Annales Mathematicae et Informaticae
37 *(2010) pp. 7–11*
http://ami.ektf.hu

Non-integerness of class of hyperharmonic numbers

Rachid Aït Amrane[a], Hacène Belbachir[b]

[a]ESI/ Ecole nationale Supérieure d'Informatique, Alger, Algeria

[b]USTHB/ Faculté de Mathématiques, Alger, Algeria

Submitted 7 September 2009; Accepted 10 February 2010

Abstract

Our purpose is to establish that hyperharmonic numbers – successive partial sums of harmonic numbers – satisfy a non-integer property. This gives a partial answer to Mező's conjecture.

Keywords: Harmonic numbers; Hyperharmonic numbers; Bertrand's postulate.

MSC: 11B65, 11B83.

1. Introduction

In 1915, L. Taeisinger proved that, except for H_1, the harmonic number $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is not an integer. More generally, H. Belbachir and A. Khelladi [1] proved that a sum involving negative integral powers of consecutive integers starting with 1 is never an integer.

In [3, p. 258–259], Conway and Guy defined, for a positive integer r, the hyperharmonic numbers as iterate partial sums of harmonic numbers

$$H_n^{(1)} := H_n \text{ and } H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \quad (r > 1)\,.$$

The number $H_n^{(r)}$, called the n^{th} hyperharmonic number of order r, can be expressed by binomial coefficients as follows (see [3])

$$H_n^{(r)} = \binom{n+r-1}{r-1}\left(H_{n+r-1} - H_{r-1}\right). \tag{1.1}$$

For other interesting properties of these numbers, see [2].

I. Mező, see [5], proved that $H_n^{(r)}$, for $r = 2$ and 3, is never an integer except for $H_1^{(r)}$. In his proof, he used the reduction modulo the prime 2. He conjectured that $H_n^{(r)}$ is never an integer for $r \geqslant 4$, except for $H_1^{(r)}$.

In our work, we give another proof that $H_n^{(r)}$ is not an integer for $r = 2, 3$ when $n \geqslant 2$. We also give an answer to Mező's conjecture for $r = 4$ and a partial answer for $r > 4$.

Our proof is based on Bertrand's postulate which says that for any $k \geqslant 4$, there is a prime number in $]k, 2k-2[$. See for instance [4, p. 373].

2. Results

Theorem 2.1. *For any $n \geqslant 2$, the hyperharmonic number $H_n^{(2)}$ is never an integer.*

Proof. Let $n \geqslant 2$ and assume $H_n^{(2)} \in \mathbb{N}$. We have $H_n^{(2)} = \binom{n+1}{1}(H_{n+1} - H_1) = (n+1)(H_{n+1} - 1)$, therefore $(n+1)H_{n+1} = (n+1)\left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right)$ is an integer. Let P be the greatest prime number less than or equal to n. We have $\frac{(n+1)!}{P} H_{n+1} - \frac{(n+1)!}{P} \sum_{k \neq P} \frac{1}{k} = \frac{(n+1)!}{P^2}$. The left hand side of the equality is an integer while the right hand side is not. Indeed, by Bertrand's postulate, the prime P is coprime to any k, $k \leqslant n+1$, contradiction. □

Theorem 2.2. *For any $n \geqslant 2$, the hyperharmonic number $H_n^{(3)}$ is never an integer.*

Proof. The arguments here are similar to those in the proof of the following theorem. □

Theorem 2.3. *For any $n \geqslant 2$, the hyperharmonic number $H_n^{(4)}$ is never an integer.*

Proof. We have $H_2^{(4)} = \frac{9}{2} \notin \mathbb{N}$, $H_3^{(4)} = \frac{37}{3} \notin \mathbb{N}$ and $H_4^{(4)} = \frac{319}{12} \notin \mathbb{N}$. Let $n \geqslant 5$ and assume that $H_n^{(4)} \in \mathbb{N}$. With the same arguments as in the proof of Theorem 1 we deduce that $(n+1)(n+2)(n+3)H_n \in \mathbb{N}$. Let P be the greatest prime less than or equal to n. Then $\frac{(n+3)!}{P} H_n - \frac{(n+3)!}{P}\left(1 + \frac{1}{2} + \cdots + \frac{1}{P-1} + \frac{1}{P+1} + \cdots + \frac{1}{n}\right) = \frac{(n+3)!}{P^2}$. The left hand side of the equality is an integer while the right hand side is not. Again, P is coprime to any k, $P < k \leqslant n+3$. Therefore, if P divides $(n+3)!$, then P would divide $(P+1)\cdots(n+3)$, thus one of the factors would be equal to $2P$, consequently $2P - 2 \leqslant n+1$, hence, by Bertrand's postulate, there would exist a prime strictly between P and $n+1$, contradicting the fact that P is the greatest prime less than or equal to n. Therefore, $H_n^{(4)} \notin \mathbb{N}$ for any $n \geqslant 2$. □

For $r \geqslant 5$, we give a class of hyperharmonic numbers satisfying the non-integer property.

Theorem 2.4. *Let $n \in \mathbb{N}$ such that $n \geqslant 2$ and that none of the integers $n+1, n+2, \ldots, n+r-4$ is a prime number, then we have $H_n^{(r)} \notin \mathbb{N}$.*

Proof. It is easy to see that $H_2^{(r)} = \frac{r+1}{2} + \frac{r}{2} \notin \mathbb{N}$, $H_3^{(r)} = \frac{(r+1)(r+2)}{6} + \frac{r(r+2)}{6} + \frac{r(r+1)}{6} \notin \mathbb{N}$ and $H_4^{(r)} = \frac{(r+1)(r+2)(r+3)}{24} + \frac{r(r+2)(r+3)}{24} + \frac{r(r+1)(r+3)}{24} + \frac{r(r+1)(r+2)}{24} \notin \mathbb{N}$. For any $n \geqslant 5$, we have by relation (1.1)

$$H_n^{(r)} = \frac{(n+1)(n+2)\cdots(n+r-1)}{(r-1)!}\left(H_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+r-1} - H_{r-1}\right).$$

Set $E_{r,n} := (r-1)!\left(H_n^{(r)} - \binom{n+r-1}{r-1}H_{r-1}\right) - (n+1)\cdots(n+r-1)\left(\frac{1}{n+1} + \cdots + \frac{1}{n+r-1}\right)$. Thus $E_{r,n} = (n+1)(n+2)\cdots(n+r-1)\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$.

Assume that $H_n^{(r)}$ is an integer. So $E_{r,n}$ is an integer as well. Let P be the greatest prime $\leqslant n$. Then we have

$$\frac{n!}{P}E_{r,n} = \frac{(n+r-1)!}{P}\left(1 + \cdots + \frac{1}{P} + \cdots + \frac{1}{n}\right),$$

and therefore

$$\frac{(n+r-1)!}{P}E_{r,n} - \frac{(n+r-1)!}{P}\sum_{k \neq P}\frac{1}{k} = \frac{(n+r-1)!}{P^2}.$$

The left side of the equality is an integer. If the right side is an integer, then P should divide $(n+2)\cdots(n+r-1)$, hence one of the integers $n, \ldots, (n+r-3)$ should be equal to $2P-2$, so either there exist a prime Q strictly between P and $n+1$ and this is a contradiction with Bertrand's postulate, either one of the integers $n+k$ with $1 \leqslant k \leqslant r-4$ is prime and this contradicts the assumption of the Theorem. □

It is well known that we can exhibit an arbitrary long sequence of consecutive composite integers: $m!+2$, $m!+3, \ldots,$ $m!+m$, $(m \geqslant 3)$. We will use this fact to establish that for all $r \geqslant 5$, we can find a non integer hyperharmonic number $H_n^{(r)}$.

Corollary 2.5. *Let $r \geqslant 5$, then the hyperharmonic numbers $H_{r!+1}^{(r)}$, $H_{r!+2}^{(r)}$, $H_{r!+3}^{(r)}$ and $H_{r!+4}^{(r)}$ satisfy the non-integer property.*

Proof. It suffices to use Theorem 2.4. □

The arguments used in the proof of Theorem 2.4 give more. As an illustration, we treat the case $r = 5$.

Proposition 2.6. *For any $n \geqslant 2$, the hyperharmonic number $H_n^{(5)}$ is never an integer when $n+1 \neq 2Q-3$ is prime with Q prime.*

Proof. For $n = 2$ or 3, n odd, or even with $n+1$ composite, see Theorem 2.4. For even $n \geqslant 4$ with $n+1$ prime, using notations in the proof of Theorem 2.4, if $H_n^{(5)} \in \mathbb{N}$ then $P \mid (n+2)(n+3)(n+4)$. We have $P \nmid (n+2)$, there would be a prime between P and $n = 2P-2$. We have $P \nmid (n+3)$, otherwise $n+3 = 2P$ which contradicts the fact $n+3$ is odd. Finally, if $n+4 = 2P$ i.e. $n+1 = 2P-3$, we have a contradiction. □

Example 2.7. For $n \leqslant 100$, we list the values of r, given by Theorem 2.4, such that $H_n^{(r)}$ is never an integer.

1. $H_n^{(5)} \notin \mathbb{N}$ for $n =$ 2, 3, **4**, 5, 7, 8, 9, 11, **12**, 13, 14, 15, **16**, 17, 19, 20, 21, 23, 24, 25, 26, 27, **28**, 29, 31, 32, 33, 34, 35, **36**, 37, 38, 39, **40**, 41, 43, 44, 45, **46**, 47, 48, 49, 50, 51, **52**, 53, 54, 55, 56, 57, 59, **60**, 61, 62, 63, 64, 65, **66**, 67, 68, 69, 71, **72**, 73, 74, 75, 76, 77, 79, 80, 81, 83, 84, 85, 86, 87, **88**, 89, 90, 91, 92, 93, 94, 95, **96**, 97, 98, 99, **100**.

   The bold numbers are given by Proposition 2.6.

2. $H_n^{(6)} \notin \mathbb{N}$ for $n =$ 2, 3, 7, 8, 13, 14, 19, 20, 23, 24, 25, 26, 31, 32, 33, 34, 37, 38, 43, 44, 47, 48, 49, 50, 53, 54, 55, 56, 61, 62, 63, 64, 67, 68, 73, 74, 75, 76, 79, 80, 83, 84, 85, 86, 89, 90, 91, 92, 93, 94, 97, 98.

3. $H_n^{(7)} \notin \mathbb{N}$ for $n =$ 2, 3, 7, 19, 23, 24, 25, 31, 32, 33, 37, 43, 47, 48, 49, 53, 54, 55, 61, 62, 63, 67, 73, 74, 75, 79, 83, 84, 85, 89, 90, 91, 92, 93, 97.

4. $H_n^{(8)} \notin \mathbb{N}$ for $n =$ 2, 3, 23, 24, 31, 32, 47, 48, 53, 54, 61, 62, 73, 74, 83, 84, 89, 90, 91, 92.

5. $H_n^{(9)} \notin \mathbb{N}$ for $n =$ 2, 3, 23, 31, 47, 73, 83, 89, 90, 91.

6. $H_n^{(10)} \notin \mathbb{N}$ for $n =$ 2, 3, 89, 90.

7. $H_n^{(11)} \notin \mathbb{N}$ for $n =$ 2, 3, 89.

Acknowledgements. The second author is grateful to István Mező for sending us a copy of his cited paper and pointing our attention to hyperharmonic numbers. We are also grateful to Yacine Aït Amrane for useful discussions.

References

[1] Belbachir, H., Khelladi, A., On a sum involving powers of reciprocals of an arithmetical progression, *Annales Mathematicae et Informaticae*, 34 (2007), 29–31.

[2] Benjamin, A. T.,Gaebler, D., Gaebler, R., A combinatorial approach to hyperharmonic numbers, *INTEGERS: The Electronic Journal of Combinatorial Number Theory*, 3 (2003), #A15.

[3] Conway, J. H., Guy, R. K., The book of numbers, *New York, Springer-Verlag*, 1996.

[4] Hardy, G. H., Wright, E. M., An introduction to the theory of numbers, *Oxford at the Clarendon Press*, 1979.

[5] Mező, I., About the non-integer property of hyperharmonic numbers, *Annales Univ. Sci. Budapest., Sect. Math.*, 50 (2007), 13–20.

[6] Taeisinger, L., Bemerkung über die harmonische Reihe, *Monatschefte für Mathematik und Physik*, 26 (1915), 132–134.

Rachid Aït Amrane
ESI/ Ecole nationale Supérieure d'Informatique,
BP 68M, Oued Smar,
16309, El Harrach,
Alger, Algeria
e-mail: `r_ait_amrane@esi.dz, raitamrane@gmail.com`

Hacène Belbachir
USTHB/ Faculté de Mathématiques,
BP 32, El Alia,
16111 Bab Ezzouar,
Alger, Algeria
e-mail: `hbelbachir@usthb.dz, hacenebelbachir@gmail.com`