Cofinite derivations in rings

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Abstract

A derivation \( d : R \to R \) is called cofinite if its image \( \text{Im} \ d \) is a subgroup of finite index in the additive group \( R^+ \) of an associative ring \( R \). We characterize left Artinian (respectively semiprime) rings with all non-zero inner derivations to be cofinite.

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1. Introduction

Throughout this paper \( R \) will always be an associative ring with identity. A derivation \( d : R \to R \) is said to be cofinite if its image \( \text{Im} \ d \) is a subgroup of finite index in the additive group \( R^+ \) of \( R \). Obviously, in a finite ring every derivation is cofinite. As noted in [3], only a few results are known concerning images of derivations.

We study properties of rings with cofinite non-zero derivations and prove the following

Proposition 1.1. Let \( R \) be a left Artinian ring. Then every non-zero inner derivation of \( R \) is cofinite if and only if it satisfies one of the following conditions:

1. \( R \) is finite ring;
2. \( R \) is a commutative ring;
3. \( R = F \oplus D \) is a ring direct sum of a finite commutative ring \( F \) and a skew field \( D \) with cofinite non-zero inner derivations.
Recall that a ring $R$ with 1 is called \textit{semiprime} if it does not contains non-zero nilpotent ideals. A ring $R$ with an identity in which every non-zero ideal has a finite index is called \textit{residually finite} (see [2] and [10]).

\textbf{Theorem 1.2.} Let $R$ be a semiprime ring. Then all non-zero inner derivations are cofinite in $R$ if and only if it satisfies one of the following conditions:

1. $R$ is finite ring;
2. $R$ is a commutative ring;
3. $R = F \oplus B$ is a ring direct sum, where $F$ is a finite commutative semiprime ring and $B$ is a residually finite domain generated by all commutators $xa - ax$, where $a, x \in B$.

Throughout this paper for any ring $R$, $Z(R)$ will always denote the center, $Z_0 = Z_0(R)$ the ideal generated by all central ideals of $R$, $N(R)$ the set of all nilpotent elements of $R$, $\text{Der} R$ the set of all derivations of $R$, $\text{Im} d = d(R)$ the image and $\text{Ker} d$ the kernel of $d \in \text{Der} R$, $U(R)$ the unit group of $R$, $|R : I|$ the index of a subring $I$ in the additive group $R^+$, $\partial_x(a) = xa - ax = [x, a]$ the commutator of $a, x \in R$ and $C(R)$ the commutator ideal of $R$ (i.e., generated by all $[x, a]$). If $|R : I| < \infty$, then we say that $I$ has a finite index in $R$.

Any unexplained terminology is standard as in [6], [4], [5], [8] and [11].

\section{Some examples}
We begin with some examples of derivations in associative rings.

\textbf{Example 2.1.} Let $D$ be an infinite (skew) field,

$$
A = \begin{pmatrix}
a & 0 \\
0 & 0
\end{pmatrix},
X = \begin{pmatrix}
x & y \\
z & t
\end{pmatrix} \in M_2(D).
$$

Then we obtain that

$$
\partial_A(X) = AX -XA = \begin{pmatrix}
ax - xa & ay \\
-za & 0
\end{pmatrix},
$$

and so the image $\text{Im} \partial_A$ has an infinite index in $M_2(D)^+$. 

Recall that a ring $R$ having no non-zero derivations is called \textit{differentially trivial} [1].

\textbf{Example 2.2.} Let $F[X]$ be a commutative polynomial ring over a differentially trivial field $F$. Assume that $d$ is any derivation of $F[X]$. Then for every polynomial

$$
f = \sum_{i=0}^{n} a_iX^{n-i} \in F[X]
$$
we have
\[ d(f) = \left(\sum_{i=0}^{n-1} (n - i)a_iX^{n-i-1}\right)d(X) \in d(X)F[X], \]
where \(d(X)\) is some element from \(F[X]\). This means that the image \(\text{Im} \ d \subseteq d(X)F[X]\).

a) Let \(F\) be a field of characteristic 0. If we have
\[ g = \left(\sum_{i=0}^{m} b_iX^{m-i}\right) \cdot d(X) \in d(X)F[X], \]
then the following system
\[
\begin{aligned}
(1 + m)d_0 &= b_0, \\
md_1 &= b_1, \\
& \vdots \\
2d_{m-1} &= b_{m-1}, \\
d_m &= b_m,
\end{aligned}
\]
has a solution in \(F\), i.e., there exists such polynomial
\[ h = \sum_{i=0}^{m+1} d_iX^{m+1-i} \in F[X], \]
that \(d(h) = g\). This gives that \(\text{Im} \ d = d(X)F[X]\). If \(d\) is non-zero, then the additive quotient group
\[ G = F[X]/d(X)F[X] \]
is infinite and every non-zero derivation \(d\) of a commutative Noetherian ring \(F[X]\) is not cofinite.

b) Now assume that \(F\) has a prime characteristic \(p\) and \(d(X) = X\). If \(X^{p^l} - X^{p^s} \in \text{Im} \ d\) for some positive integer \(l, s\), where \(l > s\), then
\[ X^{p^l} - X^{p^s} = d(t) \]
for some polynomial \(t = d_0X^m + d_1X^{m-1} + \cdots + d_{m-1}X + d_m \in F[X]\) and consequently
\[ X^{p^l} - X^{p^s} = md_0X^m + (m - 1)d_1X^{m-1} + \cdots + 2d_{m-1}X^2 + d_mX. \]
Let \(k\) be the smallest non-negative integer such that
\[ (m - k)d_k \neq 0. \]
Then \(p^l = m - k\), a contradiction. This means that \(|F[X] : \text{Im} \ d| = \infty\).
Example 2.3. Let
\[ H = \{ \alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}, \]
\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j \]  
be the skew field of quaternions over the field \( \mathbb{R} \) of real numbers. Then
\[ \partial_i(H) = \{ \gamma j + \delta k \mid \gamma, \delta \in \mathbb{R} \} \]
and so the index \( |H : \text{Im} \partial_i| \) is infinite. Hence the inner derivation \( \partial_i \) is not cofinite in \( H \).

Example 2.4. Let \( D = F(y) \) be the rational functions field in a variable \( y \) over a field \( F \) and \( \sigma : D \to D \) be an automorphism of the \( F \)-algebra \( D \) such that
\[ \sigma(y) = y + 1. \]

By
\[ R = D((X; \sigma)) = \left\{ \sum_{i=n}^{\infty} a_i X^i \mid a_i \in D \text{ for all } i \geq n, \ n \in \mathbb{Z} \right\} \]
we denote the ring of skew Laurent power series with a multiplication induced by the rule
\[ (aX^k)(bX^l) = a\sigma^k(b)X^{k+l} \]
for any elements \( a, b \in D \). Then we compute the commutator
\[ \left[ \sum_{i=n}^{\infty} a_i X^i, y \right] = \sum_{i=n}^{\infty} a_i X^i y - y \sum_{i=n}^{\infty} a_i X^i \]
\[ = \sum_{i=n}^{\infty} a_i \sigma^i(y)X^i - \sum_{i=n}^{\infty} a_i yX^i \]
\[ = \sum_{i=n}^{\infty} a_i (\sigma^i(y) - y)X^i = \sum_{i=n}^{\infty} ia_i X^i. \]

If now
\[ f = \sum_{i=n}^{\infty} b_i X^i \in R, \]
then there exist elements \( a_i \in D \) such that
\[ b_i = ia_i \]
for any \( i \geq n \). This implies that the image \( \text{Im} \partial_y = R \) and \( \partial_y \) is a cofinite derivation of \( R \).

Lemma 2.5. Let \( R = F[X,Y] \) be a commutative polynomial ring in two variables \( X \) and \( Y \) over a field \( F \). Then \( R \) has a non-zero derivation that is not confinite.
Proof. Let us $f = \sum \alpha_{ij}X^iY^j \in R$ and $d : R \to R$ be a derivation defined by the rules

$$d(X) = X,$$
$$d(Y) = 0,$$
$$d(f) = \sum i\alpha_{ij}X^{i-1}Y^j d(X).$$

It is clear that $\text{Im } d \subseteq XR$ and $|R : XR| = \infty$. □

In the same way we can prove the following

**Lemma 2.6.** Let $R = F[\{X_\alpha\}_{\alpha \in \Lambda}]$ be a commutative polynomial ring in variables $\{X_\alpha\}_{\alpha \in \Lambda}$ over a field $F$. If $\text{card } \Lambda \geq 2$, then $R$ has a non-zero derivation that is not cofinite.

3. Cofinite inner derivations

**Lemma 3.1.** If every non-zero inner derivation of a ring $R$ is cofinite, then for each ideal $I$ of $R$ it holds that $I \subseteq Z(R)$ or $|R : I| < \infty$.

**Proof.** Indeed, if $I$ is a non-zero ideal of $R$ and $0 \neq a \in I$, then the image $\text{Im } \partial_a \subseteq I$. □

**Remark 3.2.** If $\delta$ is a cofinite derivation of an infinite ring $R$, then $|R : \text{Ker } \delta| = \infty$.

In fact, if the kernel $\text{Ker } \delta = \{a \in R \mid \delta(a) = 0\}$ has a finite index in $R$, in view of the group isomorphism

$$R^+ / \text{Ker } \delta \cong \text{Im } \delta,$$

we conclude that $\text{Im } \delta$ is a finite group.

**Lemma 3.3.** If $I$ is a central ideal of a ring $R$, then $C(R)I = (0)$.

**Proof.** For any elements $t, r \in R$ and $i \in I$ we have

$$(rt)i = r(ti) = (ti)r = t(ir) = t(ri) = (tr)i,$$

and therefore

$$(rt - tr)i = 0.$$

Hence $C(R)I = (0)$. □

**Lemma 3.4.** Let $R$ be a non-simple ring with all non-zero inner derivations to be cofinite. If all ideals of $R$ are central, then $R$ is commutative or finite.
Proof. a) If a ring $R$ is not local, then $R = M_1 + M_2 \subseteq Z(R)$ for any two different maximal ideals $M_1$ and $M_2$ of $R$.

b) Suppose that $R$ is a local ring and $J(R) \neq (0)$, where $J(R)$ is the Jacobson ideal of $R$. Then $J(R)C(R) = (0)$, $C(R) \neq R$ and, consequently,
$$C(R)^2 = (0).$$

If we assume that $R$ is not commutative, then
$$(0) \neq C(R) < R,$$
and so there exists an element $x \in R \setminus Z(R)$ such that
$$\{0\} \neq \text{Im} \partial_x \subseteq C(R).$$

Then $|R : C(R)| < \infty$. Since $C(R) \subseteq Z(R)$, we deduce that the index $|R : Z(R)|$ is finite. By Proposition 1 of [7], the commutator ideal $C(R)$ is finite and $R$ is also finite.

**Lemma 3.5.** If $N(R) \subseteq Z(R)$, then every idempotent is central in a ring $R$.

**Proof.** If $d \in \text{Der } R$ and $e = e^2 \in R$, then we obtain $d(e) = d(e)e + ed(e)$, and this implies that
$$ed(e)e = 0 \text{ and } d(e)e, ed(e) \in N(R).$$

Then $ed(e) = e^2d(e) = ed(e)e = 0$ and $d(e)e = 0$. As a consequence, $d(e) = 0$ and so $e \in Z(R)$. \hfill \square

**Lemma 3.6.** Let $R$ be a ring with all non-zero inner derivations to be cofinite.

Then one of the following conditions holds:

1. $R$ is a finite ring;
2. $R$ is a commutative ring;
3. $R$ contains a finite central ideal $Z_0$ such that $R/Z_0$ is an infinite residually finite ring (and, consequently, $R/Z_0$ is a prime ring with the ascending chain condition on ideals).

**Proof.** Assume that $R$ is an infinite ring which is not commutative and its every non-zero inner derivation is cofinite. Then $|R : C(R)| < \infty$ and every non-zero ideal of the quotient ring $B = R/Z_0$ has a finite index. If $B$ is finite (or respectively $C(R) \subseteq Z_0$), then $|R : Z(R)| < \infty$ and, by Proposition 1 of [7], the commutator ideal $C(R)$ is finite. From this it follows that a ring $R$ is finite, a contradiction. Hence $B$ is an infinite ring and $C(R)$ is not contained in $Z_0$. Since $Z_0C(R) = (0)$, we deduce that $Z_0$ is finite. By Corollary 2.2 and Theorem 2.3 from [2], $B$ is a prime ring with the ascending chain condition on ideals. \hfill \square

Let $D(R)$ be the subgroup of $R^+$ generated by all subgroups $d(R)$, where $d \in \text{Der } R$. 

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Corollary 3.7. Let $R$ be an infinite ring that is not commutative and with all non-zero derivations (respectively inner derivations) to be cofinite. Then either $R$ is a prime ring with the ascending chain condition on ideals or $Z_0$ is non-zero finite, $Z_0D(R) = (0)$, $D(R) \cap U(R) = \emptyset$ and $D(R)$ is a subgroup of finite index in $R^+$ (respectively $Z_0C(R) = (0)$, $C(R) \cap U(R) = \emptyset$ and $|R : C(R)| < \infty$).

Proof. We have $Z_0 \neq R$, $Z_0C(R) = (0)$ and the quotient $R/Z_0$ is an infinite prime ring with the ascending chain condition on ideals by Corollary 2.2 and Theorem 2.3 from [2]. By Lemma 3.6, $Z_0$ is finite. Assume that $Z_0 \neq (0)$. If $d$ is a non-zero derivation of $R$, then $Z_0d(R) \subseteq Z_0$ and so $Z_0d(R) = (0)$.

If we assume that $A = \text{ann}_l d(R)$ is infinite, then $A/Z_0$ is an infinite left ideal of $B$ with a non-zero annihilator, a contradiction with Lemma 2.1.1 from [6]. This gives that $A$ is finite and, consequently, $A = Z_0$.

Finally, if $u \in D(R) \cap U(R)$, then $Z_0 = uZ_0 = (0)$, a contradiction. \qed

Corollary 3.8. Let $R$ be a ring that is not prime. If $R$ contains an infinite subfield, then it has a non-zero derivation that is not cofinite.

Proof of Proposition 1.1. ($\Leftarrow$) It is clear.

($\Rightarrow$) Assume that $R$ is an infinite ring which is not commutative and its every non-zero inner derivation is cofinite. Then $Z_0 \neq R$ and $R/Z_0$ is an infinite prime ring by Lemma 3.6. Then $J(R) \subseteq Z_0$. Then

\[ R/Z_0 = \bigoplus_{i=1}^{m} M_{n_i}(D_i) \]

is a ring direct sum of finitely many full matrix rings $M_{n_i}(D_i)$ over skew fields $D_i$ ($i = 1, \ldots, m$) and so by applying Example 2.1 and Remark 3.2, we have that $R/Z_0 = F_1 \oplus D_1$ is a ring direct sum of a finite commutative ring $F_1$ and an infinite skew field $D_1$ that is not commutative. As a consequence of Proposition 1 from [8, §3.6] and Lemma 3.5,

\[ R = F \oplus D \]

is a ring direct sum of a finite ring $F$ and an infinite ring $D$. Then $F = Z_0$. \qed

4. Semiprime rings with cofinite inner derivations

Lemma 4.1. Let $R$ be a prime ring. If $R$ contains a non-zero proper commutative ideal $I$, then $R$ is commutative.

Proof. Assume that $C(R) \neq (0)$. Then for any elements $u \in R$ and $a, b \in I$ we have

\[ abu = a(bu) = (bu)a = b(ua) = uab \]

and so $ab \in Z(R)$. This gives that

\[ I^2 \subseteq Z(R) \]
and therefore
\[ I^2 C(R) = (0). \]
Since \( I^2 \neq (0) \), we obtain a contradiction with Lemma 2.1.1 of [6]. Hence \( R \) is commutative.

**Lemma 4.2.** Let \( R \) be a reduced ring (i.e. \( R \) has no non-zero nilpotent elements). If \( R \) contains a non-zero proper commutative ideal \( I \) such that the quotient ring \( R/I \) is commutative, then \( R \) is commutative.

**Proof.** Obviously, \( C(R) \leq I \) and \( I^2 \neq (0) \). If \( C(R) \neq (0) \), then, as in the proof of Lemma 4.1,
\[ C(R)^3 \leq I^2 C(R) = (0) \]
and thus \( C(R) = (0) \).

**Lemma 4.3.** If a ring \( R \) contains an infinite commutative ideal \( I \), then \( R \) is commutative or it has a non-zero derivation that is not cofinite.

**Proof.** Suppose that \( R \) is not commutative. If all non-zero derivations are cofinite in \( R \), then \( B = R/Z_0 \) is a prime ring by Lemma 3.6 and \( C(B) \neq (0) \). Therefore \( I^2 C(R) \subseteq Z_0 \) and, consequently, \( I \subseteq Z_0 \), a contradiction.

**Proof of Theorem 1.2.** (\( \Leftarrow \)) It is obviously.

(\( \Rightarrow \)) Suppose that \( R \) is an infinite ring which is not commutative and its every non-zero inner derivation is cofinite. Then \( B = R/Z_0 \) is a prime ring satisfying the ascending chain condition on ideals.

Assume that \( B \) is not a domain. By Proposition 2.2.14 of [11],
\[ \text{ann}_l b = \text{ann}_r a = \text{ann} b \]
is a two-sided ideal for any \( b \in B \), and by Lemma 2.3.2 from [11], each maximal right annihilator in \( B \) has the form \( \text{ann}_r a \) for some \( 0 \neq a \in B \). Then \( \text{ann}_r a \) is a prime ideal. Since \( |B : \text{ann}_r a| \) is finite, left and right ideals \( Ba, aB \) are finite and this gives a contradiction. Hence \( B \) is a domain.

Now assume that \( Z_0 \neq (0) \). In view of Corollary to Proposition 5 from [8, §3.5] we conclude that \( Z_0 \) is not nilpotent. As a consequence of Lemma 3 from [9] and Lemma 3.5,
\[ R = Z_0 \oplus B_1 \]
is a ring direct sum with a ring \( B_1 \) isomorphic to \( B \).

**Remark 4.4.** If \( R \) is a ring with all non-zero inner derivations to be cofinite and \( R/Z_0 \) is an infinite simple ring, then \( R = Z_0 \oplus B \) is a ring direct sum of a finite central ideal \( Z_0 \) and a simple non-commutative ring \( B \).

**Problem 4.5.** Characterize domains and, in particular, skew fields with all non-zero derivations (respectively inner derivations) to be cofinite.
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References


