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On weighted averages of double sequences

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Dedicated to Mátyás Arató on his eightieth birthday

1. Introduction

The well known Kolmogorov strong law of large numbers states the following. If $X_1, X_2, \dots$ are independent identically distributed (i.i.d.) random variables with finite expectation and $\mathrm{E}\, X_1 = 0$, then the average $(X_1 + \dots + X_n)/n$ converges to 0 almost surely (a.s.). However, if we consider a double sequence, then we need another condition. Actually, if (X_{ij}) is a double sequence of i.i.d. random variables with $\mathrm{E}\, X_{11} = 0$, then $\mathrm{E}\, |X_{11}| \log^+ |X_{11}| < \infty$ implies that $\left(\sum_{i=1}^m \sum_{j=1}^n X_{ij}\right)/(mn)$ converges to 0 a.s., as n, m tend to infinity (see Smythe [6]).

For a double numerical sequence x_{ij} there are different notions of convergences. One can consider a strong version of convergence when x_{ij} converges as one of the indices i, j goes to infinity (this type of convergence was used in Fazekas [1]). Another version when x_{ij} converges as both indices i, j tend to infinity. However, in the second case convergence does not imply boundedness. To avoid unpleasant situations one can assume that the sequence is bounded. In this paper we shall study the so called bounded convergence of double sequences.

We shall prove two criteria for the bounded convergence of weighted averages of double sequences. Both criteria are based on subsequences. The subsequence is constructed by a well-known method: we proceed along a non-negative, increasing, unbounded sequence and pick up a member which is about the double

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of the previous selected member of the sequence. (This method was applied e.g. in Fazekas–Klesov [2]). However, this method is not convenient for an arbitrary double sequence of weights. Therefore we apply weights of product type (it was considered e.g. in Noszály–Tómács [5]).

Our theorems can be considered as generalizations of some results in Fekete–Georgieva–Móricz [3], where harmonic averages of double sequences were considered. They obtained the following theorem.

$$\frac{1}{\ln m \ln n} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{x_{ij}}{ij} \xrightarrow{\text{b}} L, \quad \text{as} \quad m, n \to \infty \tag{1.1}$$

if and only if

$$\frac{1}{2^{m+n}} \max_{\substack{2^{2^{m-1}} < k \le 2^{2^m} \\ 2^{2^{n-1}} < l \le 2^{2^n}}} \left| \sum_{i=2^{2^{m-1}}+1}^{k} \sum_{j=2^{2^{n-1}}+1}^{l} \frac{x_{ij} - L}{ij} \right| \xrightarrow{\text{b}} 0, \text{ as } m, n \to \infty. \tag{1.2}$$

Here $\xrightarrow{\text{b}}$ means the bounded convergence. Our Theorem 2.4 is a generalization of this result for general weights.

Our results can also be considered as extensions of certain theorems of Móricz and Stadtmüller [4] where ordinary (that is not double) sequences were studied. In our proofs we apply ideas of [4].

2. Main results

Let $(x_{kl} : k, l = 1, 2, \dots)$ be a sequence of real numbers, and let $(b_k : k = 1, 2, \dots)$, $(c_l : l = 1, 2, \dots)$ be sequences of weights, that is, sequences of non-negative numbers for which

$$B_m := \sum_{k=1}^{m} b_k \to \infty, \quad \text{as} \quad m \to \infty, \tag{2.1}$$

$$C_n := \sum_{l=1}^{n} c_l \to \infty, \quad \text{as} \quad n \to \infty. \tag{2.2}$$

Let $a_{kl} := b_k c_l$, $A_{mn} := \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}$ and $S_{mn} := \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} x_{kl}$. The *weighted averages* Z_{mn} of the sequence (x_{kl}) with respect to the weights (a_{kl}) are defined by

$$Z_{mn} := \frac{1}{A_{mn}} S_{mn}$$

for n, m large enough so that $A_{mn} > 0$.

We define a sequence $m_0 = 0, m_1 = 1 < m_2 < m_3 < \dots$ of integers with the following property

$$B_{m_{i+1}-1} < 2B_{m_i} \le B_{m_{i+1}}, \quad i = 1, 2, \dots \tag{2.3}$$

Similarly, let $n_0 = 0, n_1 = 1 < n_2 < n_3 < \dots$ be a sequence of integers such that

$$C_{n_{j+1}-1} < 2C_{n_j} \le C_{n_{j+1}}, \quad j = 1, 2, \dots \tag{2.4}$$

In this paper we shall also use the following notation

$$\Delta_{st}^{mn} A := \sum_{k=s+1}^{m} \sum_{l=t+1}^{n} a_{kl}, \quad \Delta_{st}^{mn} S := \sum_{k=s+1}^{m} \sum_{l=t+1}^{n} a_{kl} x_{kl}.$$

Actually $\Delta_{st}^{mn} A$ is an increment on a rectangle (in other word two-dimensional difference) of the sequence A_{mn}. We note that

$$\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \Delta_{m_i n_j}^{m_{i+1} n_{j+1}} S$$

is called the *moving average* of the sequence (x_{kl}) with respect to the weights (a_{kl}).

Definition 2.1. Let $(y_{kl} : k, l = 1, 2, \dots)$ be a sequence of real numbers, and let y be a real number. It is said that *bounded convergence*

$$y_{kl} \xrightarrow{\text{b}} y, \quad \text{as} \quad k, l \to \infty,$$

is satisfied if
(i) the sequence $(y_{kl} : k, l = 1, 2, \dots)$ is bounded; and
(ii) for every $\varepsilon > 0$ there exist positive integers k_0, l_0, such that

$$|y_{kl} - y| < \varepsilon \quad \text{for} \quad k \ge k_0, l \ge l_0. \tag{2.5}$$

Remark 2.2. Relation (2.5) does not imply that (y_{kl}) is bounded. For example if $y_{1l} = l$ for $l \ge 1$ and $y_{kl} = y$ for $k \ge 2, l \ge 1$, then (2.5) holds but (y_{kl}) is unbounded.

Theorem 2.3. *Suppose that conditions* (2.1) *and* (2.2) *are satisfied. Then for some constant* L*, we have*

$$Z_{m_i n_j} \xrightarrow{\text{b}} L, \quad \textit{as} \quad i, j \to \infty \tag{2.6}$$

if and only if

$$\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \Delta_{m_i n_j}^{m_{i+1} n_{j+1}} S \xrightarrow{\text{b}} L, \quad \textit{as} \quad i, j \to \infty, \tag{2.7}$$

where the sequences (m_i) *and* (n_j) *are defined in* (2.3) *and* (2.4)*.*

Theorem 2.4. *Assume that* $B_m/b_m \ge 1 + \delta$ *and* $C_m/c_m \ge 1 + \delta$ *for* m *being large enough where* $\delta > 0$*. Assume that conditions* (2.1), (2.2) *are satisfied. Then for some constant* L*, we have*

$$Z_{mn} \xrightarrow{\text{b}} L, \quad \textit{as} \quad m, n \to \infty \tag{2.8}$$

if and only if

$$\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \max_{\substack{m_i < m \le m_{i+1} \\ n_j < n \le n_{j+1}}} \left| \sum_{k=m_i+1}^{m} \sum_{l=n_j+1}^{n} a_{kl}(x_{kl} - L) \right| \xrightarrow{\text{b}} 0, \text{ as } i, j \to \infty, \tag{2.9}$$

where the sequences (m_i) *and* (n_j) *are defined in* (2.3) *and* (2.4).

The following two corollaries characterize the strong law of large numbers for weighted averages of a sequence of random variables with two-dimensional indices. These corollaries are consequences of Theorem 2.3 and 2.4.

Corollary 2.5. *Let* $(X_{kl} : k, l = 1, 2, \dots)$ *be a sequence of random variables. If conditions* (2.1) *and* (2.2) *are satisfied, then for some constant* L, *we have*

$$\frac{1}{A_{m_i n_j}} \sum_{k=1}^{m_i} \sum_{l=1}^{n_j} a_{kl} X_{kl} \xrightarrow{\text{b}} L, \quad \text{as} \quad i, j \to \infty \quad \text{a.s.}$$

if and only if

$$\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \sum_{k=m_i+1}^{m_{i+1}} \sum_{l=n_j+1}^{n_{j+1}} a_{kl} X_{kl} \xrightarrow{\text{b}} L, \quad \text{as} \quad i, j \to \infty \quad \text{a.s.},$$

where the sequences (m_i) *and* (n_j) *are defined in* (2.3) *and* (2.4).

Corollary 2.6. *Let* $(X_{kl} : k, l = 1, 2, \dots)$ *be a sequence of random variables. Assume that* $B_m/b_m \ge 1 + \delta$ *and* $C_m/c_m \ge 1 + \delta$ *for* m *being large enough where* $\delta > 0$. *Assume that conditions* (2.1) *and* (2.2) *are satisfied. Then for some constant* L, *we have*

$$\frac{1}{A_{mn}} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} X_{kl} \xrightarrow{\text{b}} L, \quad \text{as} \quad m, n \to \infty \quad \text{a.s.}$$

if and only if

$$\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \max_{\substack{m_i < m \le m_{i+1} \\ n_j < n \le n_{j+1}}} \left| \sum_{k=m_i+1}^{m} \sum_{l=n_j+1}^{n} a_{kl}(X_{kl} - L) \right| \xrightarrow{\text{b}} 0, \text{ as } i, j \to \infty \quad \text{a.s.},$$

where the sequences (m_i) *and* (n_j) *are defined in* (2.3) *and* (2.4).

Remark 2.7. In the above two corollaries L can be an a.s. finite random variable, as well.

Remark 2.8. The results of this section can be generalized for sequences with d-dimensional indices.

3. Proofs of Theorems 2.3 and 2.4

Proof of Theorem 2.3. Let ε be a fixed positive real number. First we prove the necessity. Assume that (2.6) is satisfied, that is, there exist integers i_0, j_0 such that

$$\left|Z_{m_i n_j} - L\right| < \varepsilon \quad \text{for all} \quad i \geq i_0, j \geq j_0,$$

furthermore $(Z_{m_i n_j})$ is a bounded sequence. So, if $i \geq i_0, j \geq j_0$, then we have

$$\left|\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \Delta_{m_i n_j}^{m_{i+1} n_{j+1}} S - L\right| = \frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \left|\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} S - L \Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A\right|$$

$$= \frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \big|(S_{m_{i+1} n_{j+1}} - LA_{m_{i+1} n_{j+1}}) - (S_{m_i n_{j+1}} - LA_{m_i n_{j+1}})$$

$$- (S_{m_{i+1} n_j} - LA_{m_{i+1} n_j}) + (S_{m_i n_j} - LA_{m_i n_j})\big|$$

$$\leq \frac{A_{m_{i+1} n_{j+1}}}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \left(|Z_{m_{i+1} n_{j+1}} - L| + |Z_{m_i n_{j+1}} - L| + |Z_{m_{i+1} n_j} - L| + |Z_{m_i n_j} - L|\right)$$

$$< 4\varepsilon \frac{A_{m_{i+1} n_{j+1}}}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} = 4\varepsilon \frac{B_{m_{i+1}}}{B_{m_{i+1}} - B_{m_i}} \frac{C_{n_{j+1}}}{C_{n_{j+1}} - C_{n_j}} \leq 16\varepsilon. \tag{3.1}$$

Now, turn to the boundedness. Similarly as above

$$\left|\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \Delta_{m_i n_j}^{m_{i+1} n_{j+1}} S\right| \leq \frac{B_{m_{i+1}}}{B_{m_{i+1}} - B_{m_i}} \frac{C_{n_{j+1}}}{C_{n_{j+1}} - C_{n_j}} \big(|Z_{m_{i+1} n_{j+1}}| + |Z_{m_i n_{j+1}}|$$

$$+ |Z_{m_{i+1} n_j}| + |Z_{m_i n_j}|\big) \leq \text{const.}, \tag{3.2}$$

because $(Z_{m_i n_j})$ is bounded. Inequalities (3.1) and (3.2) imply (2.7).

Now, we turn to sufficiency. Assume that (2.7) is satisfied, that is, there exist integers i_0, j_0 such that

$$\left|\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \Delta_{m_i n_j}^{m_{i+1} n_{j+1}} S - L\right| < \varepsilon \quad \text{for all} \quad i \geq i_0, j \geq j_0, \tag{3.3}$$

furthermore $\left(\frac{1}{\Delta_{m_i n_j}^{m_{i+1} n_{j+1}} A} \Delta_{m_i n_j}^{m_{i+1} n_{j+1}} S\right)$ is a bounded sequence. If $i \geq i_0$ and $j \geq j_0$, then $m_{i+1} > m_{i_0}$ and $n_{j+1} > n_{j_0}$, so

$$Z_{m_{i+1} n_{j+1}} - L$$

$$= \frac{1}{A_{m_{i+1} n_{j+1}}} (S_{m_{i+1} n_{j+1}} - LA_{m_{i+1} n_{j+1}}) = \frac{1}{A_{m_{i+1} n_{j+1}}} \sum_{k=1}^{m_{i+1}} \sum_{l=1}^{n_{j+1}} a_{kl}(x_{kl} - L)$$

$$= \frac{1}{A_{m_{i+1} n_{j+1}}} \Bigg(\sum_{k=1}^{m_{i_0}} \sum_{l=1}^{n_{j_0}} a_{kl}(x_{kl} - L) + \sum_{k=m_{i_0}+1}^{m_{i+1}} \sum_{l=n_{j_0}+1}^{n_{j+1}} a_{kl}(x_{kl} - L)$$

$$+\sum_{k=1}^{m_{i_0}}\sum_{l=n_{j_0}+1}^{n_{j+1}} a_{kl}(x_{kl}-L)+\sum_{k=m_{i_0}+1}^{m_{i+1}}\sum_{l=1}^{n_{j_0}} a_{kl}(x_{kl}-L)\Bigg) \tag{3.4}$$

for all $i\geq i_0, j\geq j_0$.

Consider the first term in (3.4). Since $\frac{1}{A_{m_{i+1}n_{j+1}}}\to 0$, as $i\to\infty, j\to\infty$, then there exist integers $i_1\geq i_0$ and $j_1\geq j_0$, such that

$$\frac{1}{A_{m_{i+1}n_{j+1}}}\left|\sum_{k=1}^{m_{i_0}}\sum_{l=1}^{n_{j_0}} a_{kl}(x_{kl}-L)\right|<\varepsilon \quad \text{for all} \quad i\geq i_1, j\geq j_1. \tag{3.5}$$

Now, turn to the secont term in (3.4). If $i\geq k$, then

$$\frac{B_{m_{k+1}}-B_{m_k}}{B_{m_{i+1}}}=\frac{B_{m_{k+1}}-B_{m_k}}{B_{m_{k+1}}}\frac{B_{m_{k+1}}}{B_{m_{k+2}}}\frac{B_{m_{k+2}}}{B_{m_{k+3}}}\cdot\dots\cdot\frac{B_{m_i}}{B_{m_{i+1}}}\leq\left(\frac{1}{2}\right)^{i-k}.$$

Similarly, if $j\geq l$, then

$$\frac{C_{n_{l+1}}-C_{n_l}}{C_{n_{j+1}}}\leq\left(\frac{1}{2}\right)^{j-l}.$$

Hence we get from (3.3)

$$\begin{aligned}
&\frac{1}{A_{m_{i+1}n_{j+1}}}\left|\sum_{k=m_{i_0}+1}^{m_{i+1}}\sum_{l=n_{j_0}+1}^{n_{j+1}} a_{kl}(x_{kl}-L)\right|\\
&=\frac{1}{A_{m_{i+1}n_{j+1}}}\left|\sum_{k=i_0}^{i}\sum_{l=j_0}^{j}\sum_{s=m_k+1}^{m_{k+1}}\sum_{t=n_l+1}^{n_{l+1}} a_{st}(x_{st}-L)\right|\\
&=\left|\sum_{k=i_0}^{i}\sum_{l=j_0}^{j}\frac{B_{m_{k+1}}-B_{m_k}}{B_{m_{i+1}}}\frac{C_{n_{l+1}}-C_{n_l}}{C_{n_{j+1}}}\left(\frac{1}{\Delta_{m_k n_l}^{m_{k+1}n_{l+1}}A}\Delta_{m_k n_l}^{m_{k+1}n_{l+1}}S-L\right)\right|\\
&<\varepsilon\sum_{k=i_0}^{i}\left(\frac{1}{2}\right)^{i-k}\sum_{l=j_0}^{j}\left(\frac{1}{2}\right)^{j-l}<4\varepsilon \quad \text{for all} \quad i\geq i_0, j\geq j_0.
\end{aligned} \tag{3.6}$$

For the third term in (3.4) we have

$$\begin{aligned}
&\frac{1}{A_{m_{i+1}n_{j+1}}}\left|\sum_{k=1}^{m_{i_0}}\sum_{l=n_{j_0}+1}^{n_{j+1}} a_{kl}(x_{kl}-L)\right|\\
&=\frac{1}{A_{m_{i+1}n_{j+1}}}\left|\sum_{k=0}^{i_0-1}\sum_{l=j_0}^{j}\sum_{s=m_k+1}^{m_{k+1}}\sum_{t=n_l+1}^{n_{l+1}} a_{st}(x_{st}-L)\right|\\
&=\left|\sum_{k=0}^{i_0-1}\sum_{l=j_0}^{j}\frac{B_{m_{k+1}}-B_{m_k}}{B_{m_{i+1}}}\frac{C_{n_{l+1}}-C_{n_l}}{C_{n_{j+1}}}\left(\frac{1}{\Delta_{m_k n_l}^{m_{k+1}n_{l+1}}A}\Delta_{m_k n_l}^{m_{k+1}n_{l+1}}S-L\right)\right|
\end{aligned}$$

$$\leq \frac{1}{B_{m_{i+1}}} \sum_{k=0}^{i_0-1} (B_{m_{k+1}} - B_{m_k}) \sum_{l=j_0}^{j} \left(\frac{1}{2}\right)^{j-l} \text{const.}$$

$$\leq \text{const.} \frac{1}{B_{m_{i+1}}} B_{m_{i_0}} \sum_{k=0}^{i_0-1} \frac{B_{m_{k+1}} - B_{m_k}}{B_{m_{i_0}}} \leq \text{const.} \frac{B_{m_{i_0}}}{B_{m_{i+1}}} \sum_{k=0}^{i_0-1} \left(\frac{1}{2}\right)^{i_0-1-k}$$

$$\leq \text{const.} \frac{B_{m_{i_0}}}{B_{m_{i+1}}} \to 0, \quad \text{as} \quad i \to \infty.$$

Hence, there exists $i_2 \geq i_1$ such that

$$\frac{1}{A_{m_{i+1}n_{j+1}}} \left| \sum_{k=1}^{m_{i_0}} \sum_{l=n_{j_0}+1}^{n_{j+1}} a_{kl}(x_{kl} - L) \right| < \varepsilon \quad \text{for all} \quad i \geq i_2, j \geq j_0. \tag{3.7}$$

Similarly, for the fourth term in (3.4) we obtain that there exists $j_2 \geq j_1$ such that

$$\frac{1}{A_{m_{i+1}n_{j+1}}} \left| \sum_{k=m_{i_0}+1}^{m_{i+1}} \sum_{l=1}^{n_{j_0}} a_{kl}(x_{kl} - L) \right| < \varepsilon \quad \text{for all} \quad i \geq i_0, j \geq j_2. \tag{3.8}$$

By (3.4)–(3.8), we have

$$|Z_{m_{i+1}n_{j+1}} - L| < 7\varepsilon \quad \text{for all} \quad i \geq i_2, j \geq j_2. \tag{3.9}$$

Finally, turn to the proof of boundedness.

$$|Z_{m_i n_j}| = \frac{1}{A_{m_i n_j}} \left| \sum_{k=1}^{m_i} \sum_{l=1}^{n_j} a_{kl} x_{kl} \right| = \frac{1}{A_{m_i n_j}} \left| \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \Delta_{m_k n_l}^{m_{k+1} n_{l+1}} S \right|$$

$$= \left| \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \frac{B_{m_{k+1}} - B_{m_k}}{B_{m_i}} \frac{C_{n_{l+1}} - C_{n_l}}{C_{n_j}} \frac{1}{\Delta_{m_k n_l}^{m_{k+1} n_{l+1}} A} \Delta_{m_k n_l}^{m_{k+1} n_{l+1}} S \right|$$

$$\leq \text{const.} \sum_{k=0}^{i-1} \frac{B_{m_{k+1}} - B_{m_k}}{B_{m_i}} \sum_{l=0}^{j-1} \frac{C_{n_{l+1}} - C_{n_l}}{C_{n_j}} \leq 4 \cdot \text{const.}$$

This inequality and (3.9) imply (2.6). Thus the theorem is proved. □

Proof of Theorem 2.4. Let ε be a fixed positive real number. First we prove the necessity. Assume that (2.8) is satisfied, that is, there exist integers M_0, N_0 such that

$$|Z_{mn} - L| < \varepsilon \quad \text{for all} \quad m \geq M_0, n \geq N_0, \tag{3.10}$$

furthermore (Z_{mn}) is a bounded sequence. Since we have

$$\sum_{k=m_i+1}^{m} \sum_{l=n_j+1}^{n} a_{kl}(x_{kl} - L) = A_{mn}(Z_{mn} - L) - A_{m_i n}(Z_{m_i n} - L)$$

$$- A_{mn_j}(Z_{mn_j} - L) + A_{m_i n_j}(Z_{m_i n_j} - L),$$

if $m > m_i$ and $n > n_j$, hence the ratio on the left-hand side in (2.9) is less than or equal to

$$\frac{A_{m_{i+1}n_{j+1}}}{\Delta_{m_i n_j}^{m_{i+1}n_{j+1}} A} \left(\max_{\substack{m_i < m \le m_{i+1} \\ n_j < n \le n_{j+1}}} |Z_{mn} - L| + \max_{n_j < n \le n_{j+1}} |Z_{m_i n} - L| \right.$$
$$\left. + \max_{m_i < m \le m_{i+1}} |Z_{mn_j} - L| + |Z_{m_i n_j} - L| \right). \tag{3.11}$$

There exist integers i_0, j_0 such that if $i \ge i_0$ and $j \ge j_0$, than $m_i \ge M_0$ and $n_j \ge N_0$. So (3.10) and (3.11) imply, that the ratio on the left-hand side in (2.9) is less than

$$\frac{A_{m_{i+1}n_{j+1}}}{\Delta_{m_i n_j}^{m_{i+1}n_{j+1}} A} 4\varepsilon \le 16\varepsilon \quad \text{for all} \quad i \ge i_0, j \ge j_0. \tag{3.12}$$

On the other hand, since (Z_{mn}) is a bounded sequence, so by (3.11), the ratio on the left-hand side in (2.9) is less than or equal to

$$\frac{A_{m_{i+1}n_{j+1}}}{\Delta_{m_i n_j}^{m_{i+1}n_{j+1}} A} 4 \cdot \text{const.} \le 16 \cdot \text{const.} \quad \text{for all} \quad i, j.$$

This fact and (3.12) imply (2.9).

Now we turn to sufficiency. Assume that (2.9) is satisfied. The ratio on the left-hand side in (2.9) is greater than or equal to

$$\frac{1}{\Delta_{m_i n_j}^{m_{i+1}n_{j+1}} A} \left| \sum_{k=m_i+1}^{m_{i+1}} \sum_{l=n_j+1}^{n_{j+1}} a_{kl}(x_{kl} - L) \right| = \left| \frac{1}{\Delta_{m_i n_j}^{m_{i+1}n_{j+1}} A} \Delta_{m_i n_j}^{m_{i+1}n_{j+1}} S - L \right|,$$

so (2.7) is satisfied. Now, applying Theorem 2.3, we get that (2.6) is true. In the following parts of the proof, for fixed integers m, n let i, j be integers, such that

$$m_i < m \le m_{i+1} \quad \text{and} \quad n_j < n \le n_{j+1}.$$

We have

$$Z_{mn} - L = \frac{1}{A_{mn}} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}(x_{kl} - L)$$
$$= \frac{1}{A_{mn}} \sum_{k=1}^{m_i} \sum_{l=1}^{n_j} a_{kl}(x_{kl} - L) + \frac{1}{A_{mn}} \sum_{k=m_i+1}^{m} \sum_{l=n_j+1}^{n} a_{kl}(x_{kl} - L)$$
$$+ \frac{1}{A_{mn}} \sum_{k=m_i+1}^{m} \sum_{l=1}^{n_j} a_{kl}(x_{kl} - L) + \frac{1}{A_{mn}} \sum_{k=1}^{m_i} \sum_{l=n_j+1}^{n} a_{kl}(x_{kl} - L). \tag{3.13}$$

Consider the absolute values of all terms of this sum. For the first term, from (2.6) we get that

$$\begin{aligned}
&\frac{1}{A_{mn}}\left|\sum_{k=1}^{m_i}\sum_{l=1}^{n_j} a_{kl}(x_{kl}-L)\right| \\
&= \frac{A_{m_in_j}}{A_{mn}}|Z_{m_in_j}-L| \le |Z_{m_in_j}-L| \xrightarrow{\mathrm{b}} 0, \quad \text{as} \quad m,n\to\infty. \qquad (3.14)
\end{aligned}$$

We shall use the following relations for the coefficients.

$$\begin{aligned}
&\frac{\Delta_{m_in_j}^{m_{i+1}n_{j+1}}A}{A_{mn}} = \frac{(B_{m_{i+1}}-B_{m_i})(C_{n_{j+1}}-C_{n_j})}{B_mC_n} \le \frac{B_{m_{i+1}}}{B_{m_i+1}}\frac{C_{n_{j+1}}}{C_{n_j+1}} \\
&= \frac{B_{m_{i+1}-1}}{B_{m_i+1}}\left(1+\frac{b_{m_{i+1}}}{B_{m_{i+1}-1}}\right)\frac{C_{n_{j+1}-1}}{C_{n_j+1}}\left(1+\frac{c_{n_{j+1}}}{C_{n_{j+1}-1}}\right) \\
&\le 4\left(1+\frac{b_{m_{i+1}}}{B_{m_{i+1}-1}}\right)\left(1+\frac{c_{n_{j+1}}}{C_{n_{j+1}-1}}\right) \le \text{const.} \qquad (3.15)
\end{aligned}$$

To see the above relation, we mention that

$$\frac{B_{m-1}}{b_m}+1 = \frac{B_{m-1}+b_m}{b_m} = \frac{B_m}{b_m} \ge 1+\delta,$$

because of the assumptions of the theorem. Therefore (b_m/B_{m-1}) is a bounded sequence. Similarly (c_n/C_{n-1}) is a bounded sequence, too.

Consider the second term in (3.13). From (3.15) and (2.9) we get that

$$\begin{aligned}
&\frac{1}{A_{mn}}\left|\sum_{k=m_i+1}^{m}\sum_{l=n_j+1}^{n} a_{kl}(x_{kl}-L)\right| \\
&\le \frac{\Delta_{m_in_j}^{m_{i+1}n_{j+1}}A}{A_{mn}}\frac{1}{\Delta_{m_in_j}^{m_{i+1}n_{j+1}}A}\max_{\substack{m_i<t\le m_{i+1}\\ n_j<s\le n_{j+1}}}\left|\sum_{k=m_i+1}^{t}\sum_{l=n_j+1}^{s} a_{kl}(x_{kl}-L)\right| \xrightarrow{\mathrm{b}} 0, \\
&\text{as} \quad m,n\to\infty. \qquad (3.16)
\end{aligned}$$

Now turn to the third and fourth terms on the left hand side of (3.13). With notation

$$\Phi_{it} := \frac{1}{\Delta_{m_in_{t-1}}^{m_{i+1}n_t}A}\max_{m_i<s\le m_{i+1}}\left|\sum_{k=m_i+1}^{s}\sum_{l=n_{t-1}+1}^{n_t} a_{kl}(x_{kl}-L)\right|$$

we get that

$$\frac{1}{A_{mn}}\left|\sum_{k=m_i+1}^{m}\sum_{l=1}^{n_j} a_{kl}(x_{kl}-L)\right| \le \frac{1}{A_{mn}}\sum_{t=1}^{j}\left|\sum_{k=m_i+1}^{m}\sum_{l=n_{t-1}+1}^{n_t} a_{kl}(x_{kl}-L)\right|$$

$$\leq \frac{1}{A_{mn}} \sum_{t=1}^{j} \Delta_{m_i n_{t-1}}^{m_{i+1} n_t} A\, \Phi_{it} \leq \frac{B_{m_{i+1}} - B_{m_i}}{B_{m_i+1}} \sum_{t=1}^{j} \frac{C_{n_t} - C_{n_{t-1}}}{C_{n_j+1}} \Phi_{it}. \tag{3.17}$$

But

$$\frac{B_{m_{i+1}} - B_{m_i}}{B_{m_i+1}} < \frac{b_{m_{i+1}} + B_{m_i}}{B_{m_i+1}} < 1 + \frac{B_{m_{i+1}-1}}{B_{m_i}} \frac{b_{m_{i+1}}}{B_{m_{i+1}-1}} < 1 + 2\frac{b_{m_{i+1}}}{B_{m_{i+1}-1}},$$

which is bounded as we have already seen. Furthermore, for $t = 1, 2, \ldots, j$,

$$\frac{C_{n_t} - C_{n_{t-1}}}{C_{n_j+1}} = \frac{C_{n_t} - C_{n_{t-1}}}{C_{n_t}} \frac{C_{n_t}}{C_{n_{t+1}}} \frac{C_{n_{t+1}}}{C_{n_{t+2}}} \cdots \frac{C_{n_{j-1}}}{C_{n_j}} \frac{C_{n_j}}{C_{n_j+1}} \leq \left(\frac{1}{2}\right)^{j-t+1}.$$

Hence (3.17) implies that

$$\frac{1}{A_{mn}} \left| \sum_{k=m_i+1}^{m} \sum_{l=1}^{n_j} a_{kl}(x_{kl} - L) \right| \leq \text{const.} \sum_{t=1}^{j} \left(\frac{1}{2}\right)^{j-t} \Phi_{it}. \tag{3.18}$$

By (2.9), $\Phi_{it} \xrightarrow{\text{b}} 0$. This and (3.18) imply that the expression on the left-hand side in (3.18) is bounded. Moreover, there exist i_0, j_0 such that $\Phi_{it} < \varepsilon$ and at the same time $(1/2)^t < \varepsilon$ for all $i \geq i_0, t \geq j_0$. From these facts and applying that the sequence Φ_{it} is bounded, we get

$$\begin{aligned} &\sum_{t=1}^{j} \left(\frac{1}{2}\right)^{j-t} \Phi_{it} = \sum_{t=1}^{j_0} \left(\frac{1}{2}\right)^{j-t} \Phi_{it} + \sum_{t=j_0+1}^{j} \left(\frac{1}{2}\right)^{j-t} \Phi_{it} \\ &< \text{const.} \left(\frac{1}{2}\right)^{j/2} \sum_{t=1}^{j_0} \left(\frac{1}{2}\right)^{j/2-t} + 2\varepsilon < \text{const.}\varepsilon \quad \text{for all} \quad i \geq i_0, j \geq 2j_0. \end{aligned}$$

So it follows from (3.18) that

$$\frac{1}{A_{mn}} \left| \sum_{k=m_i+1}^{m} \sum_{l=1}^{n_j} a_{kl}(x_{kl} - L) \right| \xrightarrow{\text{b}} 0, \quad \text{as} \quad m, n \to \infty. \tag{3.19}$$

By similar arguments, for the fourth term in (3.13), we have

$$\frac{1}{A_{mn}} \left| \sum_{k=1}^{m_i} \sum_{l=n_j+1}^{n} a_{kl}(x_{kl} - L) \right| \xrightarrow{\text{b}} 0, \quad \text{as} \quad m, n \to \infty. \tag{3.20}$$

Finally (3.13), (3.14), (3.16), (3.19) and (3.20) imply (2.8). Thus the theorem is proved. □

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