A note on the derived length of the group of units of group algebras of characteristic two

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In memoriam Mihály Rados (1941–2016)

Abstract

Denote by $FG$ the group algebra of a group $G$ over a field $F$, by $U(FG)$ its group of units, and by $\text{dl}(U(FG))$ the derived length of $U(FG)$. We know very little about $\text{dl}(U(FG))$, especially when $F$ has characteristic 2. In this short note, it is shown that, if $F$ is of characteristic 2, $G'$ is cyclic of order $2^n$ and the nilpotency class of $G$ is less than $n + 1$, then $\text{dl}(U(FG))$ is equal to $n$ or $n + 1$. In addition, if $n > 1$ and $G' = \text{Syl}_2(G)$, then $\text{dl}(U(FG)) = n$.

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1. Introduction

Let $FG$ be the group algebra of a group $G$ over a field $F$ of prime characteristic $p$, and let $U(FG)$ be the group of units of $FG$. It is determined in [4] when $U(FG)$ is solvable, however, we know very little about the derived length of $U(FG)$.

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Assume first that $p$ is an odd prime. For this case, the group algebras $FG$ with metabelian group of units are classified in [16], under restriction $G$ is finite, and this result is extended to torsion $G$ in [6]. In [7, 8] the finite groups $G$ are described, such that $U(FG)$ has derived length 3. According to [1], if $G$ is a finite $p$-group with cyclic commutator subgroup, then $\text{dl}(U(FG)) = \lceil \log_2(|G'| + 1) \rceil$, where $\lceil \cdot \rceil$ is the upper integer part function. The aim of [2] and [10] is to extend this result, and determine the value of $\text{dl}(U(FG))$ for arbitrary groups $G$ with $G'$ is a cyclic $p$-group, where $p$ is still an odd prime. As it turned out, if $G$ is nilpotent and torsion, then the derived length of $U(FG)$ remains $\lceil \log_2(|G'| + 1) \rceil$, but for non-nilpotent or non-torsion $G$ it can be different. However, the description is not complete yet, for the open cases we refer the reader to [10].

For $p = 2$ and finite group $G$, necessary and sufficient conditions for $U(FG)$ to be metabelian is given in [9], and independently, in [14]. This result is extended in [6] as follows: if $F$ is a field of characteristic 2, and $G$ is a nilpotent torsion group, then $U(FG)$ is metabelian exactly when $G'$ is a central elementary abelian group of order dividing 4. In [13], it is established that if $G$ is a group of maximal class of order $2^n$, then $\text{dl}(U(FG))$ is less or equal to $n - 1$. To the best of the author's knowledge, for $p = 2$ there is no other result concerning the derived length of $U(FG)$. The aim of this paper to draw the attention to this uncovered area by sharing the author's experience and an introductory result.

The group of units of a group algebra can be investigated via the Lie structure of the group algebra. For example, we can obtain an upper bound on the derived length of $U(FG)$, by the help of the strong Lie derived length of $FG$. Let $\delta^{(i)}(FG) = FG$, and for $i \geq 1$, denote by $\delta^{(i)}(FG)$ the associative ideal generated by all the Lie commutators $[x,y] = xy - yx$ with $x, y \in \delta^{(i-1)}(FG)$. $FG$ is said to be strongly Lie solvable, if there exists $i$, for which $\delta^{(i)}(FG) = 0$, and the first such $i$ is called the strong Lie derived length of $FG$, which will be denoted by $\text{dl}^{L}(FG)$. For $x, y \in U(FG)$ we have that the group commutator $[x,y] = x^{-1}y^{-1}xy$ is equal to $1 + x^{-1}y^{-1}[x,y]$, which implies that $\delta_i(U(FG)) \subseteq 1 + \delta^{(i)}(FG)$ for all $i$, where $\delta_i(U(FG))$ denotes the $i$th term of the derived series of $U(FG)$. Therefore, if $FG$ is strongly Lie solvable, then $\text{dl}(U(FG)) \leq \text{dl}^{L}(FG)$.

According to [15, Theorem 5.1], $FG$ is strongly Lie solvable if and only if either $G$ is abelian, or $G'$ is a finite $p$-group and $F$ is a field of characteristic $p$. By [11, Proposition 1], if $FG$ is strongly Lie solvable such that $G$ is nilpotent and $\gamma_3(G) \subseteq (G')^p$, then $\text{dl}^{L}(FG) = \lceil \log_2(t(G') + 1) \rceil$, where by $t(G')$ we mean the nilpotency index of the augmentation ideal of the subalgebra $FG'$.

Assume now that $G$ is a group with cyclic commutator subgroup of order $2^n$ and $F$ is a field of characteristic 2. Then $G$ is nilpotent with nilpotency class $\text{cl}(G) \leq n + 1$, so we can apply the above formulas to get

$$\text{dl}(U(FG)) \leq \text{dl}^{L}(FG) = \lceil \log_2(2^{n+1}) \rceil = n + 1.$$ 

Hence, if $n = 1$, then $\text{dl}(U(FG)) = 2$. For the case when $n > 1$ and $\text{cl}(G) \leq n$, we are able to give a lower bound on $\text{dl}(U(FG))$ as well.
Theorem 1.1. Let $F$ be a field of characteristic 2, and let $G$ be a group with cyclic commutator subgroup of order $2^n$, where $n > 1$. Then $\text{dl}(U(FG)) \geq n$, whenever $G$ has nilpotency class at most $n$.

According to [12, Theorem 1], under conditions of Theorem 1.1, $U(FG)$ is nilpotent and, by [5, Theorem 4.3], if $G' = \text{Syl}_2(G)$, then $\text{cl}(U(FG)) = 2^n - 1$. Using the well-known relation $\delta_i(U(FG)) \subseteq \gamma_2^i(U(FG))$ between terms of the derived series and the lower central series of groups, we have the following assertion.

Corollary 1.2. Let $F$ be a field of characteristic 2, and let $G$ be a group with cyclic commutator subgroup of order $2^n$, where $n > 1$. If $G' = \text{Syl}_2(G)$ and $\text{cl}(G) \leq n$, then $\text{dl}(U(FG)) = n$.

For instance, if

$$G = \langle a, b, c \mid c^{2^n} = 1, b^{-1}ab = ac, ac = ca, bc = cb \rangle,$$

with $n > 1$, and $\text{char}(F) = 2$, then $\text{dl}(U(FG)) = n$. This example also witnesses that for non-torsion $G$, $U(FG)$ can be metabelian, even if $G'$ is cyclic of order 4.

The GAP system for computational discrete algebra [17] and its package, the LAGUNA [3] open the door to compute the derived length of $U(FG)$ for $G$ of not too large size. Computing $\text{dl}(U(FG))$ for some group $G$ of order not greater than 512 and $F$ of 2 elements, it seems that $\text{dl}(U(FG))$ will always be at least $n$, even if $\text{cl}(G) = n + 1$. However, it would also be interesting to know when $\text{dl}(U(FG))$ is $n$ or when it is $n + 1$.

2. Proof of Theorem 1.1

We will use the following notations. For a normal subgroup $H$ of $G$ we denote by $\mathfrak{z}(H)$ the ideal in $FG$ generated by all elements of the form $h - 1$ with $h \in H$. For the subsets $X, Y \subseteq FG$ by $[X, Y]$ we mean the additive subgroup of $FG$ generated by all Lie commutators $[x, y]$ with $x \in X$ and $y \in Y$.

Write $G' = \langle x \mid x^{2^n} = 1 \rangle$, and assume that $n > 1$. Then for any $m > 1$, $y \in \gamma_m(G)$ and $g \in G$ we have $g^{-1}yg = y^k$, where $k$ is odd, thus $(y, g) = y^{k-1} \in \gamma_m(G)^2$. Hence, $\gamma_{m+1}(G) \subseteq \gamma_m(G)^2$ for all $m > 1$, so $G$ is nilpotent of class at most $n + 1$. Evidently, if $\gamma_3(G) \subseteq (G')^4$, then $\text{cl}(G)$ cannot exceed $n$. We show first the converse, that is, if $\text{cl}(G) \leq n$, then

$$\gamma_3(G) \subseteq (G')^4.$$  \hspace{1cm} (2.1)

This is clear, if $n = 2$. For $n \geq 3$, it is well known that the automorphism group of $G'$ is the direct product of the cyclic group $\langle \alpha \rangle$ of order 2 and the cyclic group $\langle \beta \rangle$ of order $2^{n-2}$, where the action of these automorphisms on $G'$ is given by $\alpha(x) = x^{-1}$, $\beta(x) = x^5$. Consequently, for every $g \in G$ there exists $i \geq 0$, such that either $g^{-1}xg = x^5^i$ or $g^{-1}xg = x^{-5}^i$. Assume that there is a $g \in G$ such that $g^{-1}xg = x^{-5}^i$ for some $i$, and let $y \in \gamma_m(G)$ with $m > 1$. Then $(y, g) = y^{i-5} \in \gamma_m(G)$.
\( \gamma_{m+1}(G) \), and as \(-1 - 5i \equiv 2 \pmod{4} \), we have that \( \gamma_{m+1}(G) = (\gamma_m(G))^2 \). This means that \( cl(G) = n + 1 \), which is a contradiction. Therefore, for any \( g \in G \) there exists \( i \) such that \( g^{-1}xg = x^{5i} \) and \( (x,g) = x^{-1+5i} = x^{4k} \) for some integer \( k \), which forces 2.1.

Let \( F \) be a field of characteristic 2. The next step is to show by induction that
\[ [\omega(FG')^m, FG] \subseteq \mathcal{J}(G')^{m+3} \tag{2.2} \]
for all \( m \geq 1 \). Let \( y \in G' \) and \( g \in G \). Then, using that \( \gamma_3(G) \subseteq (G')^4 \), we have
\[ [y + 1, g] = [y, g] = gg((y, g) + 1) \in \mathcal{J}(\gamma_3(G)) \subseteq \mathcal{J}(G')^4. \]
Since the Lie commutators of the form \([y + 1, g]\) span the subspace \([\omega(FG'), FG]\), (2.2) holds for \( m = 1 \). Assume now (2.2) for some \( m \geq 1 \). Then,
\[ [\omega(FG')^{m+1}, FG] \subseteq \omega(FG')^m [\omega(FG'), FG] + [\omega(FG')^m, FG] \omega(FG') \]
\[ \subseteq \mathcal{J}(G')^{m+4}, \]
as desired. Furthermore, by using (2.2), for all \( k, l \geq 1 \) we have
\[ [\mathcal{J}(G')^k, \mathcal{J}(G')^l] \]
\[ = [FG\omega(FG')^k, FG\omega(FG')^l] \]
\[ \subseteq FG[\omega(FG')^k, FG\omega(FG')^l] + [FG, FG\omega(FG')^l] \omega(FG')^k \]
\[ \subseteq FG[\omega(FG')^k, FG] \omega(FG')^l + FG[FG, \omega(FG')^l] \omega(FG')^k \]
\[ + [FG, FG] \omega(FG')^{k+l} \]
\[ \subseteq \mathcal{J}(G')^{k+l+1}. \]

At this stage, it may be worth mentioning that without the assumption \( cl(G) \leq n \) we can only claim that \( \gamma_3(G) \subseteq (G')^2 \) and \( [\omega(FG')^m, FG] \subseteq \omega(FG')^{m+1} \) instead of (2.1) and (2.2). Although those would be enough for (2.3), but not for what follows.

Denote by \( S \) the set of those \( a \in G \), for which there exists \( b \in G \), such that \( \langle (a, b) \rangle = G' \). We are going to show that for all \( k \geq 1 \) and \( a \in S \), there exists \( w_k \in \mathcal{J}(G')^{3\cdot2^{k-1}} \), such that
\[ 1 + a(x + 1)^{3\cdot2^{k-1} - 1} + w_k \in \delta_k(U(FG)). \tag{2.4} \]
This implies that \( \delta_k(U(FG)) \) contains non-identity element, while \( 3\cdot2^{k-1} - 1 < 2^n \), and then
\[ dl(U(FG)) \geq \left\lfloor \log_2 \left( \frac{2}{3}(2^n + 1) \right) \right\rfloor = n, \]
and the proof of Theorem 1.1 will be done.

Let \( a \in S \). Then there exists \( b \in G \) such that \( (a, b) = x^i \), where \( i \) is odd. By (2.2), \([x + 1, b] \in \mathcal{J}(G')^4 \), and
\[ u := (1 + a(x + 1), b) = 1 + (1 + a(x + 1))^{-1}b^{-1}[a(x + 1), b] \]
\[\equiv 1 + (1 + a(x + 1))^{-1}b^{-1}[a, b](x + 1)\]
\[\equiv 1 + (1 + a(x + 1))^{-1}a(x^i + 1)(x + 1) \pmod{I(G')^3}.\]

Since \(1 + a(x + 1)\) belongs to the normal subgroup \(1 + \mathcal{I}(G')\), so does its inverse, and
\[u \equiv 1 + a(x^i + 1)(x + 1) \pmod{I(G')^3}.\]

Using that \(x^i + 1 \equiv i(x + 1) = x + 1 \pmod{\omega(FG')^2}\), we obtain that
\[u \equiv 1 + a(x + 1)^2 \pmod{I(G')^3},\]
and (2.4) is confirmed for \(k = 1\). Assume, by induction, the truth of (2.4) for some \(k \geq 1\), and let \(a \in S\). Then there exists \(b \in G\) such that \(\langle (a, b) \rangle = G'\), and of course, \(b\) also belongs to \(S\). Moreover, \(b^{-1}a \in S\), because \((b^{-1}a, b) = (a, b)\). By the inductive hypothesis, there exist \(w_k, w_k' \in \mathcal{I}(G')^{3 \cdot 2^{k-1}}\), such that
\[u := 1 + b^{-1}a(x + 1)^{3 \cdot 2^{k-1}-1} + w_k \in \delta_k(U(FG))\]
and
\[v := 1 + b(x + 1)^{3 \cdot 2^{k-1}-1} + w_k' \in \delta_k(U(FG)).\]

According to (2.3),
\[[u, v] \equiv [b^{-1}a(x + 1)^{3 \cdot 2^{k-1}-1}, b(x + 1)^{3 \cdot 2^{k-1}-1}] \pmod{\mathcal{I}(G')^{3 \cdot 2^k}}.\]

Applying (2.2), we have that \([(x + 1)^{3 \cdot 2^{k-1}-1}, b]\) and \([b^{-1}a, (x + 1)^{3 \cdot 2^{k-1}-1}]\) belong to \(\mathcal{I}(G')^{3 \cdot 2^{k-1} + 2}\), and
\[[u, v] \equiv b^{-1}a[(x + 1)^{3 \cdot 2^{k-1}-1}, b](x + 1)^{3 \cdot 2^{k-1}-1} + b[b^{-1}a, (x + 1)^{3 \cdot 2^{k-1}-1}](x + 1)^{3 \cdot 2^{k-1}-1} + [b^{-1}a, b](x + 1)^{3 \cdot 2^{k}-2}\equiv a(x^i + 1)(x + 1)^{3 \cdot 2^{k}-2} \equiv a(x + 1)^{3 \cdot 2^{k}-1} \pmod{\mathcal{I}(G')^{3 \cdot 2^k}},\]
where \(i\) is not divisible by 2. Since \(u^{-1}, v^{-1} \in 1 + \mathcal{I}(G')\), so
\[(u, v) = 1 + u^{-1}v^{-1}[u, v] \equiv 1 + a(x + 1)^{3 \cdot 2^{k}-1} \pmod{\mathcal{I}(G')^{3 \cdot 2^k}}\]
and the induction is done.

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References


