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Experiments on the distance of two-dimensional samples

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

The distance of two-dimensional samples is studied. The distance is based on the optimal matching method. Simulation results are obtained when the samples are drawn from normal and uniform distributions.

Keywords: Optimal matching, simulation, Gaussian distribution, goodness of fit, general extreme value distribution.

MSC: 62E17, 62H10

1. Introduction

A well-known result in optimal matchings is the following (see Ajtai-Komlós-Tusnády [1]). Assume that both $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are independent identically distributed (i.i.d.) random variables with uniform distribution on the two-dimensional unit square. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be independent of each other. Let

$$t_n = \min_\pi \sum_{i=1}^{n} ||X_{\pi(i)} - Y_i||, \tag{1.1}$$

where the minimum is taken over all permutations π of the first n positive integers. Then

$$C_1(n \log n)^{1/2} < t_n < C_2(n \log n)^{1/2} \tag{1.2}$$

with probability $1 - \mathrm{o}(1)$ (Theorem in [1]). t_n is the so-called transportation cost. Talagrand in [6] explains the specific feature of the two-dimensional case. In [7] it is explained that the transportation cost is closely related to the empirical process. So the following question arises. Can t_n serve as the basis of testing goodness of fit? Therefore to find the distribution of t_n is an interesting task. That problem was suggested by G. Tusnády.

Testing multidimensional normality is an important task in statistics (see e.g. [4]). In this paper we study a particular case of this problem. We study the fit to two-dimensional standard normality. The main idea is the following. Assume that we want to test if a random sample $X_1, \dots, X_n$ is drawn from a population with distribution F. We generate another sample $Y_1, \dots, Y_n$ from the distribution F. Then we try to find for any X_i a similar member of the sample $Y_1, \dots, Y_n$. We hope that the optimal matching of the two samples gives a reasonable statistic to test the goodness of fit.

In this paper we concentrate on three cases, that is when both $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are standard normal, then both of them are uniform, finally when $X_1, \dots, X_n$ are normal and $Y_1, \dots, Y_n$ are uniform. We calculate the distances of the samples, then we find the statistical characteristics of the distances. The quantiles can serve as critical values of a goodness of fit test. Finally, we show some results on the distribution of our test statistic.

We use the classical notion of sample, i.e. $X_1, \dots, X_n$ is called a sample if $X_1, \dots, X_n$ are i.i.d. random variables.

For two given samples $X_i, Y_i \in \mathbb{R}^2$ $(i = 1, \dots, n)$ let us define the statistic T_n by

$$T_n = \min_{\pi \in S_n} \sum_{i=1}^{n} ||X_{\pi(i)} - Y_i||^2. \tag{1.3}$$

Here S_n denotes the set of permutations of $\{1, \dots, n\}$ and $||.||$ is the Euclidean norm. Formula (1.3) naturally expresses the 'distance' of two samples. We study certain properties of T_n for Gaussian and uniform samples. To this aim we made simulation studies for sample sizes $n = 2, \dots, 200$ with replication 1000 in each case. That is we generated two samples of sizes n, calculated T_n, then repeated this procedure 1000 times. Then we tried to fit the so called general extreme value (GEV) distribution (see [5], page 61) to the obtained data of size 1000. The distribution function of the general extreme value distribution is

$$F(x, \mu, \sigma, \xi) = \begin{cases} \exp\left(-\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right), & \xi \neq 0; \\ \exp\left(-\exp\left(-\frac{(x-\mu)}{\sigma}\right)\right), & \xi = 0. \end{cases} \tag{1.4}$$

Here $\mu, \sigma > 0, \xi$ are real parameters. For further details see [5].

The values of T_n are obtained by Kuhn's Hungarian algorithm as described in [3]. We mention that a previous simulation study of T_n was performed in [2].

2. Simulation results for samples with common distribution

In this section we want to determine the distribution of T_n when the samples $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ have the same distribution. In terms of testing goodness of fit the task is the following.

Let $X_1, \dots, X_n$ be a sample. We want to test the hypothesis

$$H_0 : \text{the distribution of } X_i \text{ is } F.$$

Generate another sample $Y_1, \dots, Y_n$ from distribution F and calculate the test statistics T_n. If T_n is large, then we reject H_0. (In practice $X_1, \dots, X_n$ are real life data, while $Y_1, \dots, Y_n$ are random numbers.) To create a test we have to find some information on the distribution of T_n.

To obtain the distribution of T_n by simulation, we proceed as follows. For a fixed sample size n, $2n$ two-dimensional points are generated: $X_i = (X_{i1}, X_{i2})$, $Y_i = (Y_{i1}, Y_{i2})$, $i = 1, \dots, n$, with independent coordinates. We restrict our attention to the simplest cases.

(a) Gaussian case when $X_{ij}, Y_{ij} \in \mathcal{N}(0,1)$, $i = 1, 2, \dots, n$, $j = 1, 2$, i.e. they are standard normal.

(b) Uniform case when $X_{ij}, Y_{ij} \in \mathcal{U}(0,1)$, $i = 1, 2, \dots, n$, $j = 1, 2$, i.e. they are uniformly distributed on $[0, 1]$.

All the random variables involved are independent. Graphs of descriptives and tables of $5\%, 10\%, 90\%$ and 95% quantiles for selected sample sizes are presented in figures 1, 2 and tables 1, 4.

Figure 1(a) and Figure 2(a) show the sample mean and sample standard deviation of T_n, respectively, when both X_i and Y_i comes form two-dimensional standard normal. (They are calculated for each fixed n using 1000 replications.) Figure 1(b) and Figure 2(b) concern the case when both X_i and Y_i are uniform.

Table 1 shows the sample quantiles of T_n when both X_i and Y_i are two-dimensional standard normal. Each value is calculated for fixed n using 1000 replications. The upper quantile values (at 90% or 95%) can serve as critical values for the test

$$H_0 : X_i \text{ is two-dimensional standard normal.}$$

Table 2 contains the results when both samples are two-dimensional uniform (more precisely uniform on $[0, 1] \times [0, 1]$).

3. The mixed case

With the help of previous section's tables one can construct empirical confidence intervals for the distance T_n of two samples both in the Gaussian-Gaussian and uniform-uniform cases. In what follows we present some results on the distance T_n for the Gaussian-uniform case. For this aim we performed calculations for sample sizes $n = 2, \dots, 200$ with 2000 replications in each cases. Note that here we used $\mathcal{U}(-\sqrt{3}, \sqrt{3})$ for the uniform variable because then we have $\mathbb{E}(Y_{ij}) = 0$ and $\mathbb{D}^2(Y_{ij}) = 1$.

Figure 3 and Table 3 concern the distribution of T_n when X_{ij} is standard normal and $Y_{ij} \in \mathcal{U}(-\sqrt{3}, \sqrt{3})$. That is the case when H_0 is not satisfied. If we compare the last columns (95% quantiles) of Table 3 and Table 1, then we see that our test is sensitive if the sample size is large ($n \geq 100$).

4. Fitting the *GEV*

To describe the distribution of T_n we fitted general extreme value distribution. For each fixed n we estimated the parameters of GEV from the 1000 replications. The maximum likelihood estimates of parameters ξ, μ, σ in (1.4) were obtained with MATLAB's *fitdist* procedure. Then we plotted the cumulative distribution function of the GEV. Figure 4(a), Figure 5(a) and Figure 6(a) show that the empirical distribution function of T_n fits well to the theoretical distribution function of the appropriate GEV when both X_i and Y_i are standard Gaussian. Figure 4(b), Figure 5(b) and Figure 6(b) show the same for uniformly distributed X_i and Y_i.

Figure 7 shows the empirical significance of Kolmogorov-Smirnov tests performed by *kstest*. The empirical p-values in Figure 7(a) and Figure 7(b) reveal that the fitting was succesful.

5. About the *GEV* parameters

To suspect something about the possible 'analytical form' of parameters ξ, σ, μ we made further simulations in the Gaussian case with 5000 replications for sample sizes $n = 2, \dots, 500$. After several 'trial and error' attempts we got the following experimental results.

Figure 8 concern the functional form of the parameters. Here both X_i and Y_i were Gaussian. For each fixed n we fitted $GEV(\xi(n), \sigma(n), \mu(n))$. Then we approximated $\xi(n), \sigma(n)$ and $\mu(n)$ with certain functions. For example we obtained that $\xi(n)$ can be reasonably approximated with

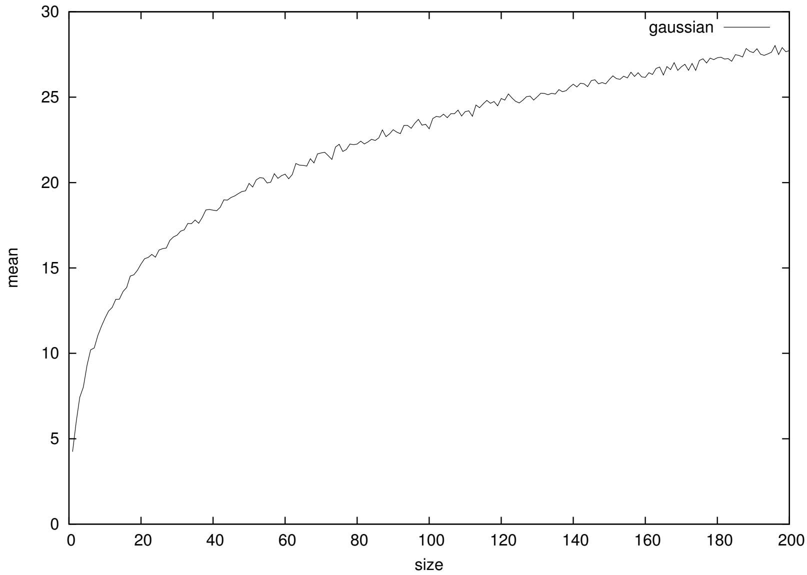
$$A/\sqrt{n} + B/\sqrt{\log(n)} + C$$

where A, B, C are given in Figure 8(a). Note that the classical goodness of fit measures (χ^2 and R^2) computed by *qtiplot* indicate tight fit.

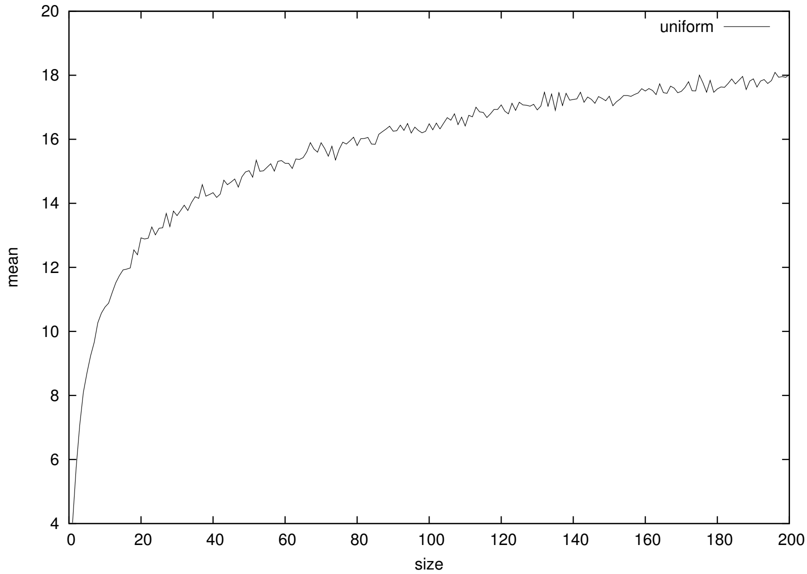
6. Tools

The Hungarian method was implemented in *C++* using the GNU g++ compiler. Most of the graphs were made with the help of the utility *gnuplot*. The fittings and the graphs of the last section were performed with *qtiplot*. MATLAB was used to compute the maximum likelihood estimators of the GEV.

7. Figures and tables

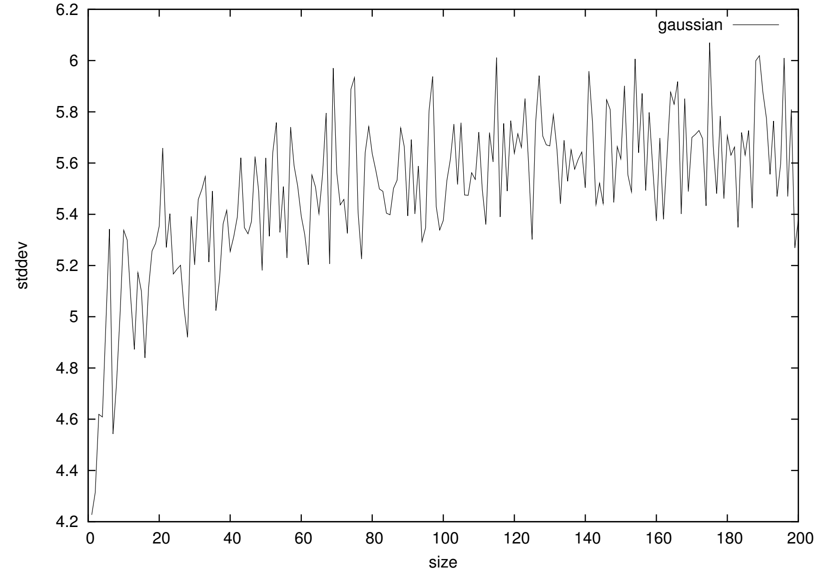


(a) Gaussian

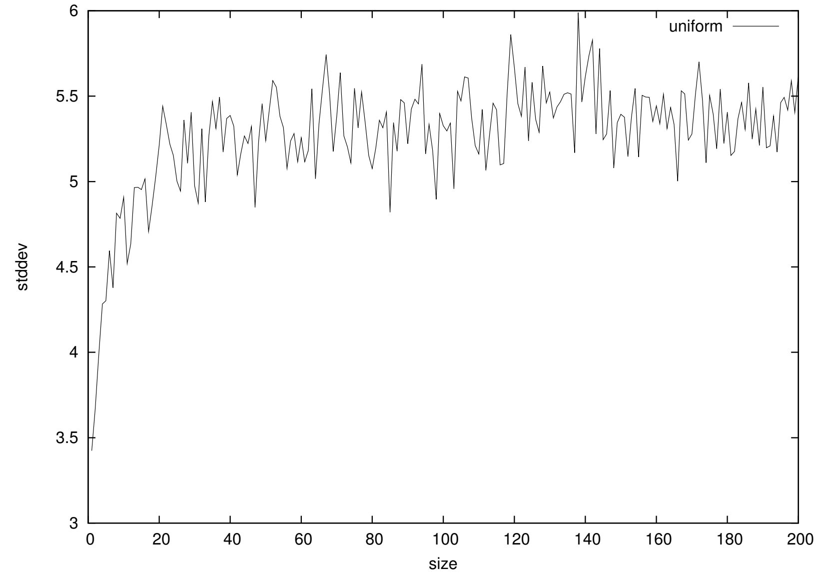


(b) uniform

Figure 1: Sample means



(a) Gaussian



(b) uniform

Figure 2: Sample standard deviations

size	mean	stddev	5%	10%	90%	95%
1	4.2412	4.2266	0.2381	0.4563	9.8020	12.2026
2	5.9673	4.3135	1.0836	1.7158	11.7292	14.2147
3	7.4222	4.6192	1.7995	2.5588	13.7402	16.0102
4	8.0206	4.6088	2.3981	3.2674	13.6933	16.0876
5	9.3045	4.9715	3.2585	4.1318	15.3954	18.3475
6	10.2078	5.3420	3.7299	4.5396	17.1818	19.4804
7	10.3123	4.5427	4.1995	5.2007	16.2449	19.0637
8	11.0368	4.7393	4.8979	5.7742	17.5610	19.8337
9	11.5844	5.0128	5.0169	5.7831	18.2823	20.9534
10	12.0570	5.3378	5.3833	6.3622	19.4668	21.8923
20	15.2328	5.3538	8.6359	9.5934	22.2431	24.9313
30	16.9197	5.2032	10.1785	11.0722	23.9962	27.0184
40	18.3806	5.2547	11.4259	12.6406	25.5429	28.0519
50	19.9502	5.6199	12.1530	13.5418	27.2146	30.6897
60	20.4902	5.3926	13.3070	14.3950	27.5160	31.5666
70	21.7366	5.5615	14.3281	15.5132	28.9196	32.1283
80	22.2543	5.6370	14.6116	15.8363	29.4256	32.3583
90	23.0996	5.3942	15.7942	16.9445	30.0369	33.0472
100	23.1510	5.3759	15.9969	17.0670	29.9957	33.3124
120	24.9210	5.6381	16.9766	18.4435	32.5778	34.7392
140	25.7610	5.5036	18.1150	19.5019	33.0206	35.3015
160	26.1585	5.3739	18.7503	20.0962	33.6252	36.3843
180	27.3072	5.7067	19.4513	20.7716	34.7677	37.3559
200	27.7257	5.3810	20.2195	21.4550	34.8416	37.1973

Table 1: Quantiles. (a) Gaussian case

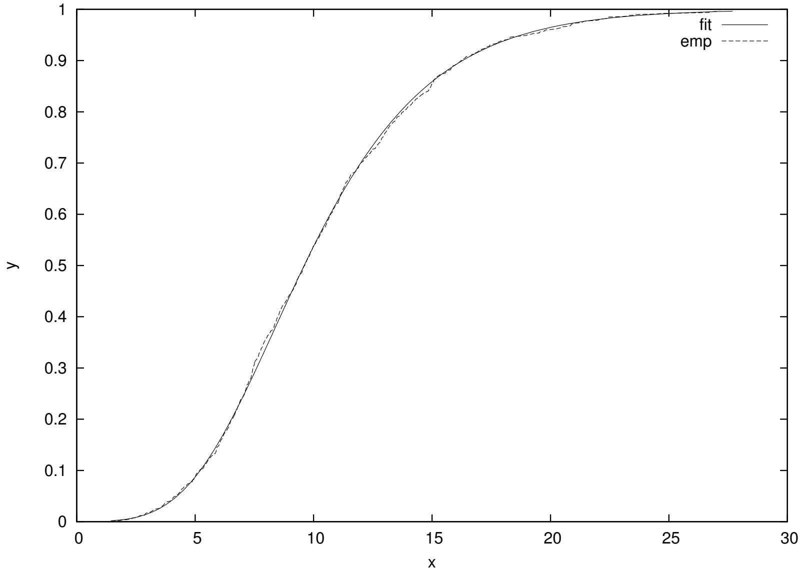
size	mean	stddev	5%	10%	90%	95%
1	4.0262	3.4236	0.1970	0.4216	9.1473	10.7406
2	5.7465	3.6683	1.1099	1.7597	10.9533	12.8762
3	7.0817	3.9889	1.9709	2.5498	12.4090	14.4607
4	8.0923	4.2833	2.5078	3.3816	13.7593	16.6473
5	8.7164	4.3022	3.0438	4.0823	14.4455	17.0613
6	9.2447	4.5952	3.4694	4.3526	15.5264	18.0257
7	9.6608	4.3776	4.1441	5.0191	15.5752	17.7570
8	10.2707	4.8150	4.3921	5.3208	16.7853	19.6646
9	10.5731	4.7847	4.6570	5.4796	16.7946	19.8556
10	10.7589	4.9081	5.0283	5.7636	16.9497	19.6734
20	12.9231	5.2142	6.8068	7.5361	19.3753	22.0094
30	13.6183	4.9743	7.7449	8.5648	20.3468	23.5615
40	14.3316	5.3870	7.9332	9.0030	21.1094	24.8500
50	15.0187	5.2400	8.8329	9.6225	21.8464	25.6322
60	15.2523	5.2565	8.9552	9.8825	21.8732	25.0060
70	15.8911	5.3841	9.5833	10.5137	22.9240	26.1478
80	15.8035	5.0754	9.7065	10.5853	22.9256	26.3499
90	16.2536	5.2216	9.9975	10.7825	22.9050	26.0921
100	16.4830	5.3264	10.2617	10.8682	24.0967	27.2588
120	17.0734	5.6685	10.4419	11.3588	24.4638	28.9698
140	17.2442	5.6141	10.7627	11.5904	24.3573	27.6821
160	17.5099	5.4433	11.1689	12.0155	24.9285	27.9288
180	17.5731	5.4062	11.1337	12.0112	24.5775	27.8075
200	18.0244	5.6245	11.4052	12.3776	25.4038	28.4148

Table 2: Quantiles. (b) uniform case

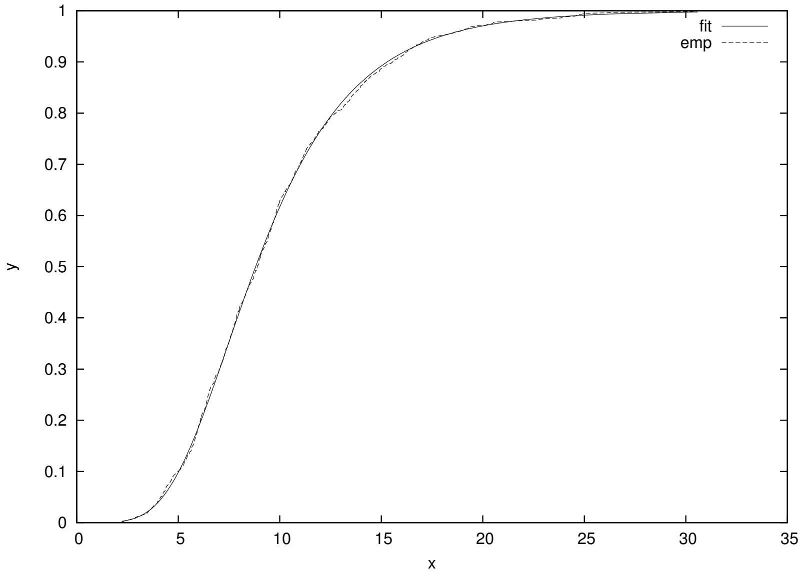
size	mean	stddev	5%	10%	90%	95%
2	5.8818	4.1502	1.1652	1.6532	11.3164	13.6333
3	7.2554	4.1646	2.0590	2.8677	13.0166	15.0618
4	7.9939	4.4994	2.7365	3.3880	13.7425	16.6208
5	8.8754	4.4742	3.2754	4.1966	14.8754	17.3966
6	9.7783	4.5869	4.0448	4.7961	15.8904	18.3626
7	10.0745	4.5939	4.3386	5.2004	16.2399	18.8547
8	10.4734	4.4532	4.5207	5.4837	16.4532	18.8462
9	11.0886	4.6638	5.2157	6.1198	17.3704	19.7215
10	11.5361	4.8266	5.4715	6.4598	17.9202	20.9748
20	14.6339	5.1743	8.0674	9.0785	21.4606	24.9300
30	16.7707	5.1719	10.0346	11.0548	23.5244	26.3029
40	18.4228	5.4695	11.1791	12.2861	25.4337	28.2022
50	20.0629	5.6343	12.4183	13.9359	27.2744	30.4001
60	21.1895	5.4223	13.4741	14.7804	28.4744	31.1559
70	22.4896	5.6856	14.7198	16.0582	29.7835	33.4986
80	24.0810	5.8618	15.6983	17.2439	31.5593	34.0761
90	25.1524	6.0496	16.9018	18.1007	33.2026	36.5343
100	26.5301	6.1983	17.9469	19.3842	34.9823	37.9294
120	28.8502	6.3829	19.9032	21.3154	37.0409	40.2436
140	30.9987	6.5694	21.6081	23.2774	39.7266	42.9169
160	33.1575	6.7402	23.3054	25.1265	42.0649	45.1835
180	34.9612	6.7718	25.3528	26.8373	43.5126	47.4540
200	37.2655	7.0713	27.1785	28.8087	46.5339	49.9597

Table 3: Quantiles. Gaussian-uniform case

Figure 3: Sample means and standard deviations, Gaussian-uniform case

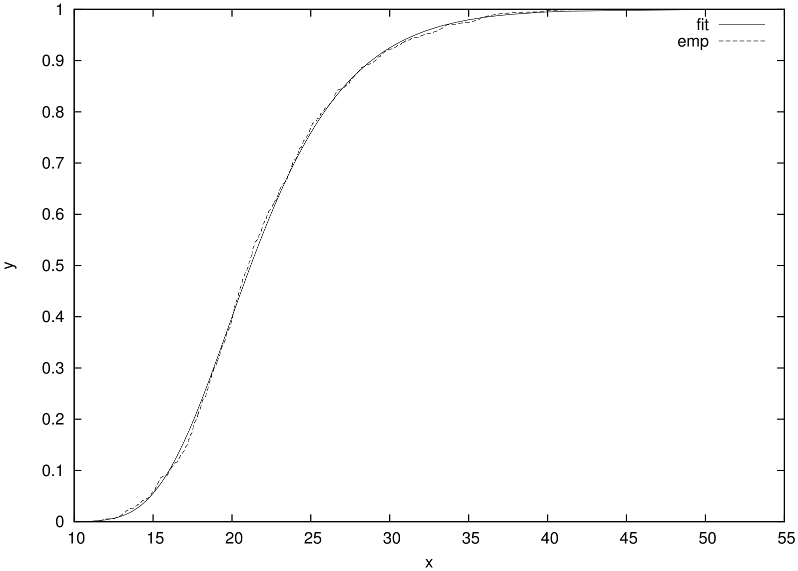


(a) Gaussian

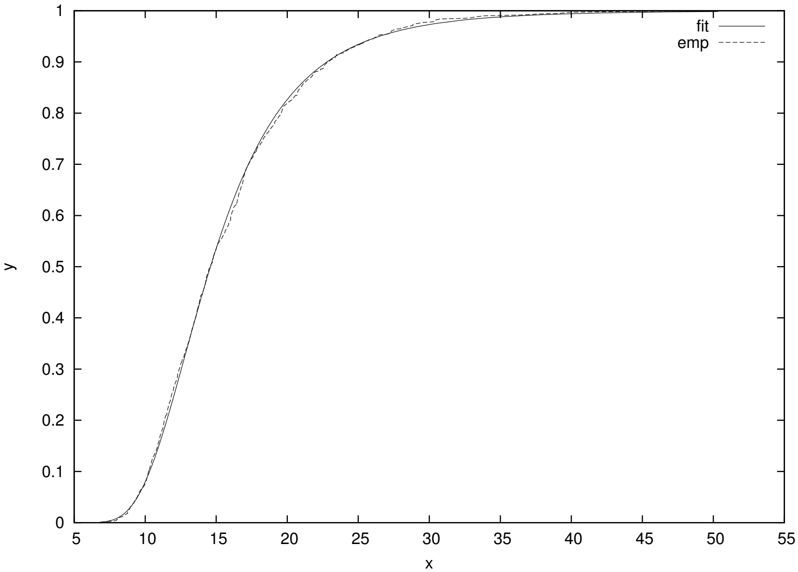


(b) uniform

Figure 4: Empirical and fitted cdf, $n = 7$

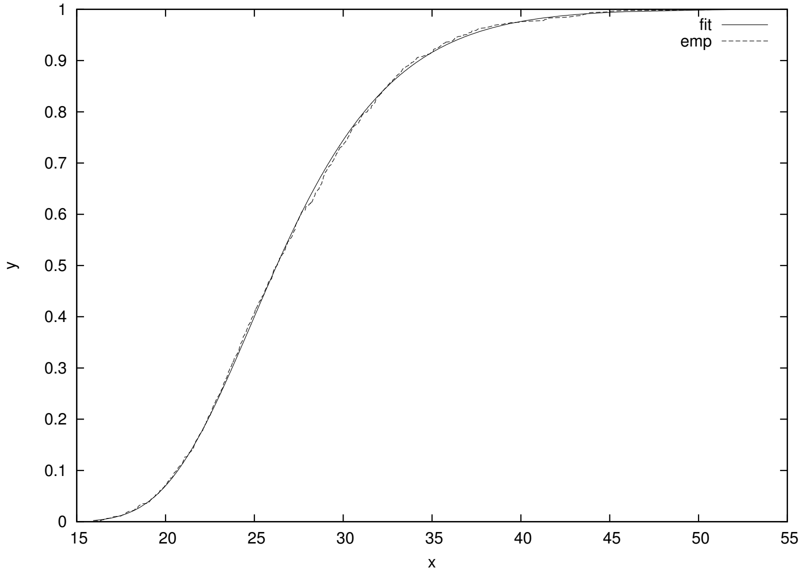


(a) Gaussian

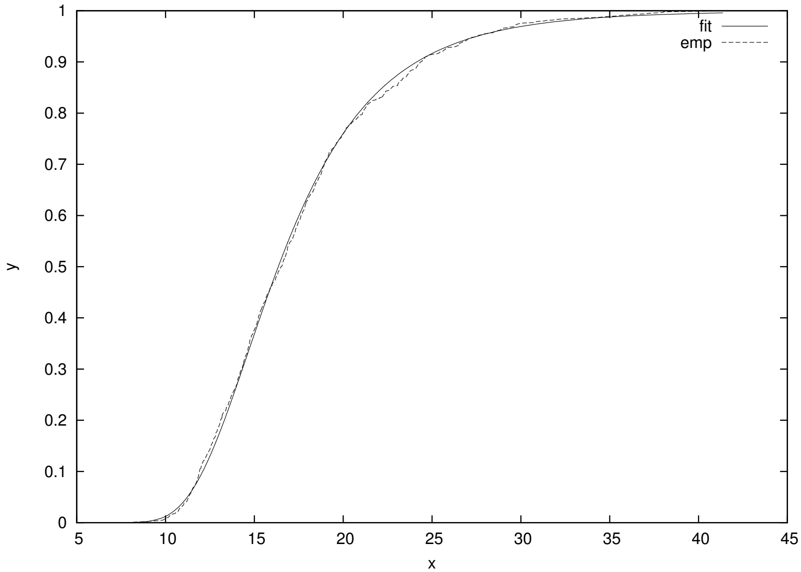


(b) uniform

Figure 5: Empirical and fitted cdf, $n = 77$

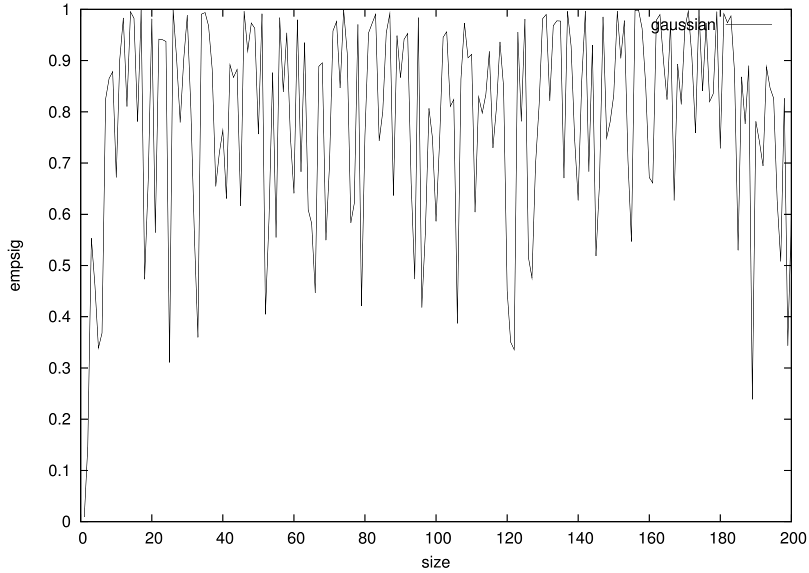


(a) Gaussian

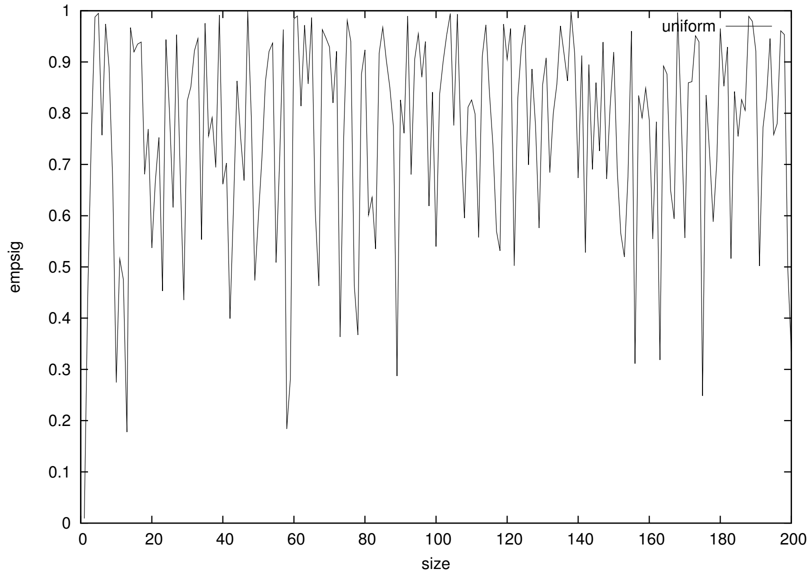


(b) uniform

Figure 6: Empirical and fitted cdf, $n = 177$

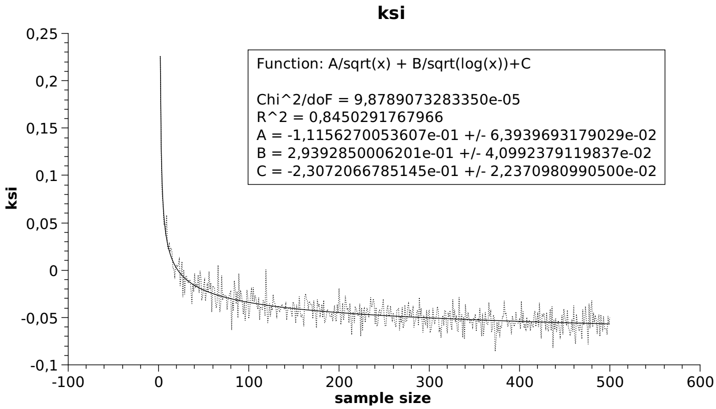


(a) Gaussian

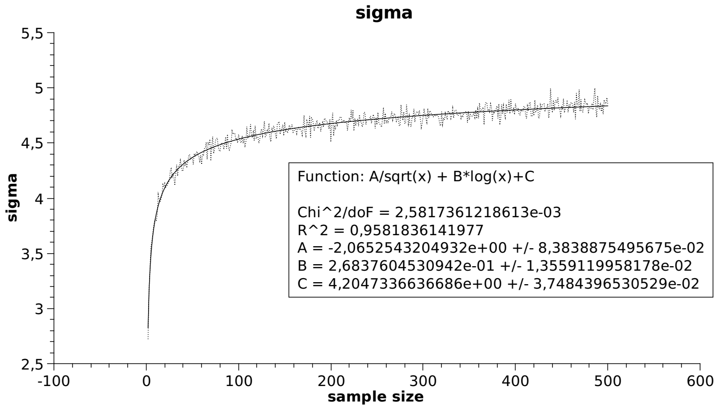


(b) uniform

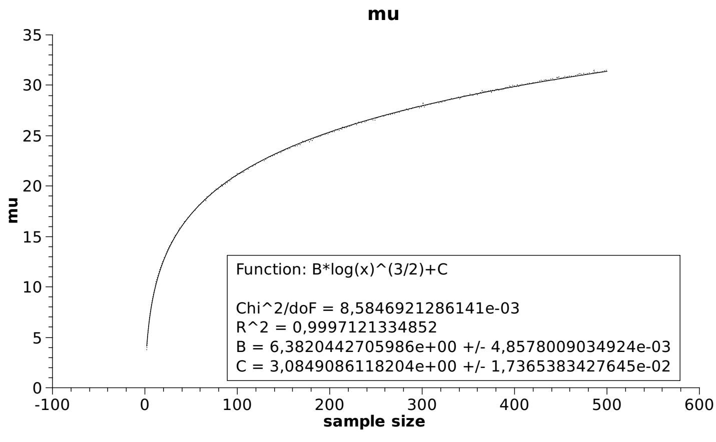
Figure 7: Empirical significance



(a) ξ



(b) σ



(c) μ

Figure 8: Estimating ξ, σ and μ

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