(n-1)F_n + 2nF_{n-1}\right], \quad n \geq 1.$$

2. Results

In this section, we present three theorems and give formulas for $\{c(n)\}_{n=0}^{\infty}$, where the formulas depend only on the initial terms and the roots of the characteristic polynomials. After each theorem, we show the special cases of the theorem in corollaries using the named sequences (Fibonacci, Pell, Jacobsthal, Mersenne, Lucas, P-Lucas, J-Lucas, M-Lucas).

In this paper –for brevity–, we use the following notations:

$$\begin{aligned} a &= (A_1 - A_2)\alpha + B_1 - B_2, \\ b &= (A_1 - A_2)\beta + B_1 - B_2, \\ c &= (A_2 - A_1)\gamma + B_2 - B_1, \\ d &= (A_2 - A_1)\delta + B_2 - B_1, \end{aligned} \tag{2.1}$$

where $abcd \neq 0$, α, β and γ, δ are distinct roots of the characteristic polynomial of $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. We suppose that all the roots are real numbers and the characteristic polynomials have no common roots.

In the following theorem, we deal with the convolution of two different R-sequences.

Theorem 2.1. *The convolution of* $G_n(0,1,A_1,B_1)$ *and* $H_n(0,1,A_2,B_2)$ *is*

$$c(n) = \sum_{k=0}^{n} G_k H_{n-k} = \frac{\frac{\alpha^{n+1}}{a} - \frac{\beta^{n+1}}{b}}{\alpha - \beta} + \frac{\frac{\gamma^{n+1}}{c} - \frac{\delta^{n+1}}{d}}{\gamma - \delta}.$$

For the well-known sequences, listed in Table 1, we can get special convolution formulas:

Corollary 2.2. *Using Theorem 2.1 the convolution of Fibonacci and Pell numbers is:*

$$c(n) = \sum_{k=0}^{n} F_k P_{n-k} = P_n - F_n.$$

Remark 2.3. In [2], (A106515) it can be found that

$$c(n) = \sum_{k=0}^{n} F_{n-k-1} P_{k+1} = P_n - F_n + P_{n+1},$$

where because of the different indices the term P_{n+1} occures, as well.

Corollary 2.4. *Using Theorem 2.1 the convolution of Fibonacci and Jacobsthal numbers is:*

$$c(n) = \sum_{k=0}^{n} F_k J_{n-k} = J_{n+1} - F_{n+1}.$$

Remark 2.5. In [2], (A094687) the formula

$$c(n) = \sum_{k=0}^{n} F_k J_{n-k} = c(n-1) + 2c(n-2) + F_{n-1}$$

can be found. After a short calculation one can easily verify that the two formulas for $c(n)$ are the same ones.

Corollary 2.6. *Using Theorem 2.1 the convolution of Fibonacci and Mersenne numbers is:*

$$c(n) = \sum_{k=0}^{n} F_k M_{n-k} = m_{n+1} - F_{n+4}.$$

Corollary 2.7. *Using Theorem 2.1 the convolution of Pell and Jacobsthal numbers is:*

$$c(n) = \sum_{k=0}^{n} P_k J_{n-k} = \frac{P_{n+1} + P_n - J_{n+2}}{2}.$$

Corollary 2.8. *Using Theorem 2.1 the convolution of Pell and Mersenne numbers is:*

$$c(n) = \sum_{k=0}^{n} P_k M_{n-k} = \frac{P_{n+2} + P_{n+1} - M_{n+2}}{2}.$$

In the following theorem, we deal with the convolution of an R-sequence and an R-Lucas sequence.

Theorem 2.9. *The convolution of $G_n(0, 1, A_1, B_1)$ and $H_n(2, A_2, A_2, B_2)$ is*

$$c(n) = \sum_{k=0}^{n} G_k H_{n-k} =$$

$$= \frac{\frac{\alpha^{n+1}(2\alpha - A_2)}{a} - \frac{\beta^{n+1}(2\beta - A_2)}{b}}{\alpha - \beta} + \frac{\frac{\gamma^{n+1}(2\gamma - A_2)}{c} - \frac{\delta^{n+1}(2\delta - A_2)}{d}}{\gamma - \delta}.$$

For the well-known sequences, listed in Table 1, we can get special convolution formulas:

Corollary 2.10. *Using Theorem 2.9 the convolution of Fibonacci and P-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} F_k p_{n-k} = p_n - 2F_{n-1}.$$

Corollary 2.11. *Using Theorem 2.9 the convolution of Fibonacci and J-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} F_k j_{n-k} = j_{n+1} - L_{n+1}.$$

Remark 2.12. This our convolution has the same form as of Griffiths and Bramham in [1].

Corollary 2.13. *Using Theorem 2.9 the convolution of Fibonacci and M-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} F_k m_{n-k} = M_{n+1} - F_{n+1}.$$

Remark 2.14. For the sequence $a(n)$ (A228078 in [2]), where $a(n+1)$ is the sum of n-th row of the Fibonacci-Pascal triangle in A228074, we get that

$$c(n) = a(n+1).$$

Corollary 2.15. *Using Theorem 2.9 the convolution of Pell and Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} P_k L_{n-k} = P_n + p_n - L_n.$$

Corollary 2.16. *Using Theorem 2.9 the convolution of Pell and J-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} P_k j_{n-k} = \frac{8P_{n+1} + p_{n+1} - 2j_{n+2}}{4}.$$

Corollary 2.17. *Using Theorem 2.9 the convolution of Pell and M-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} P_k m_{n-k} = \frac{4P_{n+2} + p_{n+1} - 2m_{n+2}}{4}.$$

Corollary 2.18. *Using Theorem 2.9 the convolution of Jacobsthal and Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} J_k L_{n-k} = j_{n+1} - L_{n+1}.$$

Remark 2.19. The convolution of Lucas and Jacobsthal numbers was also investigated by Griffiths and Bramham in [1], the two formulas are the same ones.

Corollary 2.20. *Using Theorem 2.9 the convolution of Jacobsthal and P-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} J_k p_{n-k} = 2(P_{n+1} - J_{n+1}).$$

Corollary 2.21. *Using Theorem 2.9 the convolution of Mersenne and Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} M_k L_{n-k} = 3m_{n+1} - L_{n+4} - 2.$$

Corollary 2.22. *Using Theorem 2.9 the convolution of Mersenne and P-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} M_k p_{n-k} = \frac{3p_{n+1} + p_n - M_{n+3} - 1}{2}.$$

In the following theorem, we deal with the convolution of two different R-Lucas sequences.

Theorem 2.23. *The convolution of $G_n(2, A_1, A_1, B_1)$ and $H_n(2, A_2, A_2, B_2)$ is*

$$\begin{aligned} c(n) &= \sum_{k=0}^{n} G_k H_{n-k} = \\ &= \frac{\frac{\alpha^{n+1}(2\alpha - A_1)(2\alpha - A_2)}{a} - \frac{\beta^{n+1}(2\beta - A_1)(2\beta - A_2)}{b}}{\alpha - \beta} \\ &\quad + \frac{\frac{\gamma^{n+1}(2\gamma - A_1)(2\gamma - A_2)}{c} - \frac{\delta^{n+1}(2\delta - A_1)(2\delta - A_2)}{d}}{\gamma - \delta}. \end{aligned}$$

For the well-known sequences, listed in Table 1, we can get special convolution formulas:

Corollary 2.24. *Using Theorem 2.23 the convolution of Lucas and P-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} L_k p_{n-k} = 2F_{n+1} - 6F_n + 2P_{n+1} + 6P_n.$$

Corollary 2.25. *Using Theorem 2.23 the convolution of Lucas and J-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} L_k j_{n-k} = 9J_{n+1} - 5F_{n+1}.$$

Corollary 2.26. *Using Theorem 2.23 the convolution of Lucas and M-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} L_k m_{n-k} = 3M_{n+1} - L_{n+1} + 2.$$

Corollary 2.27. *Using Theorem 2.23 the convolution of P-Lucas and J-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} p_k j_{n-k} = 2P_{n+2} + p_{n+1} - 2j_{n+1}.$$

Corollary 2.28. *Using Theorem 2.23 the convolution of P-Lucas and M-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} p_k m_{n-k} = 2P_{n+2} + 4P_{n+1} - M_{n+2} - 1.$$

3. Proofs

In the following proofs, we use the method of partial-fraction decomposition, the generating functions of second order linear recursive sequences and the idea used by Griffiths and Bramham in [1], that is $c(n)$ is the coefficient of x^n in

$$g(x)h(x) = \sum_{n=0}^{\infty} G_n x^n \cdot \sum_{n=0}^{\infty} H_n x^n = \sum_{n=0}^{\infty} c(n) x^n,$$

where $g(x)$, $h(x)$ are the generating functions of sequences $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively.

Proof of Theorem 2.1. Using (1.4), the generating functions of the sequences $G_n(0,1,A_1,B_1)$ and $H_n(0,1,A_2,B_2)$ are

$$g(x) = \frac{x}{1 - A_1 x - B_1 x^2} = \frac{x}{(1-\alpha x)(1-\beta x)}$$

and

$$h(x) = \frac{x}{1 - A_2 x - B_2 x^2} = \frac{x}{(1-\gamma x)(1-\delta x)},$$

where α, β and γ, δ are the roots of the characteristic polynomial of $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. The generating functions can be written as (by the method of partial-fraction decomposition)

$$g(x) = \frac{1}{\alpha - \beta}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right)$$

and

$$h(x) = \frac{1}{\gamma - \delta}\left(\frac{1}{1-\gamma x} - \frac{1}{1-\delta x}\right).$$

From this it follows that

$$g(x)h(x)(\alpha-\beta)(\gamma-\delta)$$

$$
\begin{aligned}
&= \left(\frac{1}{1-\alpha x}-\frac{1}{1-\beta x}\right)\left(\frac{1}{1-\gamma x}-\frac{1}{1-\delta x}\right)\\
&= \frac{1}{(1-\alpha x)(1-\gamma x)}-\frac{1}{(1-\alpha x)(1-\delta x)}-\frac{1}{(1-\beta x)(1-\gamma x)}+\frac{1}{(1-\beta x)(1-\delta x)}\\
&= \frac{\frac{\alpha}{\alpha-\gamma}}{1-\alpha x}-\frac{\frac{\gamma}{\alpha-\gamma}}{1-\gamma x}-\frac{\frac{\alpha}{\alpha-\delta}}{1-\alpha x}+\frac{\frac{\delta}{\alpha-\delta}}{1-\delta x}-\frac{\frac{\beta}{\beta-\gamma}}{1-\beta x}+\frac{\frac{\gamma}{\beta-\gamma}}{1-\gamma x}+\frac{\frac{\beta}{\beta-\delta}}{1-\beta x}-\frac{\frac{\delta}{\beta-\delta}}{1-\delta x}\\
&= \frac{\frac{\alpha(\gamma-\delta)}{(A_1-A_2)\alpha+B_1-B_2}}{1-\alpha x}-\frac{\frac{\beta(\gamma-\delta)}{(A_1-A_2)\beta+B_1-B_2}}{1-\beta x}+\frac{\frac{\gamma(\alpha-\beta)}{(A_2-A_1)\gamma+B_2-B_1}}{1-\gamma x}-\frac{\frac{\delta(\alpha-\beta)}{(A_2-A_1)\delta+B_2-B_1}}{1-\delta x}.
\end{aligned}
$$

Now using that $c(n)$ is the coefficient of x^n in $g(x)h(x)$ and e.g.,

$$\frac{1}{1-\alpha x}=\sum_{n=0}^{\infty}(\alpha x)^n \quad (0<|\alpha x|<1),$$

we get

$$
\begin{aligned}
c(n) = \frac{1}{\alpha-\beta}&\left(\frac{\alpha^{n+1}}{(A_1-A_2)\alpha+B_1-B_2}-\frac{\beta^{n+1}}{(A_1-A_2)\beta+B_1-B_2}\right)\\
&+\frac{1}{\gamma-\delta}\left(\frac{\gamma^{n+1}}{(A_2-A_1)\gamma+B_2-B_1}-\frac{\delta^{n+1}}{(A_2-A_1)\delta+B_2-B_1}\right). \qquad \square
\end{aligned}
$$

We remark that the corollaries can be obtained from Table 1 if we use the values of A_1, B_1, A_2, B_2 and the Binet formula (1.2), e.g., the proof of Corollary 2.2:

Proof of Corollary 2.2. Now $G_n = F_n(0,1,1,1)$ and $H_n = P_n(0,1,2,1)$.

$$\alpha,\beta=\frac{1\pm\sqrt{5}}{2},\quad \gamma,\delta=1\pm\sqrt{2}.$$

By (2.1), we get that

$$
\begin{aligned}
a &= -\alpha,\\
b &= -\beta,\\
c &= \gamma,\\
d &= \delta.
\end{aligned}
$$

Applying Theorem 2.1 and (1.2), we get the result

$$c(n)=\frac{\frac{\alpha^{n+1}}{a}-\frac{\beta^{n+1}}{b}}{\alpha-\beta}+\frac{\frac{\gamma^{n+1}}{c}-\frac{\delta^{n+1}}{d}}{\gamma-\delta}=\frac{-\alpha^n+\beta^n}{\alpha-\beta}+\frac{\gamma^n-\delta^n}{\gamma-\delta}=P_n-F_n. \qquad \square$$

Proof of Theorem 2.9. Using (1.4), the generating functions of the sequences $G_n(0,1,A_1,B_1)$ and $H_n(2,A_2,A_2,B_2)$ are

$$g(x)=\frac{x}{1-A_1x-B_1x^2}=\frac{x}{(1-\alpha x)(1-\beta x)}$$

and

$$h(x) = \frac{2 - A_2x}{1 - A_2x - B_2x^2} = \frac{2 - A_2x}{(1 - \gamma x)(1 - \delta x)},$$

where α, β and γ, δ are the roots of the characteristic polynomial of $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. The generating functions could be written as (by the method of partial-fraction decomposition)

$$g(x) = \frac{1}{\alpha - \beta}\left(\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right)$$

and

$$h(x) = \frac{1}{\gamma - \delta}\left(\frac{2\gamma - A_2}{1 - \gamma x} - \frac{2\delta - A_2}{1 - \delta x}\right).$$

From this it follows that

$$\begin{aligned}
&g(x)h(x)(\alpha-\beta)(\gamma-\delta)\\
&= \left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right)\left(\frac{2\gamma - A_2}{1-\gamma x} - \frac{2\delta - A_2}{1-\delta x}\right)\\
&= \frac{2\gamma - A_2}{(1-\alpha x)(1-\gamma x)} - \frac{2\delta - A_2}{(1-\alpha x)(1-\delta x)} - \frac{2\gamma - A_2}{(1-\beta x)(1-\gamma x)} + \frac{2\delta - A_2}{(1-\beta x)(1-\delta x)}\\
&= \frac{\frac{\alpha(2\delta-A_2)}{\alpha-\gamma}}{1-\alpha x} - \frac{\frac{\gamma(2\delta-A_2)}{\alpha-\gamma}}{1-\gamma x} - \frac{\frac{\alpha(2\delta-A_2)}{\alpha-\delta}}{1-\alpha x} + \frac{\frac{\delta(2\delta-A_2)}{\alpha-\delta}}{1-\delta x}\\
&\quad - \frac{\frac{\beta(2\delta-A_2)}{\beta-\gamma}}{1-\beta x} + \frac{\frac{\gamma(2\delta-A_2)}{\beta-\gamma}}{1-\gamma x} + \frac{\frac{\beta(2\delta-A_2)}{\beta-\delta}}{1-\beta x} - \frac{\frac{\delta(2\delta-A_2)}{\beta-\delta}}{1-\delta x}\\
&= \frac{\frac{\alpha(\gamma-\delta)(2\alpha-A_2)}{(A_1-A_2)\alpha+B_1-B_2}}{1-\alpha x} - \frac{\frac{\beta(\gamma-\delta)(2\beta-A_2)}{(A_1-A_2)\beta+B_1-B_2}}{1-\beta x} + \frac{\frac{\gamma(\alpha-\beta)(2\gamma-A_2)}{(A_2-A_1)\gamma+B_2-B_1}}{1-\gamma x} - \frac{\frac{\delta(\alpha-\beta)(2\delta-A_2)}{(A_2-A_1)\delta+B_2-B_1}}{1-\delta x}.
\end{aligned}$$

Now using that $c(n)$ is the coefficient of x^n in $g(x)h(x)$ and e.g.,

$$\frac{1}{1-\alpha x} = \sum_{n=0}^{\infty}(\alpha x)^n \quad (0 < |\alpha x| < 1),$$

we get

$$\begin{aligned}
c(n) = &\frac{1}{\alpha-\beta}\left(\frac{\alpha^{n+1}(2\alpha - A_2)}{(A_1 - A_2)\alpha + B_1 - B_2} - \frac{\beta^{n+1}(2\beta - A_2)}{(A_1 - A_2)\beta + B_1 - B_2}\right)\\
&+ \frac{1}{\gamma-\delta}\left(\frac{\gamma^{n+1}(2\gamma - A_2)}{(A_2 - A_1)\gamma + B_2 - B_1} - \frac{\delta^{n+1}(2\delta - A_2)}{(A_2 - A_1)\delta + B_2 - B_1}\right). \qquad \square
\end{aligned}$$

We remark that the corollaries can be obtained from Table 1 if we use the values of A_1, B_1, A_2, B_2 and the Binet formulas ((1.2) or (1.3)), e.g., the proof of Corollary 2.10:

Proof of Corollary 2.10. Now $G_n = F_n(0,1,1,1)$ and $H_n = p_n(2,2,2,1)$.

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}, \quad \gamma, \delta = 1 \pm \sqrt{2}.$$

By (2.1), we get that

$$\begin{aligned} a &= -\alpha, \\ b &= -\beta, \\ c &= \gamma, \\ d &= \delta. \end{aligned}$$

Applying Theorem 2.9, (1.2) and (1.3), we get the result

$$\begin{aligned} c(n) &= \frac{\frac{\alpha^{n+1}(2\alpha - A_2)}{a} - \frac{\beta^{n+1}(2\beta - A_2)}{b}}{\alpha - \beta} + \frac{\frac{\gamma^{n+1}(2\gamma - A_2)}{c} - \frac{\delta^{n+1}(2\delta - A_2)}{d}}{\gamma - \delta} \\ &= \frac{\alpha^n(1 - \sqrt{5}) - \beta^n(1 + \sqrt{5})}{\alpha - \beta} + \frac{\gamma^n 2\sqrt{2} + \delta^n 2\sqrt{2}}{\gamma - \delta} \\ &= \frac{\alpha^{n-1}(-2) - \beta^{n-1}(-2)}{\alpha - \beta} + \gamma^n + \delta^n = p_n - 2F_{n-1}. \end{aligned}$$ □

Proof of Theorem 2.23. Using (1.4), the generating functions of the sequences $G_n(2, A_1, A_1, B_1)$ and $H_n(2, A_2, A_2, B_2)$ are

$$g(x) = \frac{2 - A_1 x}{1 - A_1 x - B_1 x^2} = \frac{2 - A_1 x}{(1 - \alpha x)(1 - \beta x)}$$

and

$$h(x) = \frac{2 - A_2 x}{1 - A_2 x - B_2 x^2} = \frac{2 - A_2 x}{(1 - \gamma x)(1 - \delta x)},$$

where α, β and γ, δ are the roots of the characteristic polynomial of $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. The generating functions could be written as (by the method of partial-fraction decomposition)

$$g(x) = \frac{1}{\alpha - \beta}\left(\frac{2\alpha - A_1}{1 - \alpha x} - \frac{2\beta - A_1}{1 - \beta x}\right)$$

and

$$h(x) = \frac{1}{\gamma - \delta}\left(\frac{2\gamma - A_2}{1 - \gamma x} - \frac{2\delta - A_2}{1 - \delta x}\right).$$

From this it follows that

$$\begin{aligned} &g(x)h(x)(\alpha - \beta)(\gamma - \delta) \\ &= \left(\frac{2\alpha - A_1}{1 - \alpha x} - \frac{2\beta - A_1}{1 - \beta x}\right)\left(\frac{2\gamma - A_2}{1 - \gamma x} - \frac{2\delta - A_2}{1 - \delta x}\right) \end{aligned}$$

$$= \frac{(2\alpha - A_1)(2\gamma - A_2)}{(1-\alpha x)(1-\gamma x)} - \frac{(2\alpha - A_1)(2\delta - A_2)}{(1-\alpha x)(1-\delta x)}$$
$$- \frac{(2\beta - A_1)(2\gamma - A_2)}{(1-\beta x)(1-\gamma x)} + \frac{(2\beta - A_1)(2\delta - A_2)}{(1-\beta x)(1-\delta x)}$$
$$= \frac{\frac{\alpha(2\alpha-A_1)(2\gamma-A_2)}{\alpha-\gamma}}{1-\alpha x} - \frac{\frac{\gamma(2\alpha-A_1)(2\gamma-A_2)}{\alpha-\gamma}}{1-\gamma x} - \frac{\frac{\alpha(2\alpha-A_1)(2\delta-A_2)}{\alpha-\delta}}{1-\alpha x} + \frac{\frac{\delta(2\alpha-A_1)(2\delta-A_2)}{\alpha-\delta}}{1-\delta x}$$
$$- \frac{\frac{\beta(2\beta-A_1)(2\gamma-A_2)}{\beta-\gamma}}{1-\beta x} + \frac{\frac{\gamma(2\beta-A_1)(2\gamma-A_2)}{\beta-\gamma}}{1-\gamma x} + \frac{\frac{\beta(2\beta-A_1)(2\delta-A_2)}{\beta-\delta}}{1-\beta x} - \frac{\frac{\delta(2\beta-A_1)(2\delta-A_2)}{\beta-\delta}}{1-\delta x}$$
$$= \frac{\frac{\alpha(\gamma-\delta)(2\alpha-A_1)(2\alpha-A_2)}{(A_1-A_2)\alpha+B_1-B_2}}{1-\alpha x} - \frac{\frac{\beta(\gamma-\delta)(2\beta-A_1)(2\beta-A_2)}{(A_1-A_2)\beta+B_1-B_2}}{1-\beta x}$$
$$+ \frac{\frac{\gamma(\alpha-\beta)(2\gamma-A_1)(2\gamma-A_2)}{(A_2-A_1)\gamma+B_2-B_1}}{1-\gamma x} - \frac{\frac{\delta(\alpha-\beta)(2\delta-A_1)(2\delta-A_2)}{(A_2-A_1)\delta+B_2-B_1}}{1-\delta x}.$$

Now using that $c(n)$ is the coefficient of x^n in $g(x)h(x)$ and e.g.,

$$\frac{1}{1-\alpha x} = \sum_{n=0}^{\infty} (\alpha x)^n \quad (0 < |\alpha x| < 1),$$

we get

$$c(n) = \frac{1}{\alpha-\beta}\left(\frac{\alpha^{n+1}(2\alpha - A_1)(2\alpha - A_2)}{(A_1 - A_2)\alpha + B_1 - B_2} - \frac{\beta^{n+1}(2\beta - A_1)(2\beta - A_2)}{(A_1 - A_2)\beta + B_1 - B_2}\right)$$
$$+ \frac{1}{\gamma-\delta}\left(\frac{\gamma^{n+1}(2\gamma - A_1)(2\gamma - A_2)}{(A_2 - A_1)\gamma + B_2 - B_1} - \frac{\delta^{n+1}(2\delta - A_1)(2\delta - A_2)}{(A_2 - A_1)\delta + B_2 - B_1}\right). \quad \square$$

We remark that the corollaries can be obtained from Table 1 if we use the values of A_1, B_1, A_2, B_2 and the Binet formula (1.2), e.g., the proof of Corollary 2.24:

Proof of Corollary 2.24. Now $G_n = L_n(2,1,1,1)$ and $H_n = p_n(2,2,2,1)$.

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}, \quad \gamma, \delta = 1 \pm \sqrt{2}.$$

By (2.1), we get that

$$a = -\alpha,$$
$$b = -\beta,$$
$$c = \gamma,$$
$$d = \delta.$$

Applying Theorem 2.1, (1.1) and (1.2), we get the result

$$c(n) = \frac{\frac{\alpha^{n+1}(2\alpha-A_1)(2\alpha-A_2)}{a} - \frac{\beta^{n+1}(2\beta-A_1)(2\beta-A_2)}{b}}{\alpha - \beta}$$

$$+\frac{\frac{\gamma^{n+1}(2\gamma-A_1)(2\gamma-A_2)}{c}-\frac{\delta^{n+1}(2\delta-A_1)(2\delta-A_2)}{d}}{\gamma-\delta}$$

$$=\frac{-\alpha^n(4\alpha^2-6\alpha+2)+\beta^n(4\beta^2-6\beta+2)}{\alpha-\beta}$$

$$+\frac{\gamma^n(4\gamma^2-6\gamma+2)-\delta^n(4\delta^2-6\delta+2)}{\gamma-\delta}$$

$$=\frac{-\alpha^n(-2\alpha+6)+\beta^n(-2\beta+6)}{\alpha-\beta}$$

$$+\frac{\gamma^n(2\gamma+6)-\delta^n(2\delta+6)}{\gamma-\delta}=2F_{n+1}-6F_n+2P_{n+1}+6P_n. \qquad \square$$

4. Concluding remarks

In this paper, we have dealt the case, when there are no common roots of the characteristic polynomials and we have shown formulas for the convolution of R-sequences and R-Lucas sequences. In the future, we would like to continue working on the cases, when there are one or two common roots.

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