A new recursion relationship for Bernoulli Numbers

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Abstract

We give an elementary proof of a generalization of the Seidel-Kaneko and Chen-Sun formula involving the Bernoulli numbers.

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MSC: 11B68, 11B83

1. Introduction

The Bernoulli Numbers $B_n, n = 0, 1, 2, \ldots$ are defined by the exponential generating function:

$$B(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \quad (1.1)$$

As (1.1) implies that $B(-z) = z + B(z)$, we have:

$$(-1)^n B_n = B_n + \delta^n_1, \text{ for } n \geq 0. \quad (1.2)$$

where the notation $\delta^n_1$ is the classical Kronecker symbol which equals 1 if $n = i$ and 0 otherwise. Consequently, we have $B_1 = -\frac{1}{2}$, and $B_n = 0$, when $n$ is odd and $n \geq 3$. Let us define $\epsilon_n := \frac{1 + (-1)^n}{2}$, thus:

$$\epsilon_n B_n = B_n + \frac{1}{2} \delta^n_1, \text{ for } n \geq 0. \quad (1.3)$$
Note that the Bernoulli polynomials can be defined by the following function:

\[ B(x, z) := \frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}. \]

Thus, we have:

\[ \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \left( \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} x^n \frac{z^n}{n!} \right). \]

Therefore the polynomial \( B_n(x) \) satisfies the following equality:

\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k. \quad (1.4) \]

We note also that:

\[ B(x + 1, z) - B(x, z) = \sum_{n=0}^{\infty} (B_n(x + 1) - B_n(x)) \frac{z^n}{n!} = ze^{xz}. \]

Consequently, we deduce the following property of \( B_n(x) \):

\[ B_n(x + 1) - B_n(x) = nx^{n-1}, \text{ for } n \geq 1. \quad (1.5) \]

In this paper, we are extending the well-known following formulae involving Bernoulli Numbers. First, the Seidel formula (1877) [4], re-discovered later by Kaneko [3] (1995):

\[ \sum_{k=0}^{n} \binom{n+1}{k} (n+k+1) B_{n+k} = 0, \text{ for } n \geq 1. \]

And secondly, the Chen-Sun formula [1] (2009):

\[ \sum_{k=0}^{n+3} \binom{n+3}{k} (n+k+3)(n+k+2)(n+k+1) B_{n+k} = 0. \quad (1.6) \]

Our main result consists on the following:

**Theorem 1.1.** For given odd natural \( q \) and for natural number \( n \geq 0 \), we have the equality:

\[ \sum_{k=0}^{n+q} \binom{n+q}{k} (n+k+q)(n+k+q-1) \cdots (n+k+1) B_{n+k} = 0. \quad (1.7) \]

Obviously, this result gives the Seidel-Kaneko formula when \( q = 1 \), and the Chen-Sun formula when \( q = 3 \).
2. Proof of the main result

For a given odd number $q$ and for an integer number $n \geq 0$, we consider the polynomials:

$$H(x) = \frac{1}{2} x^{n+q} (x - 1)^{n+q},$$

and

$$K(x) = \sum_{k=0}^{n+q} \frac{\epsilon_{n+k}}{(n + q + k + 1)} \binom{n + q}{k} \left( B_{n+q+k+1} - B_{n+q+k+1} \right). \quad (2.1)$$

By the binomial theorem, we deduce:

$$H(x) = \frac{1}{2} \sum_{k=0}^{n+q} (-1)^{n+k+1} \binom{n + q}{k} x^{n+q+k}, \quad (2.2)$$

and

$$H(x + 1) = \frac{1}{2} \sum_{k=0}^{n+q} \binom{n + q}{k} x^{n+q+k}. \quad (2.3)$$

Thus, by using the equality property (1.5), we verify that:

$$K(x + 1) - K(x) = H(x + 1) - H(x) = \sum_{k=0}^{n+q} \epsilon_{n+k} \binom{n + q}{k} x^{n+q+k}. \quad (2.4)$$

Moreover

$$K(0) = H(0) = 0. \quad (2.5)$$

Then, (2.2), (2.3), (2.4) and (2.5) imply:

$$K(x) = H(x).$$

If $[x^n]P(x)$ denotes the coefficient of $x^n$ in the polynomial $P(x)$, we can write:

$$[x^{q+1}] K(x) = [x^{q+1}] H(x). \quad (2.6)$$

So, from (1.4)

$$[x^{q+1}] K(x) = \sum_{k=0}^{n} \frac{\epsilon_{n+k} B_{n+k}}{(n + q + k + 1)} \binom{n + q}{k} \binom{n + q + k + 1}{q + 1}. \quad (2.7)$$

and from (2.2), we have:

$$[x^{q+1}] H(x) = \frac{1}{2} \binom{n + q}{1 - q}. \quad (2.8)$$
From (1.3), we know that:

$$\epsilon_{n+k}B_{n+k} = B_{n+k} + \frac{1}{2}\delta_{1-n}^k.$$  \hfill (2.9)

Since

$$\sum_{k=0}^{n+q} \frac{\delta_{1-n}^k}{2(n+q+k+1)} \binom{n+q}{k} \binom{n+q+k+1}{q+1} = \frac{1}{2(q+1)} \binom{n+q}{1-n} \binom{q+1}{q}$$

$$= \frac{1}{2} \binom{n+q}{1-n}.$$  \hfill (2.10)

We deduce, from (2.7), (2.9) and (2.10) that:

$$[x^{q+1}]K(x) = \sum_{k=0}^{n+q} \frac{B_{n+k}}{x^{q+k+1}} \binom{n+q}{k} \binom{n+q+k+1}{q+1} + \frac{1}{2} \binom{n+q}{1-n}. \hfill (2.11)$$

It follows from (2.6), (2.8) and (2.11) that:

$$\sum_{k=0}^{n+q} \frac{1}{(n+q+k+1)} \binom{n+q}{k} \binom{n+q+k+1}{q+1} B_{n+k} = 0, \hfill (2.12)$$

and by multiplying by $(q+1)!$, we obtain, finally, the aimed result which is:

$$\sum_{k=0}^{n+q} \binom{n+q}{k} (n+k+q)(n+k+q-1)\ldots(n+k+1)B_{n+k} = 0.$$

This ends our proof.

References


