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A note on the derived length of the group of units of group algebras of characteristic two*

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In memoriam Mihály Rados (1941–2016)

Abstract

Denote by FG the group algebra of a group G over a field F, by $U(FG)$ its group of units, and by $\mathrm{dl}(U(FG))$ the derived length of $U(FG)$. We know very little about $\mathrm{dl}(U(FG))$, especially when F has characteristic 2. In this short note, it is shown that, if F is of characteristic 2, G' is cyclic of order 2^n and the nilpotency class of G is less than $n+1$, then $\mathrm{dl}(U(FG))$ is equal to n or $n+1$. In addition, if $n>1$ and $G' = \mathrm{Syl}_2(G)$, then $\mathrm{dl}(U(FG)) = n$.

Keywords: Group ring, group of units, derived length

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1. Introduction

Let FG be the group algebra of a group G over a field F of prime characteristic p, and let $U(FG)$ be the group of units of FG. It is determined in [4] when $U(FG)$ is solvable, however, we know very little about the derived length of $U(FG)$.

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Assume first that p is an odd prime. For this case, the group algebras FG with metabelian group of units are classified in [16], under restriction G is finite, and this result is extended to torsion G in [6]. In [7, 8] the finite groups G are described, such that $U(FG)$ has derived length 3. According to [1], if G is a finite p-group with cyclic commutator subgroup, then $\mathrm{dl}(U(FG)) = \lceil \log_2(|G'|+1)\rceil$, where $\lceil\cdot\rceil$ is the upper integer part function. The aim of [2] and [10] is to extend this result, and determine the value of $\mathrm{dl}(U(FG))$ for arbitrary groups G with G' is a cyclic p-group, where p is still an odd prime. As it turned out, if G is nilpotent and torsion, then the derived length of $U(FG)$ remains $\lceil \log_2(|G'|+1)\rceil$, but for non-nilpotent or non-torsion G it can be different. However, the description is not complete yet, for the open cases we refer the reader to [10].

For $p = 2$ and finite group G, necessary and sufficient conditions for $U(FG)$ to be metabelian is given in [9], and independently, in [14]. This result is extended in [6] as follows: if F is a field of characteristic 2, and G is a nilpotent torsion group, then $U(FG)$ is metabelian exactly when G' is a central elementary abelian group of order dividing 4. In [13], it is established that if G is a group of maximal class of order 2^n, then $\mathrm{dl}(U(FG))$ is less or equal to $n-1$. To the best of the author's knowledge, for $p = 2$ there is no other result concerning the derived length of $U(FG)$. The aim of this paper to draw the attention to this uncovered area by sharing the author's experience and an introductory result.

The group of units of a group algebra can be investigated via the Lie structure of the group algebra. For example, we can obtain an upper bound on the derived length of $U(FG)$, by the help of the strong Lie derived length of FG. Let $\delta^{(0)}(FG) = FG$, and for $i \geq 1$, denote by $\delta^{(i)}(FG)$ the associative ideal generated by all the Lie commutators $[x,y] = xy - yx$ with $x, y \in \delta^{(i-1)}(FG)$. FG is said to be strongly Lie solvable, if there exists i, for which $\delta^{(i)}(FG) = 0$, and the first such i is called the strong Lie derived length of FG, which will be denoted by $\mathrm{dl}^L(FG)$. For $x, y \in U(FG)$ we have that the group commutator $(x,y) = x^{-1}y^{-1}xy$ is equal to $1 + x^{-1}y^{-1}[x,y]$, which implies that $\delta_i(U(FG)) \subseteq 1 + \delta^{(i)}(FG)$ for all i, where $\delta_i(U(FG))$ denotes the ith term of the derived series of $U(FG)$. Therefore, if FG is strongly Lie solvable, then $\mathrm{dl}(U(FG)) \leq \mathrm{dl}^L(FG)$.

According to [15, Theorem 5.1], FG is strongly Lie solvable if and only if either G is abelian, or G' is a finite p-group and F is a field of characteristic p. By [11, Proposition 1], if FG is strongly Lie solvable such that G is nilpotent and $\gamma_3(G) \subseteq (G')^p$, then $\mathrm{dl}^L(FG) = \lceil \log_2(t(G')+1)\rceil$, where by $t(G')$ we mean the nilpotency index of the augmentation ideal of the subalgebra FG'.

Assume now that G is a group with cyclic commutator subgroup of order 2^n and F is a field of characteristic 2. Then G is nilpotent with nilpotency class $\mathrm{cl}(G) \leq n+1$, so we can apply the above formulas to get

$$\mathrm{dl}(U(FG)) \leq \mathrm{dl}^L(FG) = \lceil \log_2(2^n+1)\rceil = n+1.$$

Hence, if $n = 1$, then $\mathrm{dl}(U(FG)) = 2$. For the case when $n > 1$ and $\mathrm{cl}(G) \leq n$, we are able to give a lower bound on $\mathrm{dl}(U(FG))$ as well.

Theorem 1.1. *Let F be a field of characteristic 2, and let G be a group with cyclic commutator subgroup of order 2^n, where $n > 1$. Then* $\mathrm{dl}(U(FG)) \geq n$*, whenever G has nilpotency class at most n.*

According to [12, Theorem 1], under conditions of Theorem 1.1, $U(FG)$ is nilpotent and, by [5, Theorem 4.3], if $G' = \mathrm{Syl}_2(G)$, then $\mathrm{cl}(U(FG)) = 2^n - 1$. Using the well-known relation $\delta_i(U(FG)) \subseteq \gamma_{2^i}(U(FG))$ between terms of the derived series and the lower central series of groups, we have the following assertion.

Corollary 1.2. *Let F be a field of characteristic 2, and let G be a group with cyclic commutator subgroup of order 2^n, where $n > 1$. If $G' = \mathrm{Syl}_2(G)$ and $\mathrm{cl}(G) \leq n$, then* $\mathrm{dl}(U(FG)) = n$.

For instance, if

$$G = \langle a, b, c \mid c^{2^n} = 1, b^{-1}ab = ac, ac = ca, bc = cb\rangle,$$

with $n > 1$, and $\mathrm{char}(F) = 2$, then $\mathrm{dl}(U(FG)) = n$. This example also witnesses that for non-torsion G, $U(FG)$ can be metabelian, even if G' is cyclic of order 4.

The GAP system for computational discrete algebra [17] and its package, the LAGUNA [3] open the door to compute the derived length of $U(FG)$ for G of not too large size. Computing $\mathrm{dl}(U(FG))$ for some group G of order not greater than 512 and F of 2 elements, it seems that $\mathrm{dl}(U(FG))$ will always be at least n, even if $\mathrm{cl}(G) = n + 1$. However, it would also be interesting to know when $\mathrm{dl}(U(FG))$ is n or when it is $n + 1$.

2. Proof of Theorem 1.1

We will use the following notations. For a normal subgroup H of G we denote by $\mathfrak{I}(H)$ the ideal in FG generated by all elements of the form $h - 1$ with $h \in H$. For the subsets $X, Y \subseteq FG$ by $[X, Y]$ we mean the additive subgroup of FG generated by all Lie commutators $[x, y]$ with $x \in X$ and $y \in Y$.

Write $G' = \langle x \mid x^{2^n} = 1\rangle$, and assume that $n > 1$. Then for any $m > 1$, $y \in \gamma_m(G)$ and $g \in G$ we have $g^{-1}yg = y^k$, where k is odd, thus $(y, g) = y^{k-1} \in \gamma_m(G)^2$. Hence, $\gamma_{m+1}(G) \subseteq \gamma_m(G)^2$ for all $m > 1$, so G is nilpotent of class at most $n + 1$. Evidently, if $\gamma_3(G) \subseteq (G')^4$, then $\mathrm{cl}(G)$ cannot exceed n. We show first the converse, that is, if $\mathrm{cl}(G) \leq n$, then

$$\gamma_3(G) \subseteq (G')^4. \tag{2.1}$$

This is clear, if $n = 2$. For $n \geq 3$, it is well known that the automorphism group of G' is the direct product of the cyclic group $\langle\alpha\rangle$ of order 2 and the cyclic group $\langle\beta\rangle$ of order 2^{n-2}, where the action of these automorphisms on G' is given by $\alpha(x) = x^{-1}$, $\beta(x) = x^5$. Consequently, for every $g \in G$ there exists $i \geq 0$, such that either $g^{-1}xg = x^{5^i}$ or $g^{-1}xg = x^{-5^i}$. Assume that there is a $g \in G$ such that $g^{-1}xg = x^{-5^i}$ for some i, and let $y \in \gamma_m(G)$ with $m > 1$. Then $(y, g) = y^{-1-5^i} \in$

$\gamma_{m+1}(G)$, and as $-1-5^i \equiv 2 \pmod 4$, we have that $\gamma_{m+1}(G) = (\gamma_m(G))^2$. This means that $\mathrm{cl}(G) = n+1$, which is a contradiction. Therefore, for any $g \in G$ there exists i such that $g^{-1}xg = x^{5^i}$ and $(x,g) = x^{-1+5^i} = x^{4k}$ for some integer k, which forces 2.1.

Let F be a field of characteristic 2. The next step is to show by induction that

$$[\omega(FG')^m, FG] \subseteq \mathfrak{I}(G')^{m+3} \tag{2.2}$$

for all $m \geq 1$. Let $y \in G'$ and $g \in G$. Then, using that $\gamma_3(G) \subseteq (G')^4$, we have

$$[y+1, g] = [y,g] = gy((y,g)+1) \in \mathfrak{I}(\gamma_3(G)) \subseteq \mathfrak{I}(G')^4.$$

Since the Lie commutators of the form $[y+1,g]$ span the subspace $[\omega(FG'), FG]$, (2.2) holds for $m=1$. Assume now (2.2) for some $m \geq 1$. Then,

$$\begin{aligned}[\omega(FG')^{m+1}, FG] &\subseteq \omega(FG')^m[\omega(FG'), FG] + [\omega(FG')^m, FG]\omega(FG') \\ &\subseteq \mathfrak{I}(G')^{m+4},\end{aligned}$$

as desired. Furthermore, by using (2.2), for all $k, l \geq 1$ we have

$$\begin{aligned}&[\mathfrak{I}(G')^k, \mathfrak{I}(G')^l] \\ &\quad = [FG\omega(FG')^k, FG\omega(FG')^l] \\ &\quad \subseteq FG[\omega(FG')^k, FG\omega(FG')^l] + [FG, FG\omega(FG')^l]\omega(FG')^k \\ &\quad \subseteq FG[\omega(FG')^k, FG]\omega(FG')^l + FG[FG, \omega(FG')^l]\omega(FG')^k \\ &\qquad + [FG,FG]\omega(FG')^{k+l} \\ &\quad \subseteq \mathfrak{I}(G')^{k+l+1}.\end{aligned} \tag{2.3}$$

At this stage, it may be worth mentioning that without the assumption $\mathrm{cl}(G) \leq n$ we can only claim that $\gamma_3(G) \subseteq (G')^2$ and $[\omega(FG')^m, FG] \subseteq \omega(FG')^{m+1}$ instead of (2.1) and (2.2). Although those would be enough for (2.3), but not for what follows.

Denote by S the set of those $a \in G$, for which there exists $b \in G$, such that $\langle (a,b) \rangle = G'$. We are going to show that for all $k \geq 1$ and $a \in S$, there exists $w_k \in \mathfrak{I}(G')^{3\cdot 2^{k-1}}$, such that

$$1 + a(x+1)^{3\cdot 2^{k-1}-1} + w_k \in \delta_k(U(FG)). \tag{2.4}$$

This implies that $\delta_k(U(FG))$ contains non-identity element, while $3\cdot 2^{k-1} - 1 < 2^n$, and then

$$\mathrm{dl}(U(FG)) \geq \left\lceil \log_2\left(\frac{2}{3}(2^n+1)\right)\right\rceil = n,$$

and the proof of Theorem 1.1 will be done.

Let $a \in S$. Then there exists $b \in G$ such that $(a,b) = x^i$, where i is odd. By (2.2), $[x+1, b] \in \mathfrak{I}(G')^4$, and

$$u := (1 + a(x+1), b) = 1 + (1 + a(x+1))^{-1}b^{-1}[a(x+1), b]$$

$$\equiv 1 + (1 + a(x+1))^{-1} b^{-1} [a, b](x+1)$$
$$\equiv 1 + (1 + a(x+1))^{-1} a(x^i + 1)(x+1) \pmod{\mathfrak{I}(G')^3}.$$

Since $1 + a(x+1)$ belongs to the normal subgroup $1 + \mathfrak{I}(G')$, so does its inverse, and

$$u \equiv 1 + a(x^i + 1)(x+1) \pmod{\mathfrak{I}(G')^3}.$$

Using that $x^i + 1 \equiv i(x+1) = x + 1 \pmod{\omega(FG')^2}$, we obtain that

$$u \equiv 1 + a(x+1)^2 \pmod{\mathfrak{I}(G')^3},$$

and (2.4) is confirmed for $k = 1$. Assume, by induction, the truth of (2.4) for some $k \geq 1$, and let $a \in S$. Then there exists $b \in G$ such that $\langle (a,b) \rangle = G'$, and of course, b also belongs to S. Moreover, $b^{-1}a \in S$, because $(b^{-1}a, b) = (a, b)$. By the inductive hypothesis, there exist $w_k, w'_k \in \mathfrak{I}(G')^{3 \cdot 2^{k-1}}$, such that

$$u := 1 + b^{-1}a(x+1)^{3 \cdot 2^{k-1} - 1} + w_k \in \delta_k(U(FG))$$

and

$$v := 1 + b(x+1)^{3 \cdot 2^{k-1} - 1} + w'_k \in \delta_k(U(FG)).$$

According to (2.3),

$$[u, v] \equiv [b^{-1}a(x+1)^{3 \cdot 2^{k-1} - 1}, b(x+1)^{3 \cdot 2^{k-1} - 1}] \pmod{\mathfrak{I}(G')^{3 \cdot 2^k}}.$$

Applying (2.2), we have that $[(x+1)^{3 \cdot 2^{k-1} - 1}, b]$ and $[b^{-1}a, (x+1)^{3 \cdot 2^{k-1} - 1}]$ belong to $\mathfrak{I}(G')^{3 \cdot 2^{k-1} + 2}$, and

$$\begin{aligned} [u, v] &\equiv b^{-1}a[(x+1)^{3 \cdot 2^{k-1} - 1}, b](x+1)^{3 \cdot 2^{k-1} - 1} \\ &\quad + b[b^{-1}a, (x+1)^{3 \cdot 2^{k-1} - 1}](x+1)^{3 \cdot 2^{k-1} - 1} + [b^{-1}a, b](x+1)^{3 \cdot 2^k - 2} \\ &\equiv a(x^i + 1)(x+1)^{3 \cdot 2^k - 2} \equiv a(x+1)^{3 \cdot 2^k - 1} \pmod{\mathfrak{I}(G')^{3 \cdot 2^k}}, \end{aligned}$$

where i is not divisible by 2. Since $u^{-1}, v^{-1} \in 1 + \mathfrak{I}(G')$, so

$$(u, v) = 1 + u^{-1}v^{-1}[u, v] \equiv 1 + a(x+1)^{3 \cdot 2^k - 1} \pmod{\mathfrak{I}(G')^{3 \cdot 2^k}}$$

and the induction is done.

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References

[1] Bagiński, C., A note on the derived length of the unit group of a modular group algebra, *Comm. Algebra* 30 (2002), 4905–4913.

[2] Balogh, Z., Li, Y., On the derived length of the group of units of a group algebra, *J. Algebra Appl.* 6 (2007), No. 6, 991–999.

[3] Bovdi, V., Konovalov, A., Rossmanith, R., Schneider, C., LAGUNA — Lie AlGebras and UNits of group Algebras, Version 3.7.0; 11 November 2014 (`http://www.cs.st-andrews.ac.uk/~alexk/laguna/`).

[4] Bovdi, A., Group algebras with a solvable group of units, *Comm. Algebra* 36 (2008), No. 2, 315–324.

[5] Bovdi, A., Kurdics, J., Lie properties of group algebra and the nilpotency class of the group of units, *J. Algebra* 212 (1999), No. 1, 28–64.

[6] Catino, F., Spinelli, E., On the derived length of the unit group of a group algebra, *J. Group Theory* 13 (2010), No. 4, 577–588.

[7] Chandra, H., Sahai, M., On group algebras with unit groups of derived length three in characteristic three, *Publ. Math.* 82 (2013), No. 3–4, 697–708.

[8] Chandra, H., Sahai, M., Group algebras with unit groups of derived length three, *J. Algebra Appl.* 9 (2010), No. 2, 305–314.

[9] Coleman, D.B., Sandling, R., Mod 2 group algebras with metabelian unit groups. *J. Pure Appl. Algebra* 131 (1998), 25–36.

[10] Juhász, T., The derived length of the unit group of a group algebra – The case $G' = \mathrm{Syl}_p(G)$, *J. Algebra Appl.*, In press, DOI: `http://dx.doi.org/10.1142/S0219498817501420`.

[11] Juhász, T., On the derived length of Lie solvable group algebras, *Publ. Math.* 68 (2006), No. 1–2, 243–256.

[12] Khripta, I.I., The nilpotency of the multiplicative group of a group ring, *Mat. Zametki* 11 (1972), 191–200.

[13] Konovalov, A., Wreath products in the unit group of modular group algebras of 2-groups of maximal class, PhD Thesis, Mathematical Institute, Ukrainian National Academy of Sciences, Kiev, 1995.

[14] Kurdics, J., On group algebras with metabelian unit groups, *Period. Math. Hung.* 32 (1996), No. 1–2, 57–64.

[15] Sehgal, S.K., Topics in group rings, Marcel Dekker, New York, 1978.

[16] Shalev, A., Meta-abelian unit groups of group algebras are usually abelian, *J. Pure Appl. Algebra* 72 (1991), 295–302.

[17] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.6; 2016 (`http://www.gap-system.org`).