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A note on Tribonacci-coefficient polynomials

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Abstract

This paper shows, that the Tribonacci-coefficient polynomial $P_n(x) = T_2x^n + T_3x^{n-1} + \cdots + T_{n+1}x + T_{n+2}$ has exactly one real zero if n is odd, and $P_n(x)$ does not vanish otherwise. This improves the result in [1], which provides the upper bound 3 or 2 on the number of zeros of $P_n(x)$, respectively.

Keywords: linear recurrences, zeros of the polynomials with special coefficients

MSC: 11C08, 11B39

1. Introduction

The Fibonacci-coefficient polynomials $\mathcal{F}_n(x) = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}$, $n \in \mathbb{N}^+$ were defined in [2]. The authors determined the number of real zeros of $\mathcal{F}_n(x)$. Generally, but with specific initial values, for binary recurrences and for linear recursive sequences of order $k \geq 2$ the question of the number of real zeros was investigated in [3] and [1], respectively.

As usual, the Tribonacci sequence is defined by the initial values $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$, and by the recurrence relation $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ $(n \geq 3)$. The Corollary 2 of Theorem 1 in [1] states that the possible number of negative zeros of the polynomial

$$P_n(x) = T_2x^n + T_3x^n + \cdots + T_{n+1}x + T_{n+2}$$

does not exceed three. More precisely, $P_n(x)$ possesses 0 or 2 negative zeros if n is even, and 1 or 3 negative zeros when n is odd. Obviously, there is no positive zero of $P_n(x)$, since all coefficients are positive.

The following theorem gives that the number of negative zeros is 0 or 1 depending on the parity of n.

Theorem 1.1. *The polynomial $P_n(x)$ has no real zero if n is even, while $P_n(x)$ possesses exactly one real zero, which is negative, if n is odd.*

In the proof, at the beginning we partially follow the approach of [1].

2. Proof of Theorem 1.1

Proof. Let $f(x) = x^3 - x^2 - x - 1$ denote the characteristic polynomial of the Tribonacci sequence. It is known, that $f(x)$ has one positive real zeros and a pair of complex conjugate zeros. Put

$$Q_n(x) = f(x)P_n(x) = x^{n+3} - T_{n+3}x^2 - (T_{n+2} + T_{n+1})x - T_{n+2}$$

(see Lemma 1 in [1]). Applying the Descartes' rule of signs, $Q_n(x)$ has one positive real zero, which obviously belongs to $f(x)$. (It hangs together with $P_n(x)$ possesses no positive real roots.)

To examine the negative roots, put $q_n(x) = Q_n(-x)$. In order to use Descartes' result again, we must distinguish two cases based on the parity of n.

First suppose that n is even. Now

$$q_n(x) = -x^{n+3} - T_{n+3}x^2 + (T_{n+2} + T_{n+1})x - T_{n+2},$$

and the number of changes of coefficients' signs predicts 2 or 0 positive zeros of $q_n(x)$. We are going to exclude the case of 2 zeros.

Clearly, $q_n(0) = -T_{n+2} < 0$, $q_n(1) = -T_{n+3} + T_{n+1} - 1 < 0$. Further, we have

$$q_n'(x) = -(n+3)x^{n+2} - 2T_{n+3}x + (T_{n+2} + T_{n+1}).$$

The values $q_n'(0) = T_{n+2} + T_{n+1} > 0$, $q_n'(1) = -(n+3) - 2T_{n+3} + T_{n+2} + T_{n+1} < 0$ show that the function $q_n(x)$ strictly monotone increasing locally in 0, while strictly monotone decreasing in 1. Since $q_n''(x) = -T_2(n+3)(n+2)x^{n+1} - 2T_{n+3}$ is negative for all non-negative $x \in \mathbb{R}$, then $q_n(x)$ is concave on $\mathbb{R}^+$. Consequently, if exist, the positive zeros of the polynomial $q_n(x)$ are in the interval $(0;1)$.

Therefore, to show that $q_n(x)$ does not cross the x-axes it is sufficient to prove that intersection point of the tangent lines $e\ :\ y = (T_{n+2} + T_{n+1})x - T_{n+2}$ and $f\ :\ y = (-(n+3) - 2T_{n+3} + T_{n+2} + T_{n+1})(x-1) - T_{n+3} + T_{n+1} - 1$ is under the x-axes. To reduce the calculations we simply justify that $x_0 > x_1$, where x_0 is defined by $e \cap x$-axes and x_1 is given by $f \cap x$-axes (see Figure 1).

First, $(T_{n+2} + T_{n+1})x - T_{n+2} = 0$ implies

$$x_0 = \frac{T_{n+2}}{T_{n+2} + T_{n+1}} > \frac{T_{n+2}}{T_{n+2} + T_{n+2}} = \frac{1}{2}.$$

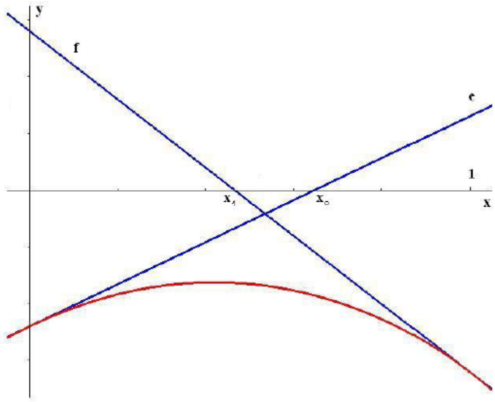

Figure 1

On the other hand,

$$x_1 = \frac{T_{n+3} - T_{n+1} + 1}{-(n+3) - 2T_{n+3} + T_{n+2} + T_{n+1}} + 1 \leq \frac{1}{2} \tag{2.1}$$

holds if $n \geq 5$. Indeed, (2.1) is equivalent to

$$\frac{1}{2} \leq \frac{T_{n+3} - T_{n+1} + 1}{(n+3) + 2T_{n+3} - T_{n+2} - T_{n+1}},$$

where both the numerator and the denominator are positive. Hence $n+1 \leq T_{n+2} - T_{n+1}$ remains to show, and it can be easily deduced, for example, by induction if $n \geq 5$.

The case $n = 4$ can be separately investigated. Now $T_5 = 4$, $T_6 = 7$, and $11x - 7 = 0$ provides $x_0 = 7/11$. Moreover, $T_7 = 13$ and $-22(x-1) - 10 = 0$ gives $x_1 = 6/11$. Thus $x_1 < x_0$.

Assume now, that n is odd. We partially repeat the procedure of the previous case.

The polynomial

$$q_n(x) = x^{n+3} - T_{n+3}x^2 + (T_{n+2} + T_{n+1})x - T_{n+2}$$

may have 3 or 1 positive zeros (by Descartes' rule of signs again).

Obviously, $q_n(0) = -T_{n+2} < 0$ and $q_n(1) = -T_{n+3} + T_{n+1} + 1 < 0$. Now

$$q_n'(x) = (n+3)x^{n+2} - 2T_{n+3}x + (T_{n+2} + T_{n+1}),$$

which together with $q_n'(0) = T_{n+2} + T_{n+1} > 0$, $q_n'(1) = (n+3) - 2T_{n+3} + T_{n+2} + T_{n+1} < 0$ implies the same monotonity behaviour in $(0;1)$ as before.

Since the equation $q_n''(x) = (n+3)(n+2)x^{n+1} - 2T_{n+3} = 0$ holds if and only if

$$x_{inf} = \sqrt[n+1]{\frac{T_{n+3}}{\binom{n+3}{2}}},$$

then $q_n(x)$ is concave on the interval $(0; x_{inf})$, and convex for $x > x_{inf}$. However, $x_{inf} > 1$ if $n \geq 9$, and in this case we can show that $q_n(x)$ does not intersect the x-axes in the interval $(0;1)$ but there is exactly one zero if $x > 1$. The second part is an immediate consequence of the existence of unique positive inflection point $x_{inf} > 1$. Concentrating on the interval $(0;1)$, similarly to the previous part $e\ :\ y = (T_{n+2} + T_{n+1})x - T_{n+2}$ and $f\ :\ y = ((n+3) - 2T_{n+3} + T_{n+2} + T_{n+1})(x - 1) - T_{n+3} + T_{n+1} + 1$ intersect each other under the x-axes, because of $x_0 > \frac{1}{2}$ holds again, and

$$x_1 = \frac{T_{n+3} - T_{n+1} - 1}{(n+3) - 2T_{n+3} + T_{n+2} + T_{n+1}} + 1 \leq \frac{1}{2}$$

follows, since $-(n+1) \leq T_{n+2} - T_{n+1}$.

For $n = 3$ or 5 or 7 we can easily check the required property. Thus the proof is complete. □

References

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