

A note on Golomb's method and the continued fraction method for Egyptian fractions

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Abstract

In this note we prove directly that Golomb's method and the continued fraction method are essentially the same, in the sense that they give the same Egyptian fraction expansions of positive rational numbers. Furthermore, we show their connection with the Farey sequence method.

Keywords: Egyptian fractions, Golomb's method, continued fraction method, Farey sequence method

MSC: 11D68

1. Introduction

It is well-known that every positive rational number can be expressed as a sum of distinct unit fractions (reciprocals of natural numbers). Ancient Egyptians already used such representations of rational numbers, for this reason we call a sum of distinct unit fractions an Egyptian fraction. We note that sometimes unit fractions themselves are called Egyptian fractions.

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Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, it is enough to give an algorithm for finding an Egyptian fraction expansion of rational numbers between 0 and 1. There are several methods to do this, many of them are summarized in [6].

Probably the oldest such algorithm is the greedy method, which subtracts always the largest possible unit fraction from the current rational number. Sometimes it is referred to as Fibonacci method or Fibonacci-Sylvester method, because it was first described by Leonardo Pisano, better known as Fibonacci [5], and later it was rediscovered by J. J. Sylvester [10].

The splitting method is based on successive application of the identity $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$. It was shown by L. Beeckmans [2] that this algorithm terminates after a finite number of steps, however it was stated previously without proof by P. J. Campbell [4].

We still mention here by name the method of S. W. Golomb [7] and the continued fraction method due to M. N. Bleicher [3], as they are the main subject of this paper. We should remark that there is a confusion in the literature, in [3] a further variant of the latter method is presented, and the author calls the modified version the continued fraction method. However, in [6] the original algorithm is called the continued fraction method, as will be in this note.

Writing her BSc thesis, the first author observed that surprisingly Golomb's method and the continued fraction method always give the same Egyptian fraction expansions. Before proving this statement directly, we discuss these methods briefly. After that, we present their connection with the Farey sequence method, and a possible usage of them in teaching basic number theory.

2. Golomb's method and the continued fraction method

Golomb's method Let $a < b$ be positive integers with $\gcd(a, b) = 1$, and consider the rational number $0 < \frac{a}{b} < 1$. If $a = 1$, then it is a unit fraction. Otherwise, since a and b are coprime, there exist a multiplicative inverse $0 < a' < b$ of a modulo b and a natural number r such that $aa' = br + 1$. Then

$$\frac{a}{b} = \frac{r}{a'} + \frac{1}{a'b}.$$

Now it follows from $aa' > br > ar$ and $aa' = br + 1$, that $0 < \frac{r}{a'} < 1$ and $\gcd(r, a') = 1$, and we can apply the above procedure for $\frac{r}{a'}$.

On the other hand, we have $aa' > br > a'r$, hence $r < a$, which guarantees the finiteness of the method. The algorithm is also correct, it gives distinct unit fractions in the Egyptian fraction expansion, which can be proved by induction showing that the unit fractions have denominators at most $b(b-1)$.

Continued fraction method Let $0 < \frac{a}{b} < 1$ be again a rational number with coprime natural numbers a and b . Suppose that the finite simple continued fraction

expansion of $\frac{a}{b}$ is $\langle c_0, c_1, \dots, c_n \rangle$, where $c_0 = 0$ and c_1, \dots, c_n are positive integers. As it is well-known, $\frac{a}{b}$ can be represented by a finite simple continued fraction in exactly two ways, but it is indifferent which of them is used.

As usual, define two sequences $(a_k)_{k=-2}^n$ and $(b_k)_{k=-2}^n$ recursively:

$$a_{-2} = 0, \quad a_{-1} = 1, \quad a_k = c_k a_{k-1} + a_{k-2} \quad (k = 0, 1, \dots, n)$$

$$b_{-2} = 1, \quad b_{-1} = 0, \quad b_k = c_k b_{k-1} + b_{k-2} \quad (k = 0, 1, \dots, n)$$

Then $a_n = a$ and $b_n = b$.

Primary and secondary convergents satisfy equations

$$\frac{a_k}{b_k} - \frac{a_{k-1}}{b_{k-1}} = \frac{(-1)^{k+1}}{b_{k-1}b_k}$$

for $1 \leq k \leq n$, and

$$\frac{a_{k-2} + la_{k-1}}{b_{k-2} + lb_{k-1}} - \frac{a_{k-2} + (l-1)a_{k-1}}{b_{k-2} + (l-1)b_{k-1}} = \frac{(-1)^k}{(b_{k-2} + (l-1)b_{k-1})(b_{k-2} + lb_{k-1})}$$

for $2 \leq k \leq n$, $1 \leq l \leq c_k$. Details about these and other properties of continued fractions can be found in [8, 9].

Using the above identities, we can describe the continued fraction method. If n is odd, then

$$\frac{a_n}{b_n} = \frac{a_{n-1}}{b_{n-1}} + \frac{1}{b_{n-1}b_n}, \tag{2.1}$$

and apply the method for $\frac{a_{n-1}}{b_{n-1}}$.

If n is even, then

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{a_{n-2} + c_n a_{n-1}}{b_{n-2} + c_n b_{n-1}} = \frac{a_{n-2}}{b_{n-2}} + \sum_{l=1}^{c_n} \left(\frac{a_{n-2} + la_{n-1}}{b_{n-2} + lb_{n-1}} - \frac{a_{n-2} + (l-1)a_{n-1}}{b_{n-2} + (l-1)b_{n-1}} \right) \\ &= \frac{a_{n-2}}{b_{n-2}} + \sum_{l=1}^{c_n} \frac{1}{(b_{n-2} + (l-1)b_{n-1})(b_{n-2} + lb_{n-1})}, \end{aligned} \tag{2.2}$$

and apply the method for $\frac{a_{n-2}}{b_{n-2}}$.

We note that the first case (odd subscript) is used at most once, while the correctness of the algorithm can be proved by induction on n showing that the denominators of the unit fractions do not exceed $b_n(b_n - 1)$.

Proof that these methods give the same Egyptian fraction expansions

If n is odd, then it follows from $a_n b_{n-1} - a_{n-1} b_n = 1$ that $0 < b_{n-1} < b_n$ is the multiplicative inverse of a_n modulo b_n , hence one step of Golomb's method gives (2.1), exactly the same sum as the continued fraction method.

If n is even, then $(a_{n-2} + la_{n-1})(b_{n-2} + (l-1)b_{n-1}) - (a_{n-2} + (l-1)a_{n-1})(b_{n-2} + lb_{n-1}) = 1$ implies that $0 < b_{n-2} + (l-1)b_{n-1} < b_{n-2} + lb_{n-1}$ is the multiplicative

inverse of $a_{n-2} + la_{n-1}$ modulo $b_{n-2} + lb_{n-1}$, hence applying Golomb's method for $\frac{a_{n-2} + la_{n-1}}{b_{n-2} + lb_{n-1}}$, it gives

$$\frac{a_{n-2} + la_{n-1}}{b_{n-2} + lb_{n-1}} = \frac{a_{n-2} + (l-1)a_{n-1}}{b_{n-2} + (l-1)b_{n-1}} + \frac{1}{(b_{n-2} + (l-1)b_{n-1})(b_{n-2} + lb_{n-1})}$$

($l = c_n, c_n - 1, \dots, 1$). It shows that after c_n steps of Golomb's method, we get (2.2) from $\frac{a_n}{b_n}$.

3. Example

As an example, we calculate the Egyptian fraction expansions of the rational number $\frac{47}{64}$ both by Golomb's method and by the continued fraction method.

Golomb's method Golomb's method gives the result through the following steps:

The multiplicative inverse of 47 modulo 64 is 15, hence $\frac{47}{64} = \frac{11}{15} + \frac{1}{960}$.

The multiplicative inverse of 11 modulo 15 is 11, hence $\frac{11}{15} = \frac{8}{11} + \frac{1}{165}$.

The multiplicative inverse of 8 modulo 11 is 7, hence $\frac{8}{11} = \frac{5}{7} + \frac{1}{77}$.

The multiplicative inverse of 5 modulo 7 is 3, hence $\frac{5}{7} = \frac{2}{3} + \frac{1}{21}$.

Finally, the multiplicative inverse of 2 modulo 3 is 2, hence $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.

Summarizing these equations, it follows that the Egyptian fraction expansion by Golomb's method is

$$\frac{47}{64} = \frac{1}{2} + \frac{1}{6} + \frac{1}{21} + \frac{1}{77} + \frac{1}{165} + \frac{1}{960}.$$

Continued fraction method The Euclidean algorithm gives the finite simple continued fraction expansion $\frac{47}{64} = \langle 0, 1, 2, 1, 3, 4 \rangle$ and the sequences $(a_k)_{k=0}^5 = (0, 1, 2, 3, 11, 47)$, $(b_k)_{k=0}^5 = (1, 1, 3, 4, 15, 64)$. Then the continued fraction method works as follows:

First, by application of the odd subscript case we obtain $\frac{47}{64} = \frac{11}{15} + \frac{1}{960}$.

Thereafter we apply the even subscript case twice to get $\frac{11}{15} = \frac{2}{3} + \frac{1}{21} + \frac{1}{77} + \frac{1}{165}$ and $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.

Consequently, the Egyptian fraction expansion by this method is

$$\frac{47}{64} = \frac{1}{2} + \frac{1}{6} + \frac{1}{21} + \frac{1}{77} + \frac{1}{165} + \frac{1}{960},$$

which is the very same as above.

4. Connection with the Farey sequence method

Our observation could be verified also through Farey sequences. Denote by \mathcal{F}_n the Farey sequence of order n , that is the list of all reduced rational numbers in $[0, 1]$, having denominators less than or equal to n , in increasing order. The main properties of Farey sequences can be found in [8, 9].

In [1], see also [3], the Farey sequence method is presented to obtain an Egyptian fraction expansion of a positive rational number. Let $0 < \frac{a}{b} < 1$ be a rational number, where a and b are positive integers with $\gcd(a, b) = 1$. If $\frac{c}{d}$ is the preceding fraction in \mathcal{F}_b , then

$$\frac{a}{b} = \frac{c}{d} + \frac{1}{db},$$

where $d < b$, and we can continue the method on $\frac{c}{d}$.

We have to notice that in practice this form of the Farey sequence method is only an algorithm in principle, because it says nothing about how to find the preceding fraction in \mathcal{F}_b .

Then it is straightforward that Golomb's method coincides with the Farey sequence method, since $ad = bc + 1$ and $0 < d < b$ is the multiplicative inverse of a modulo b .

On the other hand, the Farey sequence method gives the same result as the continued fraction method, which can be deduced from the following fact: For odd n , the preceding fraction of $\frac{a}{b} = \langle c_0, c_1, \dots, c_n \rangle$ ($c_0 = 0$) in \mathcal{F}_b is $\langle c_0, c_1, \dots, c_{n-1} \rangle$. While for even n , the preceding fraction is $\langle c_0, c_1, \dots, c_{n-2} \rangle$ if $c_n = 1$, furthermore $\langle c_0, c_1, \dots, c_{n-1}, c_n - 1 \rangle$ if $c_n \geq 2$. Thus the preceding fraction is a primary or secondary convergent, which is already mentioned in a half sentence in [3].

5. Teaching possibilities

Elementary number theory textbooks, lecture notes (see e.g. [8, 9]) and undergraduate courses often deal with Farey sequences and continued fractions. Our experiences show that these topics are rather popular among university students. Because of their interesting properties, they are also suitable to be the subject of popular science lectures or mathematics study circles for advanced secondary school students. At a higher level, in the theory of diophantine approximation, both Farey sequences and continued fractions are used to give alternative proofs of Hurwitz's theorem. Nevertheless, these topics are always handled in separate chapters, we can hardly find any sources about their connection.

Thanks to the simplicity of the necessary notions and the historical background, we think that Egyptian fractions also give a rewarding topic to popularize mathematics. On the other hand, at university level, as the lecturer's material or as the subject of students' project work, it can be an unordinary base to introduce both Farey sequences and continued fractions, as well as their properties. And it allows us not only to study them separately, but one can find out their close connection, as we have done above.

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