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Tiling approach to obtain identities for generalized Fibonacci and Lucas numbers

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Abstract

In Proofs that Really Count [2], Benjamin and Quinn have used “square and domino tiling” interpretation to provide tiling proofs of many Fibonacci and Lucas formulas. We explore this approach in order to provide tiling proofs of some generalized Fibonacci and Lucas identities.

Keywords: Generalized Fibonacci and Lucas numbers; Tiling proofs.

MSC: 05A19, 11B39, 11B37.

1. Introduction

Let U_n and V_n denote the generalized Fibonacci and Lucas numbers defined, respectively, by

$$U_n = aU_{n-1} + bU_{n-2} \quad (n \geq 2)\,, \tag{1.1}$$

with the initial conditions $U_0 = 1,\ U_1 = a$, and by

$$V_n = aV_{n-1} + bV_{n-2} \quad (n \geq 2)\,, \tag{1.2}$$

with the initial conditions $V_0 = 2,\ V_1 = a$, where a and b are non-negative integers.

In [1], the generalized Fibonacci number U_n is interpreted as the number of ways to tile a $1 \times n$ board with cells labeled $1, 2, \ldots, n$ using colored squares (1×1 tiles) and dominoes (1×2 tiles), where there are a different colors for squares and b different colors for dominoes. In fact, there is one way to tile a empty board ($U_0 = 1$), since a board of length one can be covered by one colored square ($U_1 = a$), so this satisfy the initial Fibonacci conditions. Now for $n \geq 2$, if the first tile is a square, then there are a possibilities to color the square and U_{n-1} ways to tile $1 \times (n-1)$ board. If the first tile is a domino, then there are b choices for the domino and U_{n-2} ways to tile $1 \times (n-2)$ board. This gives the relation (1.1).

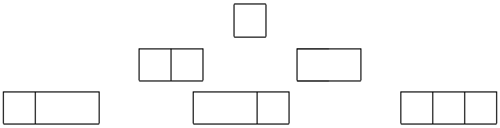

Figure 1: Tilings of length 1, 2 and 3 using squares and dominoes

Similarly, the generalized Lucas numbers count the number of ways to tile a circular $1 \times n$ board with squares and dominoes (termed $1 \times n$ bracelet). We call a $1 \times n$ bracelet in-phase if there is no domino occupying cells n and 1, and out-of phase if there is a domino occupying cells n and 1. The empty bracelet can be either in-phase or out-of phase, then $V_0 = 2$. Since a 1×1 bracelet can be tiled only by a square $V_1 = a$. For $n \geq 2$, a $1 \times n$ bracelet can be obtained from a $1 \times (n-1)$ bracelet by adding a square to the left of the first tile or from a $1 \times (n-2)$ bracelet by adding a domino to the left of the first tile. Then for $n \geq 2$ we have the relation (1.2).

Benjamin and Quinn, have used this approach to provide tiling proofs of many Fibonacci relations. Our goal is to use this interpretation to provide tiling proofs for the following two identities:

$$U_n - \sum_{k=0}^{m-1} \binom{n-k}{k} b^k a^{n-2k} = b^m \sum_{0 \leq j \leq k \leq n-2m} U_{n-k-2m} \frac{a^k}{k!} \begin{bmatrix} k \\ j \end{bmatrix} m^j, \tag{1.3}$$

where $\begin{bmatrix} k \\ j \end{bmatrix}$ are the Stirling numbers of the first kind.

$$2U_{n+m-1} = V_m U_{n-1} + V_n U_{m-1}. \tag{1.4}$$

To prove these identities we need the following Lemma.

Lemma 1.1 ([2])**.** *The number of* $1 \times n$ *tilings using exactly* k *colored dominoes is*

$$\binom{n-k}{k} b^k a^{n-2k}, \quad (k = 0, 1, \ldots, \lfloor n/2 \rfloor). \tag{1.5}$$

2. Combinatorial identities

Our first identity generalizes identity (1) given in [3]. It counts the number of ways to tile a $1 \times (n+2)$ board with at least one colored domino

$$U_{n+2} - a^{n+2} = b \sum_{k=0}^{n} U_k a^{n-k} \quad (n \geq 0). \tag{2.1}$$

Note that for $a = b = 1$, relation (2.1) gives the well known Lucas identity

$$f_{n+2} - 1 = \sum_{k=0}^{n} f_k,$$

where f_n is the shifted Fibonacci number defined recurrently by

$$f_n = f_{n-1} + f_{n-2} \quad (n \geq 2)\,, \tag{2.2}$$

with the initials $f_0 = f_1 = 1$.

The following identity counts the number of $1 \times n$ tilings with at least m colored dominoes.

Identity 1. *For $m \geq 1$ and $n \geq 2m$, we have*

$$U_n - \sum_{k=0}^{m-1} \binom{n-k}{k} b^k a^{n-2k} = b^m \sum_{0 \leq j \leq k \leq n-2m} U_{n-k-2m} \frac{a^k}{k!} {k \brack j} m^j.$$

Proof. The left hand side counts the number of tilings of length n excluding the tilings with exactly $0, 1, \ldots, m-1$ dominoes. Now, let $k+1$, $k+2$ $(0 \leq k \leq n-2m)$ be the position of the *m-th* (from the right to the left) domino (see figure 2), then there are U_k ways to tile the first k cells, b ways to color the domino at position $k+1$, $k+2$, and there are $\binom{n-m-k-1}{m-1} b^{m-1} a^{n-2m-k}$ ways to tiles cells from $k+3$ to n with exactly $m-1$ dominoes. Hence there are $\binom{n-m-k-1}{m-1} U_k b^m a^{n-2m-k}$ possible ways to tile an $1 \times n$ board with the *m-th* domino at the positions $k+1$, $k+2$. Summing over all $0 \leq k \leq n-2m$, we obtain

$$b^m \sum_{k=0}^{n-2m} U_k a^{n-k-2m} \binom{n-k-m-1}{m-1} = b^m \sum_{k=0}^{n-2m} U_{n-k-2m} a^k \binom{k+m-1}{m-1}. \tag{2.3}$$

Now, we express the binomial coefficient in terms of Stirling numbers of the first kind: $\binom{k+m-1}{m-1} = \frac{(m+k-1)\cdots(m+1)m}{k!} = \sum_{j=0}^{k} {k \brack j} \frac{m^j}{k!}$, this gives the right hand side of the identity. ☐

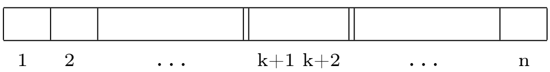

Figure 2: A $1 \times n$ tiling with the m-th domino at cells $k+1$, $k+2$

Remark 2.1. We can consider the intermediate identity (2.3), as given in the proof without using Stirling numbers.

Corollary 2.2. *Let $a = b = 1$, using relation (2.3) we have for $m = 1, 2, 3$ respectively*

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad \textit{(E. Lucas, 1878)}$$

$$\sum_{k=0}^{n} k f_k = n f_{n+2} - f_{n+3} + 3 \quad \textit{(Brother. U. Alfred, 1965)}$$

$$\sum_{k=0}^{n} k^2 f_k = (n^2+2) f_{n+2} - (2n-3) f_{n+3} - 13 \quad \textit{(Brother. U. Alfred, 1965)}$$

Now, we give tiling proof for the relation (1.4), for an algebraic proof, see for instance (V16a, pp 26, [5]).

Identity 2. *For $m \geq 1$ and $n \geq 1$, we have*

$$2U_{n+m-1} = V_m U_{n-1} + V_n U_{m-1}.$$

Proof. The left hand side counts the number of ways to tile a $1 \times (n+m-1)$ board. For the right hand side we suppose that we have a $1 \times (n+m-1)$ tiling. There is two cases:

Case 1. The $1 \times (n+m-1)$ tiling is breakable at m-th cell (there is not a domino covering positions m and $m+1$), then the $1 \times (n+m-1)$ tiling can be split into a $1 \times m$ tiling and a $1 \times (n-1)$ tiling. Now we attach the right side of the m-th cell to the left side of the first cell of the $1 \times m$ tiling, thus we form a in-phase $1 \times m$ bracelet. We denote the number of ways to tile an in-phase m-bracelet by V'_m.

Case 2. The $1 \times (n+m-1)$ tiling is not breakable at the m-th cell (there is a domino covering positions m and $m+1$), then it is breakable at $(m-1)$-th cell. In this case, we create a $1 \times (m-1)$ tiling and an out-of phase $1 \times n$ bracelet. We denote the number of ways to tile an out-phase $1 \times n$ bracelet by V''_n.

Now, we apply the same approach for the n-th cell, by considering either $1 \times (n+m-1)$ tiling is breakable at n-th cell or not. So, we obtain

$$\begin{aligned} 2U_{n+m-1} &= V'_m U_{n-1} + U_{m-1} V''_n + V'_n U_{m-1} + U_{n-1} V''_m \\ &= U_{n-1}(V'_m + V''_m) + U_{m-1}(V'_n + V''_n). \end{aligned}$$

We conclude by the fact that $V'_m + V''_m = V_m$ and $V'_n + V''_n = V_n$. □

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