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A symmetric algorithm for hyper-Fibonacci and hyper-Lucas numbers

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Abstract

In this work we study some combinatorial properties of hyper-Fibonacci, hyper-Lucas numbers and their generalizations by using a symmetric algorithm obtained by the recurrence relation $a_n^k = ua_n^{k-1} + va_{n-1}^k$. We point out that this algorithm can be applied to hyper-Fibonacci, hyper-Lucas and hyper-Horadam numbers.

Keywords: Hyper-Fibonacci numbers; hyper-Lucas numbers

MSC: 11B37; 11B39; 11B65

1. Introduction

The sequence of Fibonacci numbers is one of the most well known sequence, and it has many applications in mathematics, statistics, and physics. The Fibonacci numbers are defined by the second order linear recurrence relation: $F_{n+1} = F_n + F_{n-1}$ $(n \geq 1)$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas

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numbers are defined by $L_{n+1} = L_n + L_{n-1}$ $(n \geq 1)$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. There are some elementary identities for F_n and L_n. Two of them are $F_s + L_s = 2F_{s+1}$ and $F_s - L_s = 2F_{s-1}$. These will be generalized in section 2 (see Theorem 2.5).

The Fibonacci sequence can be generalized to the second order linear recurrence $W_n(a, b; p, q)$, or briefly W_n, defined by

$$W_{n+1} = pW_n + qW_{n-1},$$

where $n \geq 1$, $W_0 = a$ and $W_1 = b$. This sequence was introduced by Horadam [7]. Some of the special cases are:

i) The Fibonacci number $F_n = W_n(0, 1; 1, 1)$,

ii) The Lucas number $L_n = W_n(2, 1; 1, 1)$,

iii) The Pell number $P_n = W_n(0, 1; 2, 1)$.

In [4], Dil and Mező introduced the "hyper-Fibonacci" numbers $F_n^{(r)}$ and "hyper-Lucas" numbers $L_n^{(r)}$. These are defined as

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)} \quad \text{with} \quad F_n^{(0)} = F_n,\ F_0^{(r)} = 0,\ F_1^{(r)} = 1,$$

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)} \quad \text{with} \quad L_n^{(0)} = L_n,\ L_0^{(r)} = 2,\ L_1^{(r)} = 2r + 1,$$

where r is a positive integer, moreover F_n and L_n are the ordinary Fibonacci and Lucas numbers, respectively. The generating functions of hyper-Fibonacci and hyper-Lucas numbers are [4]:

$$\sum_{n=0}^{\infty} F_n^{(r)} t^n = \frac{t}{(1 - t - t^2)(1 - t)^r}, \quad \sum_{n=0}^{\infty} L_n^{(r)} t^n = \frac{2 - t}{(1 - t - t^2)(1 - t)^r}.$$

Also, the hyper-Fibonacci and hyper-Lucas numbers have the recurrence relations $F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}$ and $L_n^{(r)} = L_{n-1}^{(r)} + L_n^{(r-1)}$, respectively. The first few values of $F_n^{(r)}$ and $L_n^{(r)}$ are as follows [2]:

$$F_n^{(1)}: 0, 1, 2, 4, 7, 12, 20, 33, 54, \ldots, \qquad F_n^{(2)}: 0, 1, 3, 7, 14, 26, 46, 79, \ldots$$

$$L_n^{(1)}: 2, 3, 6, 10, 17, 28, 46, 75, \ldots, \qquad L_n^{(2)}: 2, 5, 11, 21, 38, 66, 112, \ldots.$$

Now we introduce the hyper-Horadam numbers $W_n^{(r)}$ defined by

$$W_n^{(r)} = W_{n-1}^{(r)} + W_n^{(r-1)} \quad \text{with} \quad W_n^{(0)} = W_n,\ W_0^{(n)} = W_0 = a$$

where W_n is the nth Horadam number. Some of the special cases of hyper-Horadam number $W_n^{(r)}$ are as follows:

i) If $W_n^{(0)} = F_n = W_n(0,1;1,1)$ and $W_0^{(n)} = W_0 = F_0 = 0$, then $W_n^{(r)}$ is the hyper-Fibonacci number, that is, $W_n^{(r)} = F_n^{(r)}$.

ii) If $W_n^{(0)} = L_n = W_n(2,1;1,1)$ and $W_0^{(n)} = W_0 = L_0 = 2$, then $W_n^{(r)}$ is the hyper-Lucas number, that is, $W_n^{(r)} = L_n^{(r)}$.

iii) If $W_n^{(0)} = P_n = W_n(0,1;2,1)$ and $W_0^{(n)} = W_0 = P_0 = 0$, then $W_n^{(r)}$ is the hyper-Pell number, that is, $W_n^{(r)} = P_n^{(r)}$.

The paper is organized as follows: In Section 2 we give some combinatorial properties of the hyper-Fibonacci and hyper-Lucas numbers by using a symmetric algorithm. In Section 3 we generalize the symmetric algorithm introduced in section 2 and, in addition, we generalize the hyper-Horadam numbers as well.

2. A symmetric algorithm

The Euler–Seidel algorithm and its analogues are useful in the study of recurrence relations of some numbers and polynomials [2, 3, 4, 5]. Let (a_n) and (a^n) be two real initial sequences. Then the infinite matrix, which is called symmetric infinite matrix in [4], with entries a_n^k corresponding to these sequences is determined recursively by the formulas

$$\begin{aligned} a_n^0 &= a_n, \quad a_0^n = a^n \quad (n \geq 0)\,, \\ a_n^k &= a_n^{k-1} + a_{n-1}^k \quad (n \geq 1, k \geq 1)\,, \end{aligned}$$

i.e., in matrix form

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n^{k-1} & \cdot & \cdot & \cdot \\ & & & & \downarrow & & & \\ \cdot & \cdot & \cdot & a_{n-1}^k \rightarrow & a_n^k & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The entries a_n^k (where k is the row index, n is the column index) have the following symmetric relation [4]:

$$a_n^k = \sum_{i=1}^{k} \binom{n+k-i-1}{n-1} a_0^i + \sum_{s=1}^{n} \binom{n+k-s-1}{k-1} a_s^0. \tag{2.1}$$

Dil and Mező [4], by using the relation (2.1), obtained an explicit formula for hyperharmonic numbers, general generating functions of the Fibonacci and Lucas numbers. By using relation (2.1) and the following well known identity [6, p. 160]

$$\sum_{t=a}^{c} \binom{t}{a} = \binom{c+1}{a+1}, \tag{2.2}$$

we have some new findings contained in the following theorems.

Theorem 2.1. *If $n \geq 1$, $r \geq 1$ and $m \geq 0$, then*

$$F_n^{(m+r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} F_s^{(m)}.$$

Proof. Let $a_n^0 = F_{n+1}^{(m)}$ and $a_0^n = F_1^{(m+n)} = 1$ be given for $n \geq 1$. If we calculate the elements of the corresponding infinite matrix by using the recursive formula (2.1), it turns out that they equal to

$$\begin{pmatrix} F_1^{(m)} & F_2^{(m)} & F_3^{(m)} & F_4^{(m)} & \cdots \\ F_1^{(m+1)} & F_2^{(m+1)} & F_3^{(m+1)} & F_4^{(m+1)} & \cdots \\ F_1^{(m+2)} & F_2^{(m+2)} & F_3^{(m+2)} & F_4^{(m+2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.3}$$

From relation (2.1) it follows that

$$\begin{aligned} a_{n+1}^{r+1} &= \sum_{i=1}^{r+1} \binom{n+r-i+1}{n} + \sum_{s=1}^{n+1} \binom{n+r-s+1}{r} F_{s+1}^{(m)} \\ &= \sum_{i=0}^{r} \binom{n+r-i}{n} + \sum_{s=0}^{n} \binom{n+r-s}{r} F_{s+2}^{(m)} \\ &= \sum_{k=0}^{r} \binom{n+k}{n} + \sum_{b=0}^{n} \binom{r+b}{r} F_{n-b+2}^{(m)}, \end{aligned}$$

where $k = r - i$ and $b = n - s$. From (2.2), we have

$$a_{n+1}^{r+1} = \binom{n+r+1}{n+1} + \sum_{b=0}^{n} \binom{r+b}{r} F_{n-b+2}^{(m)} = \sum_{b=0}^{n+1} \binom{r+b}{r} F_{n-b+2}^{(m)}.$$

Then the matrix (2.3) yields

$$a_{n-1}^{r} = F_n^{(m+r)} = \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} F_{n-b}^{(m)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} F_s^{(m)}.$$

Thus the proof is completed. □

We then can easily deduce an expression for the hyper-Fibonacci numbers which contains the ordinary Fibonacci numbers.

Corollary 2.2. *If $n \geq 1$ and $r \geq 1$, then*

$$F_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} F_s$$

where F_s is the sth Fibonacci number.

The corresponding theorem for the hyper-Lucas numbers is as follows.

Theorem 2.3. *If $n \geq 1$, $r \geq 1$ and $m \geq 0$, then*

$$L_n^{(m+r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} L_s^{(m)}.$$

Proof. Let $a_n^0 = L_n^{(m)}$ and $a_0^n = L_0^{(m+n)} = 2$ be given for $n \geq 1$. This special case gives the following infinite matrix:

$$\begin{pmatrix} L_0^{(m)} & L_1^{(m)} & L_2^{(m)} & L_3^{(m)} & \cdots \\ L_0^{(m+1)} & L_1^{(m+1)} & L_2^{(m+1)} & L_3^{(m+1)} & \cdots \\ L_0^{(m+2)} & L_1^{(m+2)} & L_2^{(m+2)} & L_3^{(m+2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.4}$$

From the relation (2.1) we get that

$$\begin{aligned} a_n^r &= \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} 2 + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} L_s^{(m)} \\ &= 2\sum_{i=0}^{r-1} \binom{n+r-i-2}{n-1} + \sum_{s=0}^{n-1} \binom{n+r-s-2}{r-1} L_{s+1}^{(m)} \\ &= 2\sum_{k=0}^{r-1} \binom{n+k-1}{n-1} + \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} L_{n-b}^{(m)}, \end{aligned}$$

where $k = r-i-1$ and $b = n-s-1$. From (2.2), we have

$$a_n^r = 2\binom{n+r-1}{n} + \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} L_{n-b}^{(m)} = \sum_{b=0}^{n} \binom{r+b-1}{r-1} L_{n-b}^{(m)}.$$

Then the matrix (2.4) yields

$$a_n^r = L_n^{(m+r)} = \sum_{b=0}^{n} \binom{r+b-1}{r-1} L_{n-b}^{(m)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} L_s^{(m)},$$

this completes the proof. $\square$

Corollary 2.4. *If $n \geq 1$ and $r \geq 1$, then*

$$L_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} L_s,$$

where L_n is the nth Lucas number.

Theorem 2.5. *If $n \geq 1$ and $r \geq 1$, then*

i) $F_n^{(r)} + L_n^{(r)} = 2F_{n+1}^{(r)}$,

ii) $F_n^{(r)} - L_n^{(r)} = 2F_{n+1}^{(r-1)}$.

Proof. From Corollaries 2.2 and 2.4, we have

$$\begin{aligned} F_n^{(r)} + L_n^{(r)} &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} (F_s + L_s) \\ &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} (2F_{s+1}) = 2F_{n+1}^{(r)} \end{aligned}$$

and

$$\begin{aligned} F_n^{(r)} - L_n^{(r)} &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} (F_s - L_s) \\ &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} (2F_{s-1}) = 2F_{n+1}^{(r-1)}. \end{aligned}$$ □

Theorem 2.6. *If $n \geq 1$ and $r \geq 1$, then*

$$\sum_{s=0}^{r} F_n^{(s)} = F_{n+1}^{(r)} - F_{n-1}.$$

Proof. From Corollary 2.2, we have

$$\sum_{s=1}^{r} F_n^{(s)} = \sum_{s=1}^{r} \left(\sum_{t=0}^{n} \binom{n+s-t-1}{s-1} F_t \right) = \sum_{t=0}^{n} \left(F_t \sum_{s=1}^{r} \binom{n+s-t-1}{s-1} \right).$$

From (2.2), we obtain

$$\sum_{s=1}^{r} F_n^{(s)} = \sum_{t=0}^{n} \binom{n+r-t}{r-1} F_t = \sum_{t=0}^{n+1} \binom{n+r-t}{r-1} F_t - F_{n+1} = F_{n+1}^{(r)} - F_{n+1}.$$

Thus

$$\sum_{s=0}^{r} F_n^{(s)} = F_{n+1}^{(r)} - F_{n-1}.$$ □

Theorem 2.7. *If $n \geq 1$ and $r \geq 1$, then*

$$\sum_{s=0}^{r} L_n^{(s)} = L_{n+1}^{(r)} - L_{n-1}.$$

Proof. The proof is similar to the proof of Theorem 2.6. □

3. A generalized symmetric algorithm

In this section we generalize the algorithm for determining a_n^k in the symmetric infinite matrix. To this end we fix two arbitrary, nonzero real numbers u and v. Then our new algorithm reads as

$$\begin{aligned} a_n^0 &= a_n, \quad a_0^n = a^n \quad (n \geq 0), \\ a_n^k &= u a_n^{k-1} + v a_{n-1}^k \quad (n \geq 1, k \geq 1). \end{aligned}$$

That is, the symmetric infinite matrix now can be constructed in the following way:

$$\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & u a_n^{k-1} & \cdot & \cdot & \cdot \\
 & & & & \downarrow & & & \\
\cdot & \cdot & \cdot & v a_{n-1}^k \rightarrow & a_n^k & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}.$$

It can easily be seen that (2.1) generalizes to

$$a_n^k = \sum_{i=1}^{k} v^n u^{k-i} \binom{n+k-i-1}{n-1} a_0^i + \sum_{s=1}^{n} v^{n-s} u^k \binom{n+k-s-1}{k-1} a_s^0. \tag{3.1}$$

As an application, we can generalize the hyper-Horadam number as

$$W_n^{(r)}(u,v) = u W_n^{(r-1)} + v W_{n-1}^{(r)}$$

where u and v are two nonzero real parameters and the initial conditions are $W_n^{(0)}(u,v) = W_n(a,b;p,q) = W_n$ and $W_0^{(n)}(u,v) = W_0(a,b;p,q) = a$. Some special cases of the hyper-Horadam numbers $W_n^{(r)}(u,v)$ are:

i) If $W_n^{(0)}(u,v) = F_n^{(0)}(u,v) = F_n$ and $W_0^{(n)}(u,v) = F_0^{(n)}(u,v) = 0$, then we have the generalized hyper-Fibonacci numbers defined as

$$F_n^{(r)}(u,v) = u F_n^{(r-1)} + v F_{n-1}^{(r)},$$

ii) If $W_n^{(0)}(u,v) = L_n^{(0)}(u,v) = L_n$ and $W_0^{(n)}(u,v) = L_0^{(n)}(u,v) = 2$, we have the generalized hyper-Lucas number defined as

$$L_n^{(r)}(u,v) = uL_n^{(r-1)} + vL_{n-1}^{(r)},$$

iii) If $W_n^{(0)}(u,v) = P_n^{(0)}(u,v) = P_n$ and $W_0^{(n)}(u,v) = P_0^{(n)}(u,v) = 0$, we have the generalized hyper-Pell number defined as

$$P_n^{(r)}(u,v) = uP_n^{(r-1)} + vP_{n-1}^{(r)}.$$

By using (3.1), Theorem 2.1 generalizes to the following Theorem.

Theorem 3.1. *If* $n \geq 1$, $r \geq 1$ *and* $m \geq 0$, *then*

$$\begin{aligned} W_n^{(m+r)}(u,v) = a\left(\frac{v}{1-u}\right)^n &\left[1 - rB_u(r,n)\binom{n+r-1}{n-1}\right] \\ &+ u^r \sum_{s=1}^{n} v^{n-s}\binom{n+r-s-1}{r-1} W_s^{(m)}(u,v). \end{aligned}$$

where $B_u(r,n)$ *is the incomplete beta function [1].*

Proof. The incomplete beta function $B_u(r,n)$ appears when we would like to evaluate the sum

$$\sum_{k=0}^{r-1}\binom{n+k-1}{k}u^k.$$

This sum equals to

$$\frac{1}{(1-u)^n}\left[1 - rB_u(r,n)\binom{n+r-1}{n-1}\right].$$

This is the most compact form we could find. The other parts of the proof are the same as the proof of Theorem 2.1, if we use relation (3.1) and assume that $a_n^0 = W_n^{(m)}(u,v)$ and $a_0^n = W_0^{(m+n)} = a$. □

Corollary 3.2. *If* $n \geq 1$ *and* $r \geq 1$, *then*

$$\begin{aligned} W_n^{(r)}(u,v) = a\left(\frac{v}{1-u}\right)^n &\left[1 - rB_u(r,n)\binom{n+r-1}{n-1}\right] \\ &+ u^r \sum_{s=1}^{n} v^{n-s}\binom{n+r-s-1}{r-1} W_s. \end{aligned}$$

From these results we have some particular results for the hyper-Fibonacci, hyper-Lucas, hyper-Pell numbers and their generalizations such as

$$F_n^{(r)}(u,v) = u^r \sum_{s=1}^{n} v^{n-s}\binom{n+r-s-1}{r-1} F_s,$$

$$L_n^{(r)}(u,v) = 2\left(\frac{v}{1-u}\right)^n \left[1 - rB_u(r,n)\binom{n+r-1}{n-1}\right] + u^r \sum_{s=1}^{n} v^{n-s} \binom{n+r-s-1}{r-1} L_s,$$

$$P_n^{(r)}(u,v) = u^r \sum_{s=1}^{n} v^{n-s} \binom{n+r-s-1}{r-1} P_s,$$

$$P_n^{(r)} = \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} P_s,$$

where F_s, L_s and P_s is the s^{th} Fibonacci, Lucas and Pell number, respectively.

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