Annales Mathematicae et Informaticae
46 *(2016) pp. 195–204*
http://ami.ektf.hu

On the (s,t)-Pell and (s,t)-Pell-Lucas numbers by matrix methods*

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Submitted March 17, 2016 — Accepted September 5, 2016

Abstract

In this paper, we investigate some generalization of Pell and Pell-Lucas numbers, which is called (s,t)-Pell and (s,t)-Pell-Lucas numbers, and we define the 2×2 matrix W, which satisfy the relation $W^2 = 2sW + tI$. After that, we establish some identities of (s,t)-Pell and (s,t)-Pell-Lucas numbers and some sum formulas for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using this matrix.

Keywords: Fibonacci number; Lucas number; Pell number; Pell-Lucas number; (s,t)-Pell number; (s,t)-Pell-Lucas number.

MSC: 11B37; 15A15.

1. Introduction

For over several years, there are many recursive sequences that have been studied in the literatures. The famous examples of these sequences are Fibonacci, Lucas, Pell and Pell-Lucas, because they are extensively used in various research areas such as Engineering, Architecture, Nature and Art (for examples see: [2, 3, 4, 5, 6, 7]). For $n \geq 2$, the classical Fibonacci $\{F_n\}$, Lucas $\{L_n\}$, Pell $\{P_n\}$ and Pell-Lucas $\{Q_n\}$

*This research was supported by faculty of science and technology, Rajamangala University of Technology Thanyaburi (RMUTT), Thailand.

sequences are defined by $F_n = F_{n-1}+F_{n-2}$, $L_n = L_{n-1}+L_{n-2}$, $P_n = 2P_{n-1}+P_{n-2}$, and $Q_n = 2Q_{n-1}+Q_{n-2}$, with the initial conditions $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, $P_0 = 0$, $P_1 = 1$ and $Q_0 = Q_1 = 2$, respectively. For more detialed information about Fibonacci, Lucas, Pell, Pell-Lucas sequences can be found in [2, 3].

Recently, Fibonacci, Lucas, Pell and Pell-Lucas were generalized and studied by many authors in the different ways to derive many identities. In 2012, Gulec and Taskara [1] introduced a new generalization of Pell and Pell-Lucas sequence which is called (s,t)-Pell and (s,t)-Pell-Lucas sequences as in the definition 1.1 and by considering these sequences, they introduced the matrix sequences which have elements of (s,t)-Pell and (s,t)-Pell-Lucas sequences. Further, they obtained some properties of (s,t)-Pell and (s,t)-Pell-Lucas matrices sequences by using elementary matrix algebra.

Definition 1.1. [1] Let s,t be any real number with $s^2+t>0$, $s>0$ and $t \neq 0$. Then the (s,t)-Pell sequences $\{\mathcal{P}_n(s,t)\}_{n\in\mathbb{N}}$ and the (s,t)-Pell-Lucas sequences $\{\mathcal{Q}_n(s,t)\}_{n\in\mathbb{N}}$ are defined respectively by

$$\mathcal{P}_n(s,t) = 2s\mathcal{P}_{n-1}(s,t) + t\mathcal{P}_{n-2}(s,t), \text{ for } n \geq 2, \tag{1.1}$$

$$\mathcal{Q}_n(s,t) = 2s\mathcal{Q}_{n-1}(s,t) + t\mathcal{Q}_{n-2}(s,t), \text{ for } n \geq 2, \tag{1.2}$$

with initial conditions $\mathcal{P}_0(s,t) = 0$, $\mathcal{P}_1(s,t) = 1$ and $\mathcal{Q}_0(s,t) = 2$, $\mathcal{Q}_1(s,t) = 2s$.

In particular, if $s = \frac{1}{2}, t = 1$, then the classical Fibonacci and Lucas sequence are obtained, and if $s = t = 1$, then the classical Pell and Pell-Lucas sequences are obtained. From the definition 1.1, we have that the characteristic equation of (1.1) and (1.2) are in the form

$$x^2 = 2sx + t \tag{1.3}$$

and the root of equation (1.3) are $\alpha = s + \sqrt{s^2+t}$ and $\beta = s - \sqrt{s^2+t}$. Note that $\alpha + \beta = 2s$, $\alpha - \beta = 2\sqrt{s^2+t}$ and $\alpha\beta = -t$. Moreover, it can be seen that [1]

$$\mathcal{Q}_n(s,t) = 2s\mathcal{P}_n(s,t) + 2t\mathcal{P}_{n-1}(s,t), \text{ for all } n \geq 0. \tag{1.4}$$

In this paper, we introduce the 2×2 matrix W which satisfy the relation $W^2 = 2sW + tI$. After that, we establish some identities of (s,t)-Pell and (s,t)-Pell-Lucas numbers and some sum formulas for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using this matrix. Now, we first define (s,t)-Pell and (s,t)-Pell-Lucas numbers for negative subscript as follows:

$$\mathcal{P}_{-n}(s,t) = \frac{-\mathcal{P}_n(s,t)}{(-t)^n}, \text{ and } \mathcal{Q}_{-n}(s,t) = \frac{\mathcal{Q}_n(s,t)}{(-t)^n}, \tag{1.5}$$

for all $n \geq 1$. In the rest of this paper, for convenience we will use the symbol $\mathcal{P}_n$ and $\mathcal{Q}_n$ instead of $\mathcal{P}_n(s,t)$ and $\mathcal{Q}_n(s,t)$ respectively.

2. Main results

We begin this section with the following Lemma.

Lemma 2.1. *If X is a square matrix with $X^2 = 2sX + tI$, then*

$$X^n = \mathcal{P}_n X + t\mathcal{P}_{n-1} I \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. If $n = 0$, then the proof is obvious. It can be shown by induction that $X^n = \mathcal{P}_n X + t\mathcal{P}_{n-1} I$ for all $n \in \mathbb{N}$. Now, we will show that $X^{-n} = \mathcal{P}_{-n} X + t\mathcal{P}_{-n-1} I$ for all $n \in \mathbb{N}$. Let $Y = 2sI - X = -tX^{-1}$. Then we have

$$Y^2 = (2sI - X)^2 = 2s(2sI - X) + tI = 2sY + tI.$$

It implies that $Y^n = \mathcal{P}_n Y + t\mathcal{P}_{n-1} I$. That is $(-tX^{-1})^n = \mathcal{P}_n(2sI - X) + t\mathcal{P}_{n-1} I$. Thus

$$\begin{aligned}(-t)^n X^{-n} &= 2s\mathcal{P}_n I - \mathcal{P}_n X + t\mathcal{P}_{n-1} I \\ &= -\mathcal{P}_n X + (2s\mathcal{P}_n + t\mathcal{P}_{n-1}) I \\ &= -\mathcal{P}_n X + \mathcal{P}_{n+1} I.\end{aligned}$$

Therefore, $X^{-n} = -\frac{\mathcal{P}_n}{(-t)^n} X + \frac{\mathcal{P}_{n+1}}{(-t)^n} I = \mathcal{P}_{-n} X + t\mathcal{P}_{-(n+1)} I = \mathcal{P}_{-n} X + t\mathcal{P}_{-n-1} I$.

This complete the proof. □

By using Lemma 2.1, we obtain the Binet's formula for (s,t)-Pell and (s,t)-Pell-Lucas numbers.

Corollary 2.2 (Binet's formula)**.** *The n^{th} (s,t)-Pell and (s,t)-Pell-Lucas number are given by*

$$\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \textit{and} \quad \mathcal{Q}_n = \alpha^n + \beta^n, \quad \textit{for all} \quad n \in \mathbb{Z},$$

where $\alpha = s + \sqrt{s^2 + t}$ and $\beta = s - \sqrt{s^2 + t}$ are the roots of the characteristic equation (1.3)*.*

Proof. Take $X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, then $X^2 = 2sX + tI$. By Lemma 2.1, we have $X^n = \mathcal{P}_n X + t\mathcal{P}_{n-1} I$. It follows that

$$\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha\mathcal{P}_n + t\mathcal{P}_{n-1} & 0 \\ 0 & \beta\mathcal{P}_n + t\mathcal{P}_{n-1} \end{bmatrix}.$$

Thus, $\alpha^n = \alpha\mathcal{P}_n + t\mathcal{P}_{n-1}$ and $\beta^n = \beta\mathcal{P}_n + t\mathcal{P}_{n-1}$, which implies that

$$\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathcal{Q}_n = \alpha^n + \beta^n, \quad \text{for all} \quad n \in \mathbb{Z}.$$ □

Let us define the 2×2 matrix W as follows:

$$W = \begin{bmatrix} s & 2(s^2 + t) \\ \frac{1}{2} & s \end{bmatrix}. \tag{2.1}$$

Then it easy to see that $W^2 = 2sW + tI$. From this fact and Lemma 2.1, we get the following Lemma.

Lemma 2.3. *Let W be a matrix as in (2.1). Then* $W^n = \begin{bmatrix} \frac{1}{2}\mathcal{Q}_n & 2(s^2+t)\mathcal{P}_n \\ \frac{1}{2}\mathcal{P}_n & \frac{1}{2}\mathcal{Q}_n \end{bmatrix}$ *for all $n \in \mathbb{Z}$.*

Proof. Since $W^2 = 2sW + tI$, the proof follows from Lemma 2.1 and using $\mathcal{Q}_n = 2s\mathcal{P}_n + 2t\mathcal{P}_{n-1}$. □

Now, by using the matrix W, we obtain some identities of (s,t)-Pell and (s,t)-Pell-Lucas numbers.

Lemma 2.4. *Let m, n be any integers. Then the following results hold.*

(i) $\mathcal{Q}_n^2 - 4(s^2+t)\mathcal{P}_n^2 = 4(-t)^n$,

(ii) $2\mathcal{Q}_{m+n} = \mathcal{Q}_m\mathcal{Q}_n + 4(s^2+t)\mathcal{P}_m\mathcal{P}_n$,

(iii) $2\mathcal{P}_{m+n} = \mathcal{P}_m\mathcal{Q}_n + \mathcal{Q}_m\mathcal{P}_n$,

(iv) $2(-t)^n\mathcal{Q}_{m-n} = \mathcal{Q}_m\mathcal{Q}_n - 4(s^2+t)\mathcal{P}_m\mathcal{P}_n$,

(v) $2(-t)^n\mathcal{P}_{m-n} = \mathcal{P}_m\mathcal{Q}_n - \mathcal{Q}_m\mathcal{P}_n$,

(vi) $\mathcal{Q}_m\mathcal{Q}_n = \mathcal{Q}_{m+n} + (-t)^n\mathcal{Q}_{m-n}$,

(vii) $\mathcal{P}_m\mathcal{Q}_n = \mathcal{P}_{m+n} + (-t)^n\mathcal{P}_{m-n}$.

Proof. Since $\det(W^n) = (\det(W))^n = (-t)^n$ and $\det(W^n) = \frac{1}{4}\mathcal{Q}_n^2 - (s^2+t)\mathcal{P}_n^2$, we get that $\mathcal{Q}_n^2 - 4(s^2+t)\mathcal{P}_n^2 = 4(-t)^n$ and then (i) immediately seen. Since $W^{m+n} = W^mW^n$, we obtain

$$\begin{aligned}
&\begin{bmatrix} \frac{1}{2}\mathcal{Q}_{m+n} & 2(s^2+t)\mathcal{P}_{m+n} \\ \frac{1}{2}\mathcal{P}_{m+n} & \frac{1}{2}\mathcal{Q}_{m+n} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4}\big(\mathcal{Q}_m\mathcal{Q}_n + 4(s^2+t)\mathcal{P}_m\mathcal{P}_n\big) & (s^2+t)(\mathcal{Q}_m\mathcal{P}_n + \mathcal{P}_m\mathcal{Q}_n) \\ \frac{1}{4}(\mathcal{P}_m\mathcal{Q}_n + \mathcal{Q}_m\mathcal{P}_n) & \frac{1}{4}\big(4(s^2+t)\mathcal{P}_m\mathcal{P}_n + \mathcal{Q}_m\mathcal{Q}_n\big) \end{bmatrix}.
\end{aligned}$$

Thus, identities (ii) and (iii) are easily seen. Next, we note that $W^{m-n} = W^m(W^{-n}) = W^m(W^n)^{-1}$. Thus, we get that

$$\begin{aligned}
&\begin{bmatrix} \frac{1}{2}\mathcal{Q}_{m-n} & 2(s^2+t)\mathcal{P}_{m-n} \\ \frac{1}{2}\mathcal{P}_{m-n} & \frac{1}{2}\mathcal{Q}_{m-n} \end{bmatrix} \\
&= \frac{1}{(-t)^n}\begin{bmatrix} \frac{1}{4}\big(\mathcal{Q}_m\mathcal{Q}_n - 4(s^2+t)\mathcal{P}_m\mathcal{P}_n\big) & (s^2+t)(-\mathcal{Q}_m\mathcal{P}_n + \mathcal{P}_m\mathcal{Q}_n) \\ \frac{1}{4}(\mathcal{P}_m\mathcal{Q}_n - \mathcal{Q}_m\mathcal{P}_n) & \frac{1}{4}\big(-4(s^2+t)\mathcal{P}_m\mathcal{P}_n + \mathcal{Q}_m\mathcal{Q}_n\big) \end{bmatrix}.
\end{aligned}$$

Therefore, the identities (iv) and (v) can be derived directly. The proof of (vi) and (vii) goes on in the same fashion as above by using the property
$W^{m+n} + (-t)^nW^{m-n} = W^m(W^n + (-t)^nW^{-n})$. □

Next, we give the following Lemma for using in the next Theorems.

Lemma 2.5. *Let W be a matrix as in (2.1). Then*

$$H = W + tW^{-1} = \begin{bmatrix} 0 & 4(s^2+t) \\ 1 & 0 \end{bmatrix}.$$

Proof. Since $\det(W) = -t$, we get that $W^{-1} = -\frac{1}{t}\begin{bmatrix} s & -2(s^2+t) \\ -\frac{1}{2} & s \end{bmatrix}$. Thus, $H = \begin{bmatrix} 0 & 4(s^2+t) \\ 1 & 0 \end{bmatrix}$. □

Finally, by using matrices W and H, we obtain some sum formulas for (s,t)-Pell and (s,t)-Pell-Lucas numbers.

Theorem 2.6. *Let $n \in \mathbb{N}$ and $m, k \in \mathbb{Z}$ with $(-t)^m - \mathcal{Q}_m \neq -1$. Then*

$$\sum_{j=0}^{n} \mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k - \mathcal{Q}_{mn+m+k} + (-t)^m \big(\mathcal{Q}_{mn+k} - \mathcal{Q}_{k-m}\big)}{1 + (-t)^m - \mathcal{Q}_m}$$

and

$$\sum_{j=0}^{n} \mathcal{P}_{mj+k} = \frac{\mathcal{P}_k - \mathcal{P}_{mn+m+k} + (-t)^m \big(\mathcal{P}_{mn+k} - \mathcal{P}_{k-m}\big)}{1 + (-t)^m - \mathcal{Q}_m}$$

Proof. It is know that

$$I - (W^m)^{n+1} = (I - W^m) \sum_{j=0}^{n} (W^m)^j.$$

By Lemma 2.4 (i), we have

$$\det(I - W^m) = (1 - \frac{1}{2}\mathcal{Q}_m)^2 - (s^2+t)\mathcal{P}_m^2 = 1 + (-t)^m - \mathcal{Q}_m.$$

Since $\det(I - W^m) \neq 0$, we can write

$$\begin{aligned}(I - W^m)^{-1}\big(I - (W^m)^{n+1}\big)W^k &= \sum_{j=0}^{n} W^{mj+k} \\ &= \begin{bmatrix} \frac{1}{2}\sum_{j=0}^{n} \mathcal{Q}_{mj+k} & 2(s^2+t)\sum_{j=0}^{n} \mathcal{P}_{mj+k} \\ \frac{1}{2}\sum_{j=0}^{n} \mathcal{P}_{mj+k} & \frac{1}{2}\sum_{j=0}^{n} \mathcal{Q}_{mj+k} \end{bmatrix}. \qquad (2.2)\end{aligned}$$

Since

$$(I - W^m)^{-1} = \frac{1}{1 + (-t)^m - \mathcal{Q}_m} \begin{bmatrix} 1 - \frac{1}{2}\mathcal{Q}_m & 2(s^2+t)\mathcal{P}_m \\ \frac{1}{2}\mathcal{P}_m & 1 - \frac{1}{2}\mathcal{Q}_m \end{bmatrix}$$

$$= \frac{1}{1+(-t)^m - \mathcal{Q}_m}\Big((1-\frac{1}{2}\mathcal{Q}_m)I + \frac{1}{2}\mathcal{P}_m H\Big),$$

we have

$$\begin{aligned}
&(I-W^m)^{-1}\big(I-(W^m)^{n+1}\big)W^k \\
&= \frac{\Big((1-\frac{1}{2}\mathcal{Q}_m)I + \frac{1}{2}\mathcal{P}_m H\Big)(W^k - W^{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \\
&= \frac{\Big((1-\frac{1}{2}\mathcal{Q}_m)(W^k - W^{mn+m+k}) + \frac{1}{2}\mathcal{P}_m H(W^k - W^{mn+m+k})\Big)}{1+(-t)^m - \mathcal{Q}_m} \\
&= (1-\frac{1}{2}\mathcal{Q}_m)\begin{bmatrix} \frac{\frac{1}{2}(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} & \frac{2(s^2+t)(\mathcal{P}_k - \mathcal{P}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \\ \frac{\frac{1}{2}(\mathcal{P}_k - \mathcal{P}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} & \frac{\frac{1}{2}(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \end{bmatrix} \\
&\quad + \frac{1}{2}\mathcal{P}_m \begin{bmatrix} \frac{2(s^2+t)(\mathcal{P}_k - \mathcal{P}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} & \frac{2(s^2+t)(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \\ \frac{\frac{1}{2}(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} & \frac{2(s^2+t)(\mathcal{P}_k - \mathcal{P}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \end{bmatrix} \qquad (2.3)
\end{aligned}$$

Using (2.2) and (2.3), we obtain

$$\begin{aligned}
&\sum_{j=0}^{n} \mathcal{Q}_{mj+k} \\
&= \frac{\Big((1-\frac{1}{2}\mathcal{Q}_m)(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k}) + 2(s^2+t)\mathcal{P}_m(\mathcal{P}_k - \mathcal{P}_{mn+m+k})\Big)}{1+(-t)^m - \mathcal{Q}_m}. \qquad (2.4)
\end{aligned}$$

By Lemma 2.4 (iv), (2.4) becomes

$$\sum_{j=0}^{n} \mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k - \mathcal{Q}_{mn+m+k} + (-t)^m\big(\mathcal{Q}_{mn+k} - \mathcal{Q}_{k-m}\big)}{1+(-t)^m - \mathcal{Q}_m}.$$

On the other hand, using (2.2) and (2.3) we get

$$\sum_{j=0}^{n} \mathcal{P}_{mj+k} = \frac{\Big((1-\frac{1}{2}\mathcal{Q}_m)(\mathcal{P}_k - \mathcal{P}_{mn+m+k}) + \frac{1}{2}\mathcal{P}_m(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})\Big)}{1+(-t)^m - \mathcal{Q}_m}. \qquad (2.5)$$

By Lemma 2.4 (v), (2.5) becomes

$$\sum_{j=0}^{n} \mathcal{P}_{mj+k} = \frac{\mathcal{P}_k - \mathcal{P}_{mn+m+k} + (-t)^m\big(\mathcal{P}_{mn+k} - \mathcal{P}_{k-m}\big)}{1+(-t)^m - \mathcal{Q}_m}. \qquad \square$$

Theorem 2.7. *Let $n \in \mathbb{N}$ and $m, k \in \mathbb{Z}$ with $(-t)^m + \mathcal{Q}_m \neq -1$. If n is even, then*

$$\sum_{j=0}^{n} (-1)^j \mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k + \mathcal{Q}_{mn+m+k} + (-t)^m \big(\mathcal{Q}_{mn+k} + \mathcal{Q}_{k-m}\big)}{1 + (-t)^m + \mathcal{Q}_m}$$

and

$$\sum_{j=0}^{n} (-1)^j \mathcal{P}_{mj+k} = \frac{\mathcal{P}_k + \mathcal{P}_{mn+m+k} + (-t)^m \big(\mathcal{P}_{mn+k} + \mathcal{P}_{k-m}\big)}{1 + (-t)^m + \mathcal{Q}_m}$$

Proof. Let n is an even natural number. Then we have

$$I + (W^m)^{n+1} = (I + W^m) \sum_{j=0}^{n} (-1)^j (W^m)^j.$$

By Lemma 2.4 (i), we have

$$\det(I + W^m) = (1 + \frac{1}{2}\mathcal{Q}_m)^2 - (s^2 + t)\mathcal{P}_m^2 = 1 + \mathcal{Q}_m + (-t)^m.$$

Since $\det(I + W^m) \neq 0$, we can write

$$\begin{aligned}
&(I + W^m)^{-1}\big(I + (W^m)^{n+1}\big)W^k \\
&= \sum_{j=0}^{n} (-1)^j W^{mj+k} \\
&= \begin{bmatrix} \frac{1}{2}\sum_{j=0}^{n} (-1)^j \mathcal{Q}_{mj+k} & 2(s^2+t)\sum_{j=0}^{n} (-1)^j \mathcal{P}_{mj+k} \\ \frac{1}{2}\sum_{j=0}^{n} (-1)^j \mathcal{P}_{mj+k} & \frac{1}{2}\sum_{j=0}^{n} (-1)^j \mathcal{Q}_{mj+k} \end{bmatrix}.
\end{aligned} \tag{2.6}$$

Since

$$\begin{aligned}
(I + W^m)^{-1} &= \frac{1}{1 + \mathcal{Q}_m + (-t)^m} \begin{bmatrix} 1 + \frac{1}{2}\mathcal{Q}_m & -2(s^2+t)\mathcal{P}_m \\ -\frac{1}{2}\mathcal{P}_m & 1 + \frac{1}{2}\mathcal{Q}_m \end{bmatrix} \\
&= \frac{1}{1 + \mathcal{Q}_m + (-t)^m}\Big((1 + \frac{1}{2}\mathcal{Q}_m)I - \frac{1}{2}\mathcal{P}_m H\Big),
\end{aligned}$$

we have

$$\begin{aligned}
&(I + W^m)^{-1}\big(I + (W^m)^{n+1}\big)W^k \\
&= \frac{\Big((1 + \frac{1}{2}\mathcal{Q}_m)I - \frac{1}{2}\mathcal{P}_m H\Big)(W^k + W^{mn+m+k})}{1 + \mathcal{Q}_m + (-t)^m} \\
&= \frac{\Big((1 + \frac{1}{2}\mathcal{Q}_m)(W^k + W^{mn+m+k}) - \frac{1}{2}\mathcal{P}_m H(W^k + W^{mn+m+k})\Big)}{1 + \mathcal{Q}_m + (-t)^m}
\end{aligned}$$

$$= (1+\frac{1}{2}\mathcal{Q}_m)\begin{bmatrix} \frac{\frac{1}{2}(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} & \frac{2(s^2+t)(\mathcal{P}_k+\mathcal{P}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} \\ \frac{\frac{1}{2}(\mathcal{P}_k+\mathcal{P}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} & \frac{\frac{1}{2}(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} \end{bmatrix}$$

$$- \frac{1}{2}\mathcal{P}_m\begin{bmatrix} \frac{2(s^2+t)(\mathcal{P}_k+\mathcal{P}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} & \frac{2(s^2+t)(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} \\ \frac{\frac{1}{2}(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} & \frac{2(s^2+t)(\mathcal{P}_k+\mathcal{P}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} \end{bmatrix}. \tag{2.7}$$

Using (2.6) and (2.7), we obtain

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k}$$
$$= \frac{\Big((1+\frac{1}{2}\mathcal{Q}_m)(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k}) - 2(s^2+t)\mathcal{P}_m(\mathcal{P}_k+\mathcal{P}_{mn+m+k})\Big)}{1+\mathcal{Q}_m+(-t)^m}. \tag{2.8}$$

By Lemma 2.4 (*iv*), (2.8) becomes

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k+\mathcal{Q}_{mn+m+k}+(-t)^m\big(\mathcal{Q}_{k-m}+\mathcal{Q}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}.$$

Similarly it can be easily seen that

$$\sum_{j=0}^{n}(-1)^j\mathcal{P}_{mj+k} = \frac{\mathcal{P}_k+\mathcal{P}_{mn+m+k}+(-t)^m\big(\mathcal{P}_{k-m}+\mathcal{P}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}.$$ □

Theorem 2.8. *Let $n\in\mathbb{N}$ and $m,k\in\mathbb{Z}$ with $(-t)^m+\mathcal{Q}_m\neq -1$. If n is odd, then*

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k-\mathcal{Q}_{mn+m+k}+(-t)^m\big(\mathcal{Q}_{k-m}-\mathcal{Q}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}$$

and

$$\sum_{j=0}^{n}(-1)^j\mathcal{P}_{mj+k} = \frac{\mathcal{P}_k-\mathcal{P}_{mn+m+k}+(-t)^m\big(\mathcal{P}_{k-m}-\mathcal{P}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}$$

Proof. Let n is an odd natural number. Then we get

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k} = \sum_{j=0}^{n-1}(-1)^j\mathcal{Q}_{mj+k} - \mathcal{Q}_{mn+k}.$$

Since n is an odd natural number then $n-1$ is even. By Thorem 2.7, we have

$$\sum_{j=0}^{n-1}(-1)^j\mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k+\mathcal{Q}_{mn+k}+(-t)^m\big(\mathcal{Q}_{mn+k-m}+\mathcal{Q}_{k-m}\big)}{1+(-t)^m+\mathcal{Q}_m}$$

and

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k}$$
$$=\frac{\mathcal{Q}_k+(-t)^m\big(\mathcal{Q}_{mn+k-m}+\mathcal{Q}_{k-m}\big)-(-t)^m\mathcal{Q}_{mn+k}-\mathcal{Q}_{mn+k}\mathcal{Q}_m}{1+(-t)^m+\mathcal{Q}_m}. \tag{2.9}$$

Using Lemma 2.4 (*vi*) in (2.9), we get

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k}=\frac{\mathcal{Q}_k-\mathcal{Q}_{mn+m+k}+(-t)^m\big(\mathcal{Q}_{k-m}-\mathcal{Q}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}.$$

In a similar way, it can be seen that

$$\sum_{j=0}^{n}(-1)^j\mathcal{P}_{mj+k}=\sum_{j=0}^{n-1}(-1)^j\mathcal{P}_{mj+k}-\mathcal{P}_{mn+k}.$$

By Theorem 2.7, it follows that

$$\sum_{j=0}^{n}(-1)^j\mathcal{P}_{mj+k}$$
$$=\frac{\mathcal{P}_k+(-t)^m\big(\mathcal{P}_{mn+k-m}+\mathcal{P}_{k-m}\big)-(-t)^m\mathcal{P}_{mn+k}-\mathcal{P}_{mn+k}\mathcal{Q}_m}{1+(-t)^m+\mathcal{Q}_m}. \tag{2.10}$$

Using Lemma 2.4 (*vii*) in (2.10), we obtain

$$\sum_{j=0}^{n}(-1)^j\mathcal{P}_{mj+k}=\frac{\mathcal{P}_k-\mathcal{P}_{mn+m+k}+(-t)^m\big(\mathcal{P}_{k-m}-\mathcal{P}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}. \qquad \square$$

Acknowledgements. The authors would like to thank the faculty of science and technology, Rajamangala University of Technology Thanyaburi (RMUTT), Thailand for the financial support. Moreover, the authors would like to thank the referees for their valuable suggestions and comments which helped to improve the quality and readability of the paper.

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