The log-concavity and log-convexity properties associated to hyperpell and hyperpell-lucas sequences

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Abstract

We establish the log-concavity and the log-convexity properties for the hyperpell, hyperpell-lucas and associated sequences. Further, we investigate the $q$-log-concavity property.

Keywords: hyperpell numbers; hyperpell-lucas numbers; log-concavity; $q$-log-concavity, log-convexity.

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1. Introduction

Zheng and Liu [13] discuss the properties of the hyperfibonacci numbers $F_n^{[r]}$ and the hyperlucas numbers $L_n^{[r]}$. They investigate the log-concavity and the log convexity property of hyperfibonacci and hyperlucas numbers. In addition, they extend their work to the generalized hyperfibonacci and hyperlucas numbers.
The hyperfibonacci numbers $F_r[n]$ and hyperlucas numbers $L_r[n]$, introduced by Dil and Mező [9] are defined as follows. Put

$$F_r[n] = \sum_{k=0}^{n} F_k^{[r-1]}, \quad \text{with} \quad F_r[0] = F_n,$$

$$L_r[n] = \sum_{k=0}^{n} L_k^{[r-1]}, \quad \text{with} \quad L_r[0] = L_n,$$

where $r$ is a positive integer, and $F_n$ and $L_n$ are the Fibonacci and Lucas numbers, respectively.

Belbachir and Belkhir [1] gave a combinatorial interpretation and an explicit formula for hyperfibonacci numbers,

$$F_r[n+1] = \sum_{k=0}^{[n/2]} \binom{n + r - k}{k + r}.$$  \hfill (1.1)

Let $\{U_n\}_{n \geq 0}$ and $\{V_n\}_{n \geq 0}$ denote the generalized Fibonacci and Lucas sequences given by the recurrence relation

$$W_n+1 = pW_n + W_{n-1} \quad (n \geq 1), \quad \text{with} \quad U_0 = 0, \ U_1 = 1, \ V_0 = 2, \ V_1 = p. \quad (1.2)$$

The Binet forms of $U_n$ and $V_n$ are

$$U_n = \frac{\tau^n - (-1)^n \tau^{-n}}{\sqrt{\Delta}} \quad \text{and} \quad V_n = \tau^n + (-1)^n \tau^{-n}; \quad (1.3)$$

with $\Delta = p^2 + 4, \ \tau = (p + \sqrt{\Delta})/2$, and $p \geq 1$.

The generalized hyperfibonacci and generalized hyperlucas numbers are defined, respectively, by

$$U_r[n] := \sum_{k=0}^{n} U_k^{[r-1]}, \quad \text{with} \quad U_r[0] = U_n,$$

$$V_r[n] := \sum_{k=0}^{n} V_k^{[r-1]}, \quad \text{with} \quad V_r[0] = V_n.$$

The paper of Zheng and Liu [13] allows us to exploit other relevant results. More precisely, we propose some results on log-concavity and log-convexity in the case of $p = 2$ for the hyperpell sequence and the hyperpell-lucas sequence.

**Definition 1.1.** Hyperpell numbers $P_r[n]$ and hyperpell-lucas numbers $Q_r[n]$ are defined by

$$P_r[n] := \sum_{k=0}^{n} P_k^{[r-1]}, \quad \text{with} \quad P_r[0] = P_n,$$
\[ Q[r] := \sum_{k=0}^{n} Q[r-1]_k, \quad \text{with} \quad Q[0]_n = Q_n, \]

where \( r \) is a positive integer, and \( \{P_n\} \) and \( \{Q_n\} \) are the Pell and the Pell-Lucas sequences respectively.

Now we recall some formulas for Pell and Pell-Lucas numbers. It is well known that the Binet forms of \( P_n \) and \( Q_n \) are

\[ P_n = \frac{\alpha^n - (-1)^n\alpha^{-n}}{2\sqrt{2}} \quad \text{and} \quad Q_n = \alpha^n + (-1)^n\alpha^{-n}, \quad (1.4) \]

where \( \alpha = (1 + \sqrt{2}) \). The integers

\[ P(n, k) = 2^{n-2k} \binom{n-k}{k} \quad \text{and} \quad Q(n, k) = 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k}, \quad (1.5) \]

are linked to the sequences \( \{P_n\} \) and \( \{Q_n\} \). It is established [2] that for each fixed \( n \) these two sequences are log-concave and then unimodal. For the generalized sequence given by (1.2), also the corresponding associated sequences are log-concave and then unimodal, see [3, 4].

The sequences \( \{P_n\} \) and \( \{Q_n\} \) satisfy the recurrence relation (1.2), for \( p = 2 \), and for \( n \geq 0 \) and \( n \geq 1 \) respectively, we have

\[ P_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{n-k}{k} \quad \text{and} \quad Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k}. \quad (1.6) \]

It follows from (1.4) that the following formulas hold

\[ P_n^2 - P_{n-1}P_{n+1} = (-1)^{n+1}, \quad (1.7) \]
\[ Q_n^2 - Q_{n-1}Q_{n+1} = 8(-1)^n. \quad (1.8) \]

It is easy to see, for example by induction, that for \( n \geq 1 \)

\[ P_n \geq n \quad \text{and} \quad Q_n \geq n. \quad (1.9) \]

Let \( \{x_n\}_{n \geq 0} \) be a sequence of nonnegative numbers. The sequence \( \{x_n\}_{n \geq 0} \) is log-concave (respectively log-convex) if \( x_j^2 \geq x_{j-1}x_{j+1} \) (respectively \( x_j^2 \leq x_{j-1}x_{j+1} \)) for all \( j > 0 \), which is equivalent (see [5]) to \( x_i x_j \geq x_{i-1}x_{j+1} \) (respectively \( x_i x_j \leq x_{i-1}x_{j+1} \)) for \( j \geq i \geq 1 \).

We say that \( \{x_n\}_{n \geq 0} \) is log-balanced if \( \{x_n\}_{n \geq 0} \) is log-convex and \( \{x_n/n!\}_{n \geq 0} \) is log-concave.

Let \( q \) be an indeterminate and \( \{f_n(q)\}_{n \geq 0} \) be a sequence of polynomials of \( q \). If for each \( n \geq 1 \), \( f_n^2(q) - f_{n-1}(q)f_{n+1}(q) \) has nonnegative coefficients, we say that \( \{f_n(q)\}_{n \geq 0} \) is \( q \)-log-concave.

In section 2, we give the generating functions of hyperpell and hyperpell-lucas sequences. In section 3, we discuss their log-concavity and log-convexity. We investigate also the \( q \)-log-concavity of some polynomials related to hyperpell and hyperpell-lucas numbers.
2. The generating functions

The generating function of Pell numbers and Pell-Lucas numbers denoted $G_P(t)$ and $G_Q(t)$, respectively, are

$$G_P(t) := \sum_{n=0}^{+\infty} P_n t^n = \frac{t}{1 - 2t - t^2}, \quad (2.1)$$

and

$$G_Q(t) := \sum_{n=0}^{+\infty} Q_n t^n = \frac{2 - 2t}{1 - 2t - t^2}. \quad (2.2)$$

So, we establish the generating function of hyperpell and hyperpell-lucas numbers using respectively

$$P_n^{[r]} = P_{n-1}^{[r]} + P_{n-1}^{[r-1]} \quad \text{and} \quad Q_n^{[r]} = Q_{n-1}^{[r]} + Q_{n-1}^{[r-1]}. \quad (2.3)$$

The generating functions of hyperpell numbers and hyperlucas numbers are

$$G_P^{[r]}(t) = \sum_{n=0}^{\infty} P_n^{[r]} t^n = \frac{t}{(1 - 2t - t^2)(1-t)^r}, \quad (2.4)$$

and

$$G_Q^{[r]}(t) = \sum_{n=0}^{\infty} Q_n^{[r]} t^n = \frac{2 - 2t}{(1 - 2t - t^2)(1-t)^r}. \quad (2.5)$$

3. The log-concavity and log-convexity properties

We start the section by some useful lemmas.

**Lemma 3.1.** [12] If the sequences $\{x_n\}$ and $\{y_n\}$ are log-concave, then so is their ordinary convolution $z_n = \sum_{k=0}^{n} x_k y_{n-k}, \ n = 0, 1, ...$

**Lemma 3.2.** [12] If the sequence $\{x_n\}$ is log-concave, then so is the binomial convolution $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k, \ n = 0, 1, ...$

**Lemma 3.3.** [8] If the sequence $\{x_n\}$ is log-convex, then so is the binomial convolution $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k, \ n = 0, 1, ...$

The following result deals with the log-concavity of hyperpell numbers and hyperlucas sequences.

**Theorem 3.4.** The sequences $\{P_n^{[r]}\}_{n \geq 0}$ and $\{Q_n^{[r]}\}_{n \geq 0}$ are log-concave for $r \geq 1$ and $r \geq 2$ respectively.
Proof. We have
\[ P_n^{[1]} = \frac{1}{4} (Q_{n+1} - 2) \quad \text{and} \quad Q_n^{[1]} = 2P_{n+1}. \quad (3.1) \]

When \( n = 1 \), \( (P_n^{[1]})^2 - P_{n-1}^{[1]}P_{n+1}^{[1]} = 1 > 0 \). When \( n \geq 2 \), it follows from (3.1) and (1.8) that
\[
(P_n^{[1]})^2 - P_{n-1}^{[1]}P_{n+1}^{[1]} = \frac{1}{16} \left[ (Q_{n+1} - 2)^2 - (Q_n - 2) (Q_{n+2} - 2) \right]
= \frac{1}{16} (Q_{n+1}^2 - Q_n Q_{n+2} - 4Q_{n+1} + 2Q_n + 2Q_{n+2})
= \frac{1}{4} (2(-1)^{n-1} + Q_{n+1}) \geq 0.
\]

Then \( \{P_n^{[1]}\}_{n \geq 0} \) is log-concave. By Lemma 3.1, we know that \( \{P_n^{[r]}\}_{n \geq 0} \) \((r \geq 1)\) is log-concave.

It follows from (3.1) and (1.7) that
\[
(Q_n^{[1]})^2 - Q_{n-1}^{[1]}Q_{n+1}^{[1]} = 4 (P_{n+1}^2 - P_n P_{n+2}) = 4 (-1)^n = \pm 4 \quad (3.2)
\]

Hence \( \{Q_n^{[1]}\}_{n \geq 0} \) is not log-concave.

One can verify that
\[
Q_n^{[2]} = \frac{1}{2} (Q_{n+2} - 2) = 2P_{n+1}. \quad (3.3)
\]

Then \( \{Q_n^{[2]}\}_{n \geq 0} \) is log-concave. By Lemma 3.1, we know that \( \{Q_n^{[r]}\}_{n \geq 0} \) \((r \geq 2)\) is log-concave. This completes the proof of Theorem 3.4.

Then we have the following corollary.

**Corollary 3.5.** The sequences \( \left\{ \sum_{k=0}^{n} \binom{n}{k} P_k^{[r]} \right\}_{n \geq 0} \) and \( \left\{ \sum_{k=0}^{n} \binom{n}{k} Q_k^{[r]} \right\}_{n \geq 0} \) are log-concave for \( r \geq 1 \) and \( r \geq 2 \) respectively.

**Proof.** Use Lemma 3.2.

Now we establish the log-concavity of order two of the sequences \( \{P_n^{[1]}\}_{n \geq 0} \) and \( \{Q_n^{[2]}\}_{n \geq 0} \) for some special sub-sequences.

**Theorem 3.6.** Let be for \( n \geq 1 \)
\[
T_n := \left( P_n^{[1]} \right)^2 - P_{n-1}^{[1]}P_{n+1}^{[1]} \quad \text{and} \quad R_n := \left( Q_n^{[2]} \right)^2 - Q_{n-1}^{[2]}Q_{n+1}^{[2]}.
\]

Then \( \{T_{2n}\}_{n \geq 1}, \{R_{2n+1}\}_{n \geq 0} \) are log-concave, and \( \{T_{2n+1}\}_{n \geq 0}, \{R_{2n}\}_{n \geq 1} \) are log-convex.
Proof. Using respectively (3.3) and (1.8), we get
\[
\left( Q_n^{[2]} \right)^2 - Q_n^{[2]} Q_{n+1}^{[2]} = 2(-1)^n + Q_{n+1},
\]
and thus, for \( n \geq 1 \),
\[
T_n = \frac{1}{4} \left( 2 (-1)^{n-1} + Q_n \right) \quad \text{and} \quad R_n = 2(-1)^n + Q_{n+1}. \tag{3.4}
\]
By applying (3.4) and (1.8), for \( n \geq 1 \) we get
\[
Q_{2n}^2 - Q_{2n-2} Q_{2n+2} = -32 \quad \text{and} \quad Q_{2n+1}^2 - Q_{2n-1} Q_{2n+3} = 32. \tag{3.5}
\]
Then
\[
T_{2n}^2 - T_{2(n-1)} T_{2(n+1)} = \frac{1}{16} \left( Q_{2n+2}^2 - Q_{2n-2} Q_{2n+2} - 4Q_{2n} + 2Q_{2n-2} + 2Q_{2n+2} \right)
\]
\[
= 4(Q_{2n} - 4) > 0.
\]
and
\[
R_{2n+1}^2 - R_{2n-1} R_{2n+3} = (Q_{2n+2}^2 - Q_{2n} Q_{2n+2} - 4Q_{2n+2} + 2Q_{2n} + 2Q_{2n+4})
\]
\[
= 64(Q_{2n+2} - 4) > 0.
\]
Then \( \{T_{2n}\}_{n \geq 1} \) and \( \{R_{2n+1}\}_{n \geq 0} \) are log-concave.

Similarly by applying (3.4) and (3.5), we have
\[
T_{2n+1}^2 - T_{2n-1} T_{2n+3} = -\frac{1}{2} Q_{2n+1} < 0,
\]
and
\[
R_{2n}^2 - R_{2(n-1)} R_{2(n+1)} = -8Q_{2n+1} < 0.
\]
Then \( \{T_{2n+1}\}_{n \geq 0} \) and \( \{R_{2n}\}_{n \geq 1} \) are log-convex. This completes the proof. \( \square \)

**Corollary 3.7.** The sequences \( \left\{ \sum_{k=0}^{n} \binom{n}{k} T_{2k} \right\}_{n \geq 0} \) and \( \left\{ \sum_{k=0}^{n} \binom{n}{k} R_{2k+1} \right\}_{n \geq 0} \) are log-concave.

**Proof.** Use Lemma 3.2. \( \square \)

**Corollary 3.8.** The sequences \( \left\{ \sum_{k=0}^{n} \binom{n}{k} T_{2k+1} \right\}_{n \geq 1} \) and \( \left\{ \sum_{k=0}^{n} \binom{n}{k} R_{2k} \right\}_{n \geq 1} \) are log-convex.

**Proof.** Use Lemma 3.3. \( \square \)

**Lemma 3.9.** Let \( a_n := \sum_{k=0}^{n} \binom{n}{k} P_{k+1} \), where \( \{P_n\}_{n \geq 0} \) is the Pell sequence. Then \( \{a_n\}_{n \geq 0} \) satisfy the following recurrence relations
\[
a_n = 3a_{n-1} + \sum_{k=0}^{n-2} a_k \quad \text{and} \quad a_n = 4a_{n-1} - 2a_{n-2}.
\]
Proof. Let be \( b_n := \sum_{k=0}^{n} \binom{n}{k} P_k \), where \( \{P_n\}_{n \geq -1} \) is the Pell sequence extended to \( P_{-1} = 1 \).

Using Pascal formula and the recurrence relation of Pell sequence together into the development \( \sum_{k=0}^{n} \binom{n}{k} P_{k+1} \) we get \( a_n = 3a_{n-1} + b_{n-1} \), then by \( b_n = b_{n-1} + a_{n-1} \). By iterated use of this relation with the precedent one, we get \( a_n = 3a_{n-1} + \sum_{k=0}^{n-2} a_k \) (with \( b_0 = 0 \) and \( a_0 = 1 \)), thus \( a_n = 4a_{n-1} - 2a_{n-2} \). □

**Theorem 3.10.** The sequences \( \left\{ nQ_n^{[1]} \right\}_{n \geq 0} \) and \( \left\{ \sum_{k=0}^{n} \binom{n}{k} Q_k^{[1]} \right\}_{n \geq 0} \) are log-concave and log-convex, respectively.

**Proof.** Let be \( S_n := n^2 \left( Q_n^{[1]} \right)^2 - (n^2 - 1)Q_{n-1}^{[1]}Q_{n+1}^{[1]} \) and \( K_n := \sum_{k=0}^{n} \binom{n}{k} Q_k^{[1]} \),

with the convention that \( K_{<0} = 0 \).

From (3.2), we have

\[
S_n = 4(n^2 - 1) (-1)^n + \left( Q_n^{[1]} \right)^2 \\
= 4 \left[ (n^2 - 1) (-1)^n + P_{n+1}^2 \right] \geq 4 \left[ (n^2 - 1) (-1)^n + (n + 1)^2 \right] > 0.
\]

Then \( \left\{ nQ_n^{[1]} \right\}_{n \geq 0} \) is log-concave.

Using Lemma 3.9, we can verify that

\[ K_n = 4K_{n-1} - 2K_{n-2} \quad (3.6) \]

The associated Binet-formula is

\[
K_n = \frac{(1 + \sqrt{2}) \alpha^n - (1 - \sqrt{2}) \beta^n}{\alpha - \beta}, \quad \text{with } \alpha, \beta = 2 \pm \sqrt{2},
\]

which provides

\[
K_n^2 - K_{n-1}K_{n+1} = -2^{n+1} < 0.
\]

Then \( \left\{ \sum_{k=0}^{n} \binom{n}{k} Q_k^{[1]} \right\}_{n \geq 0} \) is log-convex. □

**Remark 3.11.** The terms of the sequence \( \{K_n\}_n \) satisfy \( K_n = 2^{(n+2)/2}P_{n+1} \) if \( n \) is even, and \( K_n = 2^{(n-1)/2}Q_{n+1} \) if \( n \) is odd.

**Theorem 3.12.** The sequences \( \left\{ n!P_n^{[1]} \right\}_{n \geq 0} \) and \( \left\{ n!Q_n^{[2]} \right\}_{n \geq 0} \) are log-balanced.

**Proof.** By Theorem 3.4, in order to prove the log-balanced property of \( \left\{ n!P_n^{[1]} \right\}_{n \geq 0} \) and \( \left\{ n!Q_n^{[2]} \right\}_{n \geq 0} \) we only need to show that they are log-convex. It follows from the proof of Theorem 3.4 that

\[
\left( P_n^{[1]} \right)^2 - P_{n-1}^{[1]}P_{n+1}^{[1]} = \frac{1}{4} \left( 2(-1)^{n-1} + Q_{n+1} \right), \quad (3.7)
\]
and from the proof of Theorem 3.6 that

\[
\left(Q_n^2\right)^2 - Q_n^2 Q_{n+1}^2 = 2 (-1)^n + Q_{n+1}. \tag{3.8}
\]

Let

\[
M_n := n \left(P_n^1\right)^2 - (n+1) P_n^1 P_{n+1}^1,
\]

\[
B_n := n \left(Q_n^2\right)^2 - (n+1) Q_n^2 Q_{n+1}^2,
\]

from (3.3), (3.7) and (3.8), we get

\[
M_n = \frac{(n+1)}{4} \left( 2 (-1)^{n-1} + Q_{n+1} \right) - \frac{1}{4} (Q_n + 2 - 2)^2,
\]

\[
B_n = (n+1) \left( 2 (-1)^n + Q_{n+1} \right) - \frac{1}{4} (Q_{n+2} - 2)^2.
\]

Clearly \(B_n \leq 0\) for \(n = 0, 1, 2\). We have by induction that for \(n \geq 1\), \(Q_n \geq n + 1\). This gives

\[
B_n \leq (Q_{n+1} - 1) \left( 2 (-1)^n + Q_{n+1} \right) - \frac{1}{4} (2Q_{n+1} + Q_n - 2)^2 < 0.
\]

Also, \(M_n \leq 0\) for \(n = 2\) and for \(n \geq 3\), \(Q_n \geq n + 6\). This gives \(n + 1 \leq Q_{n+1} - 6\), and

\[
M_n \leq \frac{1}{4} \left[ (Q_{n+1} - 6) \left( 2 (-1)^{n-1} + Q_{n+1} \right) - (Q_n + 2 - 2)^2 \right]
\]

\[
= \frac{1}{4} \left[ (-2 + 2 (-1)^{n-1}) Q_{n+1} - 4 - 12 (-1)^{n-1} \right] < 0.
\]

Hence \(\{nP_n^1\}_{n \geq 0}\) and \(\{nQ_n^2\}_{n \geq 0}\) are log-convex. As the sequences \(\{P_n^1\}_{n \geq 0}\) and \(\{Q_n^2\}_{n \geq 0}\) are log-concave, so the sequences \(\{nP_n^1\}_{n \geq 0}\) and \(\{nQ_n^2\}_{n \geq 0}\) are log-balanced. \(\Box\)

**Theorem 3.13.** Define, for \(r \geq 1\), the polynomials

\[
P_{n,r}(q) := \sum_{k=0}^{n} P_k^r q^k \quad \text{and} \quad Q_{n,r}(q) := \sum_{k=0}^{n} Q_k^r q^k.
\]

The polynomials \(P_{n,r}(q) \ (r \geq 1)\) and \(Q_{n,r}(q) \ (r \geq 2)\) are q-log-concave.

**Proof.** When \(n \geq 1, r \geq 1\),

\[
P_{n,r}^2(q) - P_{n-1,r}(q) P_{n+1,r}(q)
\]

\[
= \left( \sum_{k=0}^{n} P_k^r q^k \right)^2 - \left( \sum_{k=0}^{n-1} P_k^r q^k \right) \left( \sum_{k=0}^{n+1} P_k^r q^k \right)
\]
\[
\begin{align*}
&= \left( \sum_{k=0}^{n} P_{k}^{[r]} q^{k} \right)^{2} - \left( \sum_{k=0}^{n} P_{k}^{[r]} q^{k} - P_{n}^{[r]} q^{n} \right) \left( \sum_{k=0}^{n} P_{k}^{[r]} q^{k} + P_{n+1}^{[r]} q^{n+1} \right) \\
&= \left( P_{n}^{[r]} q^{n} - P_{n+1}^{[r]} q^{n+1} \right) \sum_{k=0}^{n} P_{k}^{[r]} q^{k} + P_{n}^{[r]} P_{n+1}^{[r]} q^{2n+1} \\
&= \sum_{k=1}^{n} \left( P_{k}^{[r]} P_{n}^{[r]} - P_{k-1}^{[r]} P_{n+1}^{[r]} \right) q^{k+n}.
\end{align*}
\]

When \( n \geq 1, r \geq 2 \), through computation, we get

\[
Q_{n,r}^{2}(q) - Q_{n-1,r}(q)Q_{n+1,r}(q) = \sum_{k=1}^{n} \left( Q_{k}^{[r]} Q_{n}^{[r]} - Q_{k-1}^{[r]} Q_{n+1}^{[r]} \right) q^{k+n} + Q_{n}^{[r]} q^{n}.
\]

As \( \{ P_{n}^{[r]} \} \) and \( \{ Q_{n}^{[r]} \} \) \((r \geq 2)\) are log-concave, then the polynomials \( P_{n,r}(q) \) \((r \geq 1)\) and \( Q_{n,r}(q) \) \((r \geq 2)\) are \( q \)-log-concave. \( \square \)

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**References**


