Algebraic relations with the infinite products generated by Fibonacci numbers

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Abstract

In this paper, we establish explicit algebraic relations among infinite products including Fibonacci and Lucas numbers with subscripts in geometric progressions. The algebraic relations given in this paper are obtained by using general criteria for the algebraic dependency of such infinite products.

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MSC: 11J81, 11J85.

1. Introduction

Let $\alpha$ and $\beta$ be real algebraic numbers with $|\alpha| > 1$ and $\alpha\beta = -1$. We define

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \quad (n \geq 0). \quad (1.1)$$

If $\alpha = (1 + \sqrt{5})/2$, we have $U_n = F_n$ and $V_n = L_n$ ($n \geq 0$), where the sequences $\{F_n\}_{n \geq 0}$ and $\{L_n\}_{n \geq 0}$ are the Fibonacci numbers and the Lucas numbers defined,
respectively, by \( F_{n+2} = F_{n+1} + F_n \ (n \geq 0) \), \( F_0 = 0 \), \( F_1 = 1 \) and by \( L_{n+2} = L_{n+1} + L_n \ (n \geq 0) \), \( L_0 = 2 \), \( L_1 = 1 \).

Throughout this paper, we adopt the following notation. Let \( d \geq 2 \) be a fixed integer and \( \zeta_m = e^{2\pi i / m} \) a primitive \( m \)-th root of unity. For \( \tau \in \mathbb{C} \) with \( |\tau| = 1 \), we define the set \( \Omega_j(\tau) := \left\{ z \in \mathbb{C} \mid z^{d^j} = \tau \text{ or } z^{d^j} = \overline{\tau} \right\} \) for \( j = 0, 1, \ldots \). Let \( S_k(\tau) \) be a subset of \( \Omega_k(\tau) \) such that for any \( \gamma \in S_k(\tau) \) the numbers \( \zeta_d \gamma \) and \( \overline{\gamma} \) belong to \( S_k(\tau) \), where \( \overline{\gamma} \) indicates the complex conjugate of \( \gamma \). Namely, \( S_k(\tau) \) satisfies \( S_k(\tau) = \zeta_d S_k(\tau) \) and \( S_k(\tau) = \overline{S_k(\tau)} \). For example, if \( d = 2 \), \( \tau = 1 \), and \( k = 3 \) we have \( \Omega_3(1) = \{ e^{k \pi i / 4} \mid 0 \leq k \leq 7 \} \) and so we can choose \( S_3(1) = \{ \pm e^{\pi i / 4}, \pm e^{3\pi i / 4} \} \).

We define the following sets that are determined depending only on \( S_k(\tau) \):

\[
\Lambda_i(\tau) = \left\{ \gamma^{d^{k-i}} \mid \gamma \in S_k(\tau) \right\} \quad (0 \leq i \leq k-1),
\]

\[
\Gamma_i(\tau) = \left\{ \gamma \in \Omega_i(\tau) \mid \gamma^d \in \Lambda_{i-1}(\tau) \right\} \setminus \Lambda_i(\tau) \quad (1 \leq i \leq k-1).
\]

Then we put

\[
E_k(\tau) = \left( \bigcup_{i=1}^{k-1} \Gamma_i(\tau) \right) \bigcup S_k(\tau) \tag{1.2}
\]

and

\[
F_k(\tau) = \begin{cases} E_k(\tau) \cup \{ \tau, \overline{\tau} \} & \text{if } \tau \notin E_k(\tau), \\ E_k(\tau) \setminus \{ \tau, \overline{\tau} \} & \text{otherwise.} \end{cases}
\]

In [1] we established necessary and sufficient conditions for the infinite products generated by each of the sequences in (1.1) to be algebraically dependent over \( \mathbb{Q} \) and obtained the following:

**Theorem 1.1.** Let \( \{ U_n \}_{n \geq 0} \) be the sequence defined by (1.1) and \( d \) be an integer greater than 1. Let \( a_1, \ldots, a_m \) be nonzero distinct real algebraic numbers. Then the numbers

\[
\prod_{k=0}^{\infty} \left( 1 + \frac{a_i}{U_{d^k}} \right) \quad (i = 1, \ldots, m)
\]

are algebraically dependent if and only if \( d \) is odd and there exist distinct \( \tau_1, \tau_2 \in \mathbb{C} \) with \( |\tau_1| = |\tau_2| = 1 \) and \( F_{k_1}(\tau_1), F_{k_2}(\tau_2) \) for some \( k_1, k_2 \geq 1 \) such that \( F_{k_1}(\tau_1) \cap F_{k_2}(\tau_2) \subset \{ \tau_1, \overline{\tau_1}, \tau_2, \overline{\tau_2} \} \) and \( \{ a_1, \ldots, a_m \} \) contains

\[-\frac{1}{\alpha - \beta} (\gamma + \overline{\gamma})
\]

for all \( \gamma \in (F_{k_1}(\tau_1) \cup F_{k_2}(\tau_2)) \setminus \{ \pm \sqrt{-1} \} \).

**Theorem 1.2.** Let \( \{ V_n \}_{n \geq 0} \) be the sequence defined by (1.1) and \( d \) be an integer greater than 1. Let \( a_1, \ldots, a_m \) be nonzero distinct real algebraic numbers. Then the numbers

\[
\prod_{k=0}^{\infty} \left( 1 + \frac{a_i}{V_{d^k}} \right) \quad (i = 1, \ldots, m)
\]
are algebraically dependent if and only if at least one of the following properties is satisfied:

1. $d = 2$ and the set $\{a_1, \ldots, a_m\}$ contains $b_1, \ldots, b_l$ ($l \geq 3$) with $b_1 < -2$
   satisfying
   $$b_2 = -b_1, \quad b_j = b_{j-1}^2 - 2 \quad (j = 3, \ldots, l-1), \quad b_l = -b_{l-1}^2 + 2.$$

2. $d = 2$ and there exist $\tau \in \mathbb{C}$ with $|\tau| = 1$ and $F_k(\tau)$ for some $k \geq 1$
   such that $\{a_1, \ldots, a_m\}$ contains $-\left(\gamma + \frac{\alpha}{2}\right)$
   for all $\gamma \in F_k(\tau) \setminus \{\pm \sqrt{-1}\}$.

3. $d \geq 4$ is even and there exist distinct $\tau_1, \tau_2 \in \mathbb{C}$ with $|\tau_1| = |\tau_2| = 1$
   and $F_{k_1}(\tau_1), F_{k_2}(\tau_2)$ for some $k_1, k_2 \geq 1$ such that $F_{k_1}(\tau_1) \cap F_{k_2}(\tau_2) \subset 
   \{\tau_1, \tau_1, \tau_2, \tau_2\}$ and $\{a_1, \ldots, a_m\}$ contains
   $$-\left(\gamma + \frac{\alpha}{2}\right)$$
   for all $\gamma \in (F_{k_1}(\tau_1) \cup F_{k_2}(\tau_2)) \setminus \{\pm \sqrt{-1}\}$.

Note that Theorems 1.1 and 1.2 above are generalizations of [2, Theorems 1 and 2], respectively.

**Corollary 1.3** (cf. [3]). Let $d \geq 2$ be a fixed integer and $a \neq 0$ a real algebraic number. Then the numbers

$$\prod_{k=1}^{\infty} \left(1 + \frac{a}{U_{dk}}\right) \quad \text{and} \quad \prod_{k=1}^{\infty} \left(1 + \frac{a}{V_{dk}}\right)$$

are transcendental, except for only two algebraic numbers

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{V_{2k}}\right) = \frac{\alpha^4 - 1}{\alpha^4 + \alpha^2 + 1}, \quad \prod_{k=1}^{\infty} \left(1 + \frac{2}{V_{2k}}\right) = \frac{\alpha^2 + 1}{\alpha^2 - 1}. \quad (1.3)$$

**Corollary 1.4.** Let $a$ be a nonzero real algebraic number with $a \neq -V_{2k} - 2$ ($k \geq 1$). Then the number

$$\prod_{k=1}^{\infty} \left(1 + \frac{a}{V_{2k} + 2}\right)$$

is transcendental, except when $a = -3, -2$; indeed

$$\prod_{k=1}^{\infty} \left(1 - \frac{2}{V_{2k} + 2}\right) = \frac{\alpha^2 - 1}{\alpha^2 + 1}, \quad \prod_{k=1}^{\infty} \left(1 - \frac{3}{V_{2k} + 2}\right) = \frac{(\alpha^2 - 1)^2}{\alpha^4 + \alpha^2 + 1}. \quad (1.4)$$
Proof. Using the equality
\[ 1 + \frac{a}{V_{2k} + 2} = \left( 1 + \frac{a + 2}{V_{2k}} \right) \left( 1 + \frac{2}{V_{2k}} \right)^{-1} \]
and the second equality in (1.3), we have
\[ \prod_{k=1}^{\infty} \left( 1 + \frac{a}{V_{2k} + 2} \right) = \frac{\alpha^2 - 1}{\alpha^2 + 1} \prod_{k=1}^{\infty} \left( 1 + \frac{a + 2}{V_{2k}} \right). \tag{1.5} \]

By Corollary 1.3 we see that the infinite product in the right-hand side of (1.5) is algebraic only if \( a = -3, -2 \). The equalities (1.4) follow immediately from (1.5) with (1.3). \qed

Applying Corollary 1.4 with \( \alpha = (1 + \sqrt{5})/2 \), we obtain the transcendence of
\[ \prod_{k=1}^{\infty} \left( 1 + \frac{a}{L_{2k} + 2} \right) \]
for any nonzero algebraic number \( a \neq -3, -2, -L_{2k} - 2 \ (k \geq 1) \), and the equalities
\[ \prod_{k=1}^{\infty} \left( 1 - \frac{2}{L_{2k} + 2} \right) = \frac{1}{\sqrt{5}}, \quad \prod_{k=1}^{\infty} \left( 1 - \frac{3}{L_{2k} + 2} \right) = \frac{1}{4}. \tag{1.6} \]

It should be noted that Corollaries 1.3 and 1.4 hold even if the number \( a \) is a nonzero complex algebraic number (see [3]).

2. Algebraic dependence relations

Theorems 1.1 and 1.2 in the introduction are useful to obtain the explicit algebraic dependence relations among the infinite products generated by the Fibonacci and Lucas numbers as well as their transcendence degrees. We exhibit such examples in this section and their proofs in the next section.

Example 2.1. Let \( a \) be a nonzero real algebraic number. The transcendental numbers
\[ s_1 = \prod_{F_{3k} \neq a, k=0}^{\infty} \left( 1 + \frac{a}{F_{3k}} \right), \quad s_2 = \prod_{k=0}^{\infty} \left( 1 - \frac{a}{F_{3k}} \right) \]
are algebraically dependent if and only if \( a = \pm 1/\sqrt{5} \). If \( a = 1/\sqrt{5} \), then
\[ s_1 s_2^{-1} = 2 + \sqrt{5}. \]
**Example 2.2.** The transcendental numbers

\[ s_1 = \prod_{k=0}^{\infty} \left( 1 + \frac{a_1}{F_{5k}} \right), \quad s_2 = \prod_{k=0}^{\infty} \left( 1 + \frac{a_2}{F_{5k}} \right), \]

\[ s_3 = \prod_{k=0}^{\infty} \left( 1 - \frac{a_1}{F_{5k}} \right), \quad s_4 = \prod_{k=0}^{\infty} \left( 1 - \frac{a_2}{F_{5k}} \right) \]

with \( a_1 = (-5 + \sqrt{5})/10, \; a_2 = (5 + \sqrt{5})/10 \) satisfy

\[ s_1s_2s_3^{-1}s_4^{-1} = 2 + \sqrt{5}, \]

while \( \text{trans.deg}_Q \mathbb{Q}(s_1, s_2, s_3, s_4) = 3. \)

**Remark 2.3.** The infinite products \( \prod_{k=0}^{\infty} (1 + a_i/F_{d^k}) \) for odd \( d \) and \( \prod_{k=1}^{\infty} (1 + a_i/L_{d^k}) \) for even \( d \) are easily expressed as the values at an algebraic number of \( \Phi_i(z) \) defined by (3.2) with \( b = 1 \), which will be shown in (3.3) of Section 3. Hence, for simplicity, we take \( k \geq 1 \) in the following examples.

**Example 2.4.** Let \( a \neq 2, -1 \) be a real algebraic number. The transcendental numbers

\[ s_1 = \prod_{k=1}^{\infty} \left( 1 + \frac{a}{L_{2^k}} \right), \quad s_2 = \prod_{k=1}^{\infty} \left( 1 - \frac{a}{L_{2^k}} \right) \]

are algebraically dependent if and only if \( a = \pm\sqrt{2} \). If \( a = \pm\sqrt{2} \), using the relation \( L_{2^k} = L_{2^{k+1}} + 2 \) \((k \geq 1)\) and the first equality in (1.6), we have

\[ s_1s_2 = \prod_{k=2}^{\infty} \left( 1 - \frac{2}{L_{2^k} + 2} \right) = \frac{5}{3} \cdot \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{3}. \]

**Example 2.5.** The transcendental numbers

\[ s_1 = \prod_{k=1}^{\infty} \left( 1 - \frac{\sqrt{3}}{L_{4^k}} \right), \quad s_2 = \prod_{k=1}^{\infty} \left( 1 + \frac{\sqrt{3}}{L_{4^k}} \right), \]

\[ s_3 = \prod_{k=1}^{\infty} \left( 1 - \frac{1}{L_{4^k}} \right), \quad s_4 = \prod_{k=1}^{\infty} \left( 1 + \frac{2}{L_{4^k}} \right) \]

satisfy

\[ s_1s_2s_3s_4^{-1} = \frac{5}{8}, \]

while \( \text{trans.deg}_Q \mathbb{Q}(s_1, s_2, s_3, s_4) = 3. \)

**Example 2.6.** The transcendental numbers

\[ s_1 = \prod_{k=1}^{\infty} \left( 1 - \frac{1}{L_{6^k}} \right), \quad s_2 = \prod_{k=1}^{\infty} \left( 1 + \frac{1}{L_{6^k}} \right), \quad s_3 = \prod_{k=1}^{\infty} \left( 1 + \frac{2}{L_{6^k}} \right), \]
\[ s_4 = \prod_{k=1}^{\infty} \left( 1 + \frac{\sqrt{3}}{L_{6^k}} \right), \quad s_5 = \prod_{k=1}^{\infty} \left( 1 - \frac{\sqrt{3}}{L_{6^k}} \right) \]
satisfy
\[ s_1 s_2 s_3 s_4^{-1} s_5^{-1} = \frac{\sqrt{5}}{2}, \]
while \( \text{trans. deg}_Q \mathbb{Q}(s_1, s_2, s_3, s_4, s_5) = 4. \)

**Example 2.7.** The transcendental numbers
\[ s_i = \prod_{k=1}^{\infty} \left( 1 + \frac{a_i}{L_{4^k}} \right) \quad (i = 1, \ldots, 8), \]
where
\[ a_1 = -(\zeta_{16}^1 + \zeta_{16}^{15}), \quad a_2 = -(\zeta_{16}^5 + \zeta_{16}^{11}), \quad a_3 = -(\zeta_{16}^7 + \zeta_{16}^9), \quad a_4 = -(\zeta_{64}^3 + \zeta_{64}^{61}), \]
\[ a_5 = -(\zeta_{64}^{13} + \zeta_{64}^{51}), \quad a_6 = -(\zeta_{64}^9 + \zeta_{64}^{45}), \quad a_7 = -(\zeta_{64}^{29} + \zeta_{64}^{35}), \quad a_8 = 2, \]
satisfy
\[ s_1 s_2 \cdots s_7 s_8^{-2} = \frac{25}{7(7 - \sqrt{2} - \sqrt{2})}. \]

**Example 2.8.** The transcendental numbers
\[ s_i = \prod_{k=1}^{\infty} \left( 1 + \frac{a_i}{L_{4^k}} \right) \quad (i = 1, \ldots, 10), \]
where
\[ a_1 = -\frac{3}{2}, \quad a_2 = \frac{\sqrt{7}}{2}, \quad a_3 = \frac{3}{2}, \quad a_4 = -\frac{\sqrt{7}}{2}, \quad a_5 = \frac{31}{16}, \]
\[ a_6 = -\frac{4}{\sqrt{5}}, \quad a_7 = \frac{2}{\sqrt{5}}, \quad a_8 = \frac{4}{\sqrt{5}}, \quad a_9 = -\frac{2}{\sqrt{5}}, \quad a_{10} = \frac{14}{25}, \]
satisfy
\[ s_1 s_2 s_3 s_4 s_5^{-1} s_6^{-1} s_7^{-1} s_8^{-1} s_9^{-1} s_{10} = \frac{3024}{3575}, \]
while \( \text{trans. deg}_Q \mathbb{Q}(s_1, s_2, \ldots, s_{10}) = 9. \)

### 3. Proofs of the examples

Let \( \{R_n\}_{n \geq 0} \) be the sequence \( \{U_n\}_{n \geq 0} \) or \( \{V_n\}_{n \geq 0} \) defined by (1.1). Let \( d \geq 2 \) be a fixed integer and \( a_1, \ldots, a_m \) nonzero real algebraic numbers. Define
\[
(p_i, b) := \begin{cases} 
((\alpha - \beta)a_i, -(-1)^d) & \text{if } R_n = U_n, \\
(a_i, (-1)^d) & \text{if } R_n = V_n,
\end{cases}
\] (3.1)
and
\[ \Phi_i(z) := \prod_{k=0}^{\infty} \left( 1 + \frac{p_i z^{dk}}{1 + b z^{2dk}} \right) \quad (i = 1, \ldots, m). \] (3.2)

Taking an integer \( N \geq 1 \) such that \( |R_{dk}| > \max\{|a_1|, \ldots, |a_m|\} \) for all \( k \geq N \), we have
\[ \Phi_i(\alpha^{-dN}) = \prod_{k=N}^{\infty} \left( 1 + \frac{p_i \alpha^{-dk}}{1 + b \alpha^{-2dk}} \right) = \prod_{k=N}^{\infty} \left( 1 + \frac{p_i}{\alpha^{dk} + b(-1)^d \beta^{dk}} \right) = \prod_{k=N}^{\infty} \left( 1 + \frac{a_i}{R_{dk}} \right) \quad (i = 1, \ldots, m), \]
so that
\[ \prod_{k=0}^{\infty} \left( 1 + \frac{a_i}{R_{dk}} \right) = \Phi_i(\alpha^{-dN}) \prod_{\substack{k=0 \\text{if } R_{dk} \neq a_i}}^{N-1} \left( 1 + \frac{a_i}{R_{dk}} \right) \quad (i = 1, \ldots, m). \] (3.3)

We note that (3.3) is valid also for \( N = 0 \) only if \( d \) is odd and \( R_{dk} \neq -a_i \) \((k \geq 0)\).

**Proof of Example 2.1.** First we show that \( s_1 \) and \( s_2 \) are algebraically dependent only if \( a = \pm 1/\sqrt{5} \), using the case of \( m = 2 \) in Theorem 1.1. If \( s_1 \) and \( s_2 \) are algebraically dependent, then \( \{\tau_1, \tau_2\} = \{1, -1\} \), since \( \mathcal{F}_k(\tau) \) consists of at least four elements if \( \tau \neq \pm 1 \). If \( d = 3 \), \( m = 2 \), and \( \{\tau_1, \tau_2\} = \{1, -1\} \), it is easily seen that \( \mathcal{F}_1(\tau_1) \cup \mathcal{F}_1(\tau_2) = \{\zeta_3, \overline{\zeta_3}, -\zeta_3, -\overline{\zeta_3}\} \) and so \( \{a_1, a_2\} = \{1/\sqrt{5}, -1/\sqrt{5}\} \).

Next we show the equality \( s_1 s_2^{-1} = 2 + \sqrt{5} \) by proving a general relation which holds for the functions \( \Phi_i(z) \) \((1 \leq i \leq d - 1)\) defined by (3.2), where \( d \geq 3 \) is an odd integer. Put
\[ p_1 = -\left( \zeta_d + \overline{\zeta_d} \right), \quad p_2 = -\left( \zeta_d^2 + \overline{\zeta_d^2} \right), \ldots, \quad p_{d-1} = -\left( \zeta_d^{d-1} + \overline{\zeta_d^{d-1}} \right) \]
in the equation (3.2) with \( b = 1 \). Then we have
\[ \Phi_1(z) \cdots \Phi_{d-1}(z) = \prod_{k=0}^{\infty} \left( \frac{1}{(1 + z^{2dk})^{d-1}} \right) \cdot \prod_{k=0}^{\infty} \left( \frac{1}{1 - \frac{1}{2} z^{2dk}} \right) \cdot \prod_{k=0}^{\infty} \left( \frac{1}{1 - z^{2dk}} \right). \]
Moreover, putting
\[ p_{d-1} + 1 = \zeta_d + \overline{\zeta_d}, \quad p_{d-1} + 2 = \zeta_d^2 + \overline{\zeta_d^2}, \ldots, \quad p_{d-1} = \zeta_d^{d-1} + \overline{\zeta_d^{d-1}}, \]
we get
\[ \Phi_{d-1}(z) \cdots \Phi_1(z) \]


\[
= \prod_{k=0}^{\infty} \left( \frac{1}{(1 + z^{2d^{k}})^{\frac{d-1}{2}}} \right) = \frac{1}{1+z} \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2d^{k}})^{\frac{d-1}{2}}}.
\]

Hence, we have

\[
\Phi(z) := \frac{\Phi_1(z) \cdots \Phi_{d-1}(z)}{\Phi_{d-1+1}(z) \cdots \Phi_{d-1}(z)} = \frac{1+z}{1-z}.
\]  

(3.4)

If \( d = 3 \), then \( p_1 = -(\zeta_3 + \overline{\zeta_3}) = 1 \), \( p_2 = \zeta_3 + \overline{\zeta_3} = -1 \), and so

\[
a_1 = \frac{1}{\alpha - \beta} p_1 = \frac{1}{\sqrt{5}}, \quad a_2 = \frac{1}{\alpha - \beta} p_2 = -\frac{1}{\sqrt{5}}
\]

by (3.1). Then, by the equation (3.3) with \( N = 0 \), we have

\[
\Phi(\alpha^{-1}) = s_1 s_2^{-1} = \frac{\alpha + 1}{\alpha - 1} = 2\alpha + 1 = 2 + \sqrt{5}.
\]


**Proof of Example 2.2.** We consider the case of \( d = 5 \) in (3.4). Then

\[
p_1 = -(\zeta_5 + \overline{\zeta_5}) = \frac{1 - \sqrt{5}}{2}, \quad p_2 = -(\zeta_5^2 + \overline{\zeta_5^2}) = \frac{1 + \sqrt{5}}{2},
\]

\[
p_3 = \zeta_5 + \overline{\zeta_5} = \frac{-1 + \sqrt{5}}{2}, \quad p_4 = \zeta_5^2 + \overline{\zeta_5^2} = \frac{-1 + \sqrt{5}}{2}.
\]

By (3.1) we have

\[
a_1 = \frac{-5 + \sqrt{5}}{10}, \quad a_2 = \frac{5 + \sqrt{5}}{10}, \quad a_3 = \frac{5 - \sqrt{5}}{10}, \quad a_4 = \frac{-5 + \sqrt{5}}{10}.
\]

Then, by the equation (3.3) with \( N = 0 \) and (3.4), we have

\[
\Phi(\alpha^{-1}) = \frac{s_1 s_2}{s_3 s_4} = \frac{\alpha + 1}{\alpha - 1} = 2 + \sqrt{5}.
\]

Finally, we prove that \( \text{trans. deg}_Q(\mathbb{Q}(s_1, s_2, s_3, s_4)) = 3 \), using Theorem 1.1. Let \( \tau_1 = 1 \), \( \tau_2 = -1 \), \( S_1(\tau_1) = \mathcal{E}_1(\tau_1) = \{ \zeta_5, \overline{\zeta_5}, \zeta_5^2, \overline{\zeta_5^2}, 1 \} \), and \( S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{ -\zeta_5, -\overline{\zeta_5}, -\zeta_5^2, -\overline{\zeta_5^2}, -1 \} \). Then \( \mathcal{F}_1(\tau_1) = \{ \zeta_5, \overline{\zeta_5}, \zeta_5^2, \overline{\zeta_5^2} \} \) and \( \mathcal{F}_1(\tau_2) = \{ -\zeta_5, -\overline{\zeta_5}, -\zeta_5^2, -\overline{\zeta_5^2} \} \). It is enough to show that \( s_1, s_2, \) and \( s_3 \) are algebraically independent, which is equivalent to the fact that \( a_1, a_2, \) and \( a_3 \) do not satisfy Theorem 1.1 with \( m = 3 \). By (1.2) with \( S_1(\tau_i) = \mathcal{E}_1(\tau_i) \) \( (i = 1, 2) \), considering the number of the elements of \( S_k(\tau_i) \) with \( k \geq 2 \) satisfying \( S_k(\tau_i) = \zeta_5 S_k(\tau_1) \) and \( S_k(\tau_i) = \overline{S_k(\tau_1)} \), we see that \( \{ a_1, a_2, a_3, a_4 \} \) is the minimal set of \( - (\gamma + \overline{\gamma}) / \sqrt{5} \) with \( \gamma \in \mathcal{F}_{k_1}(\tau_1) \cup \mathcal{F}_{k_2}(\tau_2) \setminus \{ \pm \sqrt{-1} \} \) satisfying Theorem 1.1 with \( m = 4 \).}

**Proof of Example 2.4.** First we prove directly that \( s_1 s_2 = \sqrt{5}/3 \) if \( a = \pm \sqrt{2} \). Let \( \tau = \sqrt{-1} \) and \( S_1(\tau) = \mathcal{E}_1(\tau) = \{ \zeta_8, \overline{\zeta_8}, -\zeta_8, -\overline{\zeta_8} \} \) in the property 2 of Theorem 1.2. Then \( \mathcal{F}_1(\tau) = \{ \zeta_8, \overline{\zeta_8}, -\zeta_8, -\overline{\zeta_8}, \sqrt{-1}, -\sqrt{-1} \} \). Putting

\[
p_1 = -(\zeta_8 + \overline{\zeta_8}) = -\sqrt{2}, \quad p_2 = \zeta_8 + \overline{\zeta_8} = \sqrt{2}
\]
in the equation (3.2) with \( b = 1 \), we have

\[
\Phi_1(z)\Phi_2(z) = \prod_{k=0}^{\infty} \frac{(z^{2^k} - \zeta_8)(z^{2^k} - \overline{\zeta}_8)(z^{2^k} + \zeta_8)(z^{2^k} + \overline{\zeta}_8)}{(1 + z^{2^{2k}})^2} = \prod_{k=0}^{\infty} \frac{1 + z^{2^{k+2}}}{(1 + z^{2^{k+1}})^2} = \prod_{k=0}^{\infty} \frac{1 + z^{2^{k+2}}}{1 - z^{2^{k+2}}} = 1 - z^2 \frac{1}{1 + z^2}.
\]

By the equation (3.3) with \( N = 1 \) and \( \alpha = (1 + \sqrt{5})/2 \), we get

\[
s_1s_2 = \Phi_1(\alpha^{-2})\Phi_2(\alpha^{-2}) = \frac{\alpha^4 - 1}{\alpha^4 + 1}.
\]

Hence, noting that \( \alpha^4 = (\alpha + 1)^2 = 3\alpha + 2 \), we have

\[
s_1s_2 = \frac{1}{3} \cdot \frac{3\alpha + 1}{\alpha + 1} = \frac{1}{3} (2\alpha - 1) = \frac{\sqrt{5}}{3}.
\]

Conversely, if \( s_1 \) and \( s_2 \) are algebraically dependent for some algebraic number \( a \), then by the property 2 of Theorem 1.2 with \( m = 2 \) the set \( \mathcal{F}_k(\tau) \setminus \{ \pm\sqrt{-1} \} \) must consist of four elements, which is achieved only if \( \tau = \pm\sqrt{-1} \) and \( k = 1 \).

**Proof of Example 2.5.** We use the property 3 of Theorem 1.2. Let \( \tau_1 = \zeta_3 \), \( \tau_2 = 1 \), \( S_1(\tau_1) = \mathcal{E}_1(\tau_1) = \{ \zeta_{12}, \overline{\zeta}_{12}, \zeta_{12}^{\frac{3}{2}}, \overline{\zeta}_{12}^{\frac{3}{2}}, \zeta_{12}^{\frac{5}{2}}, \overline{\zeta}_{12}^{\frac{5}{2}}, \zeta_{12}^{\frac{7}{2}}, \overline{\zeta}_{12}^{\frac{7}{2}} \} \), and \( S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{ 1, -1, \sqrt{-1}, -\sqrt{-1} \} \). Then \( \mathcal{F}_1(\tau_1) = \{ \zeta_{12}, \zeta_{12}^{\frac{3}{2}}, \zeta_{12}^{\frac{5}{2}}, \zeta_{12}^{\frac{7}{2}} \} \) and \( \mathcal{F}_1(\tau_2) = \{ -1, \sqrt{-1}, -\sqrt{-1} \} \). Putting

\[
p_1 = - (\zeta_{12} + \overline{\zeta}_{12}) = -\sqrt{3}, \quad p_2 = - (\zeta_{12}^{\frac{5}{2}} + \overline{\zeta}_{12}^{\frac{5}{2}}) = \sqrt{3}, \quad p_3 = - (\zeta_{12}^{\frac{7}{2}} + \overline{\zeta}_{12}^{\frac{7}{2}}) = -1,
\]

and \( p_4 = 2 \) in the equation (3.2) with \( b = 1 \), we have

\[
\Phi_1(z)\Phi_2(z)\Phi_3(z) = \prod_{k=0}^{\infty} \frac{(z^{4^k} - \zeta_{12})(z^{4^k} - \overline{\zeta}_{12})(z^{4^k} - \zeta_{12}^{\frac{3}{2}})(z^{4^k} - \overline{\zeta}_{12}^{\frac{3}{2}})(z^{4^k} - \zeta_{12}^{\frac{5}{2}})(z^{4^k} - \overline{\zeta}_{12}^{\frac{5}{2}})(z^{4^k} - \zeta_{12}^{\frac{7}{2}})(z^{4^k} - \overline{\zeta}_{12}^{\frac{7}{2}})}{(1 + z^{2\cdot4^k})^3} = \prod_{k=0}^{\infty} \frac{(z^{4^k+1} - \zeta_{12}^{\frac{3}{2}})(z^{4^k+1} - \overline{\zeta}_{12}^{\frac{3}{2}})}{(z^{4^k} - \zeta_{12}^{\frac{3}{2}})(z^{4^k} - \overline{\zeta}_{12}^{\frac{3}{2}})} \frac{1}{(1 + z^{2\cdot4^k})^3} = \prod_{k=0}^{\infty} \frac{1}{(z - \zeta_{12}^{\frac{4}{2}})(z - \overline{\zeta}_{12}^{\frac{4}{2}})} \frac{1}{(1 + z^{2\cdot4^k})^3},
\]

and

\[
\Phi_4(z) = \prod_{k=0}^{\infty} \frac{1 + 2z^{4^k} + z^{2\cdot4^k}}{1 + z^{2\cdot4^k}} = \prod_{k=0}^{\infty} \frac{(1 + z^{4^k})^2(1 + z^{2\cdot4^k})^2}{(1 + z^{2\cdot4^k})^3} = \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2\cdot4^k})^3} \left( 1 - z^{4^k} \right)^2 = \frac{1}{(1 - z)^2} \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2\cdot4^k})^3}. \quad (3.5)
\]
Hence, we get
\[ \Phi(z) := \Phi_1(z)\Phi_2(z)\Phi_3(z)\Phi_4^{-1}(z) = \frac{(1-z)^2}{1+z+z^2}. \]

By the equation (3.3) with \( N = 1 \) and \( \alpha = (1 + \sqrt{5})/2 \), we have
\[ s_1s_2s_3s_4^{-1} = \Phi(\alpha^{-4}) = \frac{\alpha^8 - 2\alpha^4 + 1}{\alpha^8 + \alpha^4 + 1} = \frac{7\alpha^4 - 2\alpha^4}{7\alpha^4 + \alpha^4} = \frac{5}{8}, \]
since
\[ \alpha^8 + 1 = (3\alpha + 2)^2 + 1 = 21\alpha + 14 = 7\alpha^4. \] (3.6)

To prove that trans. deg_\( \mathbb{Q} \) \( (s_1, s_2, s_3, s_4) = 3 \), it is enough to show that \( s_2, s_3, \) and \( s_4 \) are algebraically independent, which is equivalent to the fact that \( p_2, p_3, \) and \( p_4 \) do not satisfy the property 3 of Theorem 1.2 with \( m = 3 \). By (1.2) with \( S_1(\tau_i) = \mathcal{E}_1(\tau_i) \) \((i = 1, 2)\), considering the number of the elements of \( S_k(\tau_i) \) with \( k \geq 2 \) satisfying \( S_k(\tau_i) = \zeta_i S_k(\tau_i) \) and \( S_k(\tau_i) = \overline{S}_k(\tau_i) \), we see that \( \{-\sqrt{3}, \sqrt{3}, -1, 2\} \) is the minimal set of \(- (\gamma + \overline{\gamma}) \) with \( \gamma \in \mathcal{F}_k_1(\tau_1) \cup \mathcal{F}_k_2(\tau_2) \setminus \{\pm \sqrt{-1}\} \) satisfying the property 3 of Theorem 1.2 with \( m = 4 \).

**Proof of Example 2.6.** We use the property 3 of Theorem 1.2. Let \( \tau_1 = 1, \tau_2 = -1, S_1(\tau_1) = \mathcal{E}_1(\tau_1) = \{\zeta_6, \zeta_6^2, -1, \zeta_6^4, \zeta_6^5, 1\} \), and \( S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{\zeta_12, \sqrt{-1}, \zeta_12^5, \zeta_12^7, -\sqrt{-1}, \zeta_12^11\} \). Then \( \mathcal{F}_1(\tau_1) = \{\zeta_6, \zeta_6^2, -1, \zeta_6^4, \zeta_6^5\} \) and \( \mathcal{F}_1(\tau_2) = \{\zeta_12, \sqrt{-1}, \zeta_12^5, \zeta_12^7, -\sqrt{-1}, \zeta_12^11\} \).

We show the equality \( s_1s_2s_3s_4^{-1} = \sqrt{5}/2 \) by proving a general relation among the functions \( \Phi_i(z) \) defined by (3.2). Let \( d \geq 6 \) be an even integer. Putting
\[ \begin{align*}
p_0 &= -2, \quad p_1 = -(\zeta_d + \overline{\zeta_d}), \quad p_2 = -(\zeta_d^2 + \overline{\zeta_d^2}), \quad \ldots, \quad p_\frac{d}{2} = -(\zeta_d^{\frac{d}{2}} + \overline{\zeta_d^{\frac{d}{2}}}) = 2, \end{align*} \]
in the equation (3.2) with \( b = 1 \), we have
\[ \left( \Phi_0(z) \cdot \Phi_1^2(z) \cdot \Phi_2^2(z) \cdot \Phi_3^2(z) \cdot \Phi_4^{-2}(z) \right) \cdot \Phi_0^{-1}(z) = \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2d^k})^{d-1}} \frac{(z^{d+1} - 1)^2}{(z^{d^k} - 1)^2} = \frac{1}{(z - 1)^2} \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2d^k})^{d-1}}. \]

In the same way, putting
\[ \begin{align*}
p_{\frac{d}{2}+1} = -(\zeta_{2d} + \overline{\zeta_{2d}}), \quad p_{\frac{d}{2}+2} = -(\zeta_{2d}^3 + \overline{\zeta_{2d}^3}), \quad \ldots, \quad p_d = -(\zeta_{2d}^{d-1} + \overline{\zeta_{2d}^{d-1}}), \end{align*} \]
we get
\[ \left( \Phi_{\frac{d}{2}+1}^2(z) \cdot \Phi_{\frac{d}{2}+2}^2(z) \cdot \Phi_d^2(z) \cdot \Phi_{\frac{d}{2}}^{-2}(z) \right) \cdot \Phi_0^{-1}(z) = \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2d^k})^{d-1}} \frac{(z^{d+1} + 1)^2}{(z^{d^k} + 1)^2} = \frac{1}{(z + 1)^2} \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2d^k})^{d-1}}. \]
Hence, noting that Φ_i(0) = 1 (1 ≤ i ≤ d), we have
\[ \Phi(z) := \Phi_1(z)\Phi_2(z) \cdots \Phi_d(z) = \frac{1 + z}{1 - z}. \] (3.7)

Now assume that d = 6 in (3.7). Noting that \( p_5 = 0 \) and putting
\[ a_1 = p_1 = -1, \quad a_2 = p_2 = 1, \quad a_3 = p_3 = 2, \quad a_4 = p_4 = -\sqrt{3}, \quad a_5 = p_6 = \sqrt{3} \]
in the equation (3.3) with \( N = 1 \) and \( \alpha = (1 + \sqrt{5})/2 \), we have
\[ \Phi(\alpha^{-6}) = \frac{s_1s_2s_3}{s_4s_5} = \frac{\alpha^6 + 1}{\alpha^6 - 1}. \]

Since \( \alpha^6 = 8\alpha + 5 \), we get
\[ \frac{s_1s_2s_3}{s_4s_5} = \frac{\alpha^6 + 1}{\alpha^6 - 1} = \frac{1}{2}(2\alpha - 1) = \frac{\sqrt{5}}{2}. \]

The transcendence degree is obtained in the same way as in the proof of Example 2.5.

**Proof of Example 2.7.** We use the property 3 of Theorem 1.2. Let \( \tau_1 = \sqrt{-1}, \) \( \tau_2 = 1, \)
\[ S_2(\tau_1) = \{ \zeta_{64}^3, \zeta_{64}^{13}, \zeta_{64}^{19}, \zeta_{64}^{29}, \zeta_{64}^{35}, \zeta_{64}^{45}, \zeta_{64}^{51}, \zeta_{64}^{61} \}, \]
and
\[ S_1(\tau_2) = E(\tau_2) = \{ 1, -1, \sqrt{-1}, -\sqrt{-1} \}. \]

Then
\[ \Lambda_1(\tau_1) = \{ \zeta_{16}^3, \zeta_{16}^{13} \}, \quad \Gamma_1(\tau_1) = \{ \zeta_{16}^1, \zeta_{16}^5, \zeta_{16}^7, \zeta_{16}^9, \zeta_{16}^{11}, \zeta_{16}^{15} \}, \quad \Lambda_0(\tau_1) = \{ \sqrt{-1}, -\sqrt{-1} \}, \]
and so
\[ F_2(\tau_1) = \{ \zeta_{64}^3, \zeta_{64}^{13}, \zeta_{64}^{19}, \zeta_{64}^{29}, \zeta_{64}^{35}, \zeta_{64}^{45}, \zeta_{64}^{51}, \zeta_{64}^{61}, \zeta_{16}^1, \zeta_{16}^5, \zeta_{16}^7, \zeta_{16}^9, \zeta_{16}^{11}, \zeta_{16}^{15}, \sqrt{-1}, -\sqrt{-1} \}, \]
\[ F_1(\tau_2) = \{ -1, \sqrt{-1}, -\sqrt{-1} \}. \]

Putting
\[ p_1 = - (\zeta_{16}^1 + \zeta_{16}^{15}), \quad p_2 = - (\zeta_{16}^5 + \zeta_{16}^{11}), \quad p_3 = - (\zeta_{16}^7 + \zeta_{16}^9), \]
\[ p_4 = - (\zeta_{64}^3 + \zeta_{64}^{61}), \quad p_5 = - (\zeta_{64}^{13} + \zeta_{64}^{51}), \quad p_6 = - (\zeta_{64}^{19} + \zeta_{64}^{45}), \quad p_7 = - (\zeta_{64}^{29} + \zeta_{64}^{35}) \]
in the equation (3.2) with \( b = 1 \), we get
\[ \Phi_1(z)\Phi_2(z) \cdots \Phi_7(z) \]
\[ = \prod_{k=0}^{\infty} \left( \frac{1}{(1 + z^{2\cdot4^k})^6} \frac{(z^{4\cdot4^k+1} - \zeta_{16}^3)(z^{4\cdot4^k+1} - \zeta_{16}^{13})}{(z^{4^k} - \zeta_{16}^3)(z^{4^k} - \zeta_{16}^{13})} \frac{z^{2\cdot4^k} + 1}{z^{2\cdot4^k} + 1} \right). \]
\[
\frac{1}{(z^2 + 1)(z - \zeta_{16}^3)(z - \zeta_{16}^{13})} \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2\cdot4^k})^6}.
\]

Letting \( p_8 = 2 \) and using (3.5) in the proof of Example 2.5, we have

\[
\Phi(z) := \frac{\Phi_1(z)\Phi_2(z) \cdots \Phi_7(z)}{\Phi_8'(z)} = \frac{(z - 1)^4}{(z^2 + 1)(z - \zeta_{16}^3)(z - \zeta_{16}^{13})}.
\]

Putting \( a_i = p_i \) \((1 \leq i \leq 8)\) in the equation (3.3) with \( N = 1 \) and \( \alpha = (1 + \sqrt{5})/2 \) and using (3.6), we obtain

\[
\frac{s_1 \cdots s_7}{s_8^2} = \Phi(\alpha^{-4}) = \frac{(\alpha^4 - 1)^4}{(\alpha^8 + 1)(\alpha^8 + 1 - (\zeta_{16}^3 + \zeta_{16}^{13})\alpha^4)}
\]

\[
= \frac{(7\alpha^4 - 2\alpha^4)^2}{25} = \frac{7(7\alpha^4 - (\zeta_{16}^3 + \zeta_{16}^{13})\alpha^4)}{7(7 - \sqrt{2} - \sqrt{2})},
\]

since \( \zeta_{16}^3 + \zeta_{16}^{13} = 2\cos(3\pi/8) = \sqrt{2} - \sqrt{2} \).

\[\square\]

**Proof of Example 2.8.** Let \( d \geq 2 \) be an integer. Let \( \gamma \) and \( \eta \) be complex numbers with \( |\gamma| = |\eta| = 1 \). We show a general relation which holds for the functions \( \Phi_i(z) \) \((1 \leq i \leq 2d + 2)\) defined by (3.2). Putting

\[
p_1 = - (\gamma + \overline{\eta}), \quad p_2 = - (\gamma \zeta_d + \overline{\gamma \zeta_d}), \ldots, \quad p_d = - (\gamma \zeta_d^{d-1} + \overline{\gamma \zeta_d^{d-1}}),
\]

and \( p_{d+1} = - (\gamma^d + \overline{\gamma^d}) \) in the equation (3.2) with \( b = 1 \), we have

\[
\Phi_1(z) \cdots \Phi_d(z)\Phi_{d+1}^{-1}(z)
\]

\[
= \prod_{k=0}^{\infty} \left( \frac{1}{1 + z^{2d^k}d^{-1}} - \frac{1}{1 + p_{d+1}z^{d^k} + z^{2d^k}} \prod_{i=1}^d (1 + p_i z^{d^k} + z^{2d^k}) \right)
\]

\[
= \prod_{k=0}^{\infty} \left( \frac{1}{1 + z^{2d^k}d^{-1}d - \gamma^d(z^{d^k} - \overline{\gamma^d}(z^{d^k} - \overline{\gamma^d}))} \prod_{i=0}^{d-1} (z^{d^k} - \gamma_{d'}^{i})(z^{d^k} - \overline{\gamma_{d'}^{i}}) \right)
\]

\[
= \frac{1}{(z - \gamma^d)(z - \overline{\gamma^d})} \prod_{k=0}^{\infty} \frac{1}{1 + z^{2d^k}d^{-1}}.
\]

Moreover, putting

\[
p_{d+2} = - (\eta + \overline{\gamma}), \quad p_{d+3} = - (\eta \zeta_d + \overline{\eta \zeta_d}), \ldots, \quad p_{2d+1} = - (\eta \zeta_d^{d-1} + \overline{\eta \zeta_d^{d-1}}),
\]

and \( p_{2d+2} = - (\eta^d + \overline{\eta^d}) \), we get

\[
\Phi(z) := \frac{\Phi_1(z) \cdots \Phi_d(z)}{\Phi_{d+1}(z)} \cdot \frac{\Phi_{d+2}(z) \cdots \Phi_{2d+1}(z)}{\Phi_{d+2}(z) \cdots \Phi_{2d+1}(z)} = \frac{(z - \eta^d)(z - \overline{\eta^d})}{(z - \gamma^d)(z - \overline{\gamma^d})}.
\]

(3.8)
Substituting $z = \alpha^{-4}$ into (3.8) and using (3.6), we get
\[
\Phi(\alpha^{-4}) = \frac{\alpha^8 + p_{2d+2} \alpha^4 + 1}{\alpha^8 + p_{d+1} \alpha^4 + 1} = \frac{7 + p_{2d+2}}{7 + p_{d+1}}. 
\] (3.9)

For Example 2.8, we take $d = 4$ and
\[
\gamma = \frac{3 + \sqrt{-7}}{4}, \quad \eta = \frac{2 + \sqrt{-5}}{\sqrt{5}}.
\]

Noting that $\gamma^4 \neq \eta^4$ and taking $\tau_1 = \gamma^4$ and $\tau_2 = \eta^4$ in the property 3 of Theorem 1.2, we have
\[
S_1(\tau_1) = E_1(\tau_1) = \{\gamma, \sqrt{-1} \gamma, -\gamma, -\sqrt{-1} \gamma, \bar{\gamma}, -\bar{\gamma}, -\sqrt{-1} \bar{\gamma}, \sqrt{-1} \bar{\gamma}, -\bar{\gamma}, -\sqrt{-1} \gamma\},
\]
\[
S_1(\tau_2) = E_1(\tau_2) = \{\eta, \sqrt{-1} \eta, -\eta, -\sqrt{-1} \eta, \bar{\eta}, -\bar{\eta}, -\sqrt{-1} \bar{\eta}, \sqrt{-1} \bar{\eta}, -\bar{\eta}, -\sqrt{-1} \eta\},
\]
and so
\[
F_1(\tau_1) = \{\gamma, \sqrt{-1} \gamma, -\gamma, -\sqrt{-1} \gamma, \bar{\gamma}, -\bar{\gamma}, -\sqrt{-1} \bar{\gamma}, \sqrt{-1} \bar{\gamma}, \gamma^4, \eta^4\},
\]
\[
F_1(\tau_2) = \{\eta, \sqrt{-1} \eta, -\eta, -\sqrt{-1} \eta, \bar{\eta}, -\bar{\eta}, -\sqrt{-1} \bar{\eta}, \sqrt{-1} \bar{\eta}, \eta^4, \gamma^4\},
\]
since $\gamma$ and $\eta$ are not roots of unity. Then we have
\[
p_1 = -\frac{3}{2}, \quad p_2 = \frac{\sqrt{7}}{2}, \quad p_3 = \frac{3}{2}, \quad p_4 = -\frac{\sqrt{7}}{2}, \quad p_5 = \frac{31}{16},
\]
\[
p_6 = -\frac{4}{\sqrt{5}}, \quad p_7 = \frac{2}{\sqrt{5}}, \quad p_8 = \frac{4}{\sqrt{5}}, \quad p_9 = -\frac{2}{\sqrt{5}}, \quad p_{10} = \frac{14}{25},
\]
since
\[
\gamma^4 = -\frac{31 - 3\sqrt{-7}}{32}, \quad \eta^4 = -\frac{7 - 24\sqrt{-1}}{25}.
\]

Using (3.9), we get
\[
\frac{s_1 \cdots s_4}{s_5} \cdot \frac{s_{10}}{s_6 \cdots s_9} = \frac{7 + 14/25}{7 + 31/16} = \frac{3024}{3575}
\]
bys the equation (3.3) with $N = 1$. The transcendence degree is obtained in the same way as in the proof of Example 2.5. \hfill \box

References

