A common fixed point theorem via a generalized contractive condition

Abdelkrim Aliouche\textsuperscript{a}, Faycel Merghadi\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, University of Larbi Ben M’Hidi Oum-El-Bouaghi, Algeria
\textsuperscript{b}Department of Mathematics, University of Tebessa, Algeria

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\textbf{Abstract}

We prove a common fixed point theorem for mappings satisfying a generalized contractive condition which generalizes the results of [3, 4, 12, 15, 19, 20, 24] and we correct the errors of [7, 12, 20].

\textit{Keywords:} Metric space, weakly compatible mappings, common fixed point.

\textit{MSC:} 47H10, 54H25

\section{1. Introduction}

Sessa [21] defined $S$ and $T$ to be weakly commuting as a generalization of commuting if for all $x \in X$.

$$d(STx, TSx) \leq d(Tx, Sx).$$

Jungck [9] defined $S$ and $T$ to be compatible as a generalization of weakly commuting if

$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$. It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper, see [9, 21]. Jungck et al [10] defined $S$ and $T$ to be compatible mappings of type (A) if

$$\lim_{n \to \infty} d(STx_n, T^2x_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(TSx_n, S^2x_n) = 0,$$
whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). Example are given to show that the two concepts of compatibility are independent, see [10]. Recently, Pathak and Khan [16] defined \( S \) and \( T \) to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

\[
\lim_{n \to \infty} d(TSx_n, S^2x_n) \leq \frac{1}{2} \left[ \lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, T^2x_n) \right],
\]

\[
\lim_{n \to \infty} d(STx_n, T^2x_n) \leq \frac{1}{2} \left[ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, S^2x_n) \right],
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). Clearly compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true, see [16]. However, compatible mappings of type (A) and compatibility of type (B) are equivalent if \( S \) and \( T \) are continuous, see [16]. Pathak et al [17] defined \( S \) and \( T \) to be compatible mappings of type (P) if

\[
\lim_{n \to \infty} d(S^2x_n, T^2x_n) = 0,
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if \( S \) and \( T \) are continuous, see [17]. Pathak et al [18] defined \( S \) and \( T \) to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

\[
\lim_{n \to \infty} d(TSx_n, S^2x_n) \leq \frac{1}{3} \left[ \lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, S^2x_n) + \lim_{n \to \infty} d(Tt, T^2x_n) \right],
\]

\[
\lim_{n \to \infty} d(STx_n, T^2x_n) \leq \frac{1}{3} \left[ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, T^2x_n) + \lim_{n \to \infty} d(St, S^2x_n) \right],
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). Compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if \( S \) and \( T \) are continuous, see [18]. Pant [15] defined \( S \) and \( T \) to be reciprocally continuous if

\[
\lim_{n \to \infty} STx_n = St \quad \text{and} \quad \lim_{n \to \infty} TSx_n = Tt,
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). It is clear that if \( S \) and \( T \) are both continuous, then they are reciprocally continuous, but the converse is not true. Moreover, it was proved in [15] that in the setting of common fixed point theorem for compatible mappings satisfying contractive conditions, the continuity of one of the mappings \( S \) and \( T \) implies their reciprocal continuity, but not conversely.

2. Preliminaries

**Definition 2.1** (See [11]). \( S \) and \( T \) are said to be weakly compatible if they commute at their coincidence points; i.e., if \( Su = Tu \) for some \( u \in X \), then \( STu = TSu \).
Lemma 2.2 (See [9, 10, 16, 17, 18]). If $S$ and $T$ are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.

The converse is not true in general, see [4].

Definition 2.3 (See [13]). $S$ and $T$ are said to be $R$–weakly commuting if there exists an $R > 0$ such that

$$d(STx, TSx) \leq Rd(Tx, Sx) \text{ for all } x \in X. \quad (2.1)$$

Definition 2.4 (See [14]). $S$ and $T$ are pointwise $R$–weakly commuting if for all $x \in X$, there exists an $R > 0$ such that (2.1) holds.

It was proved in [14] that $R$-weakly commutativity is equivalent to commutativity at coincidence points; i.e., $S$ and $T$ are pointwise $R$-weakly commuting if and only if they are weakly compatible.

Lemma 2.5 (See [22]). For any $t \in (0, \infty)$, $\psi(t) < t$ iff $\lim_{n \to \infty} \psi^n(t) = 0$, where $\psi^n$ denotes the $n$-times repeated composition of $\psi$ with itself.

Several authors proved fixed point and common fixed point theorems for mappings satisfying contractive conditions of integral type, see [1, 3, 4, 5, 6, 7, 12, 19, 20]. The following theorem was proved by [3].

Theorem 2.6 (See [3]). Let $A, B, S$ and $T$ be self-mappings of a metric space $(X, d)$ satisfying

$$S(X) \subset B(X) \quad \text{and} \quad T(X) \subset A(X),$$

$$\int_0^{d(Sx, Ty)} \varphi(t) \, dt \leq \psi \left( \int_0^{M(x, y)} \varphi(t) \, dt \right)$$

for all $x, y \in X$, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a right continuous function such that $\psi(0) = 0$ and $\psi(s) < s$ for all $s > 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue integrable mapping which is summable and satisfies

$$\int_0^c \varphi(t) \, dt > 0,$$

$$M(x, y) = \max \left\{ d(Ax, By), d(Sx, Ax), d(Ty, By), \frac{1}{2} [d(Sx, By) + d(Ty, Ax)] \right\}.$$

If one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of $X$, then $A$ and $S$ have a coincidence point and $B$ and $T$ have a coincidence point. Further, if $S$ and $A$ as well as $T$ and $B$ are weakly compatible, then $A, B, S$ and $T$ have a unique common fixed point in $X$.


Let $A \in (0, \infty)$, $R_A^+ = [0, A)$ and $F : R_A^+ \to \mathbb{R}$ satisfying
(i) \( F(0) = 0 \) and \( F(t) > 0 \) for each \( t \in (0, A) \),

(ii) \( F \) is nondecreasing on \( R^+_A \),

(iii) \( F \) is continuous.

Define \( I[0, A] = \{ F : F \text{ satisfies (i)-(iii)} \} \).

**Lemma 2.7** (See [24]). Let \( A \in (0, \infty] \), \( F \in I[0, A] \). If \( \lim_{n \to \infty} F(\epsilon_n) = 0 \) for \( \epsilon_n \in R^+_A \), then \( \lim_{n \to \infty} \epsilon_n = 0 \).

The following examples were given in [24].

(i) Let \( F(t) = t \), then \( F \in I[0, A] \) for each \( A \in (0, \infty] \).

(ii) Suppose that \( \varphi \) is nonnegative, Lebesgue integrable on \([0, A]\) and satisfies

\[
\int_0^\epsilon \varphi(t) \, dt > 0 \quad \text{for each} \quad \epsilon \in (0, A).
\]

Let \( F(t) = \int_0^t \varphi(s) \, ds \), then \( F \in [0, A] \).

(iii) Suppose that \( \psi \) is nonnegative, Lebesgue integrable on \([0, A]\) and satisfies

\[
\int_0^\epsilon \psi(t) \, dt > 0 \quad \text{for each} \quad \epsilon \in (0, A)
\]

and \( \varphi \) is nonnegative, Lebesgue integrable on \([0, \int_0^A \psi(s) \, ds]\) and satisfies

\[
\int_0^\epsilon \varphi(t) \, dt > 0 \quad \text{for each} \quad 0 < \epsilon < \int_0^A \psi(s) \, ds.
\]

Let \( F(t) = \int_0^{\int_0^t \psi(s) \, ds} \varphi(u) \, du \), then \( F \in I[0, A] \).

(iv) If \( G \in [0, A) \) and \( F \in I[0, G(A - 0)) \), then a composition mapping \( F \circ G \in I[0, A] \). For instance, let \( H(t) = \int_0^{F(t)} \varphi(s) \, ds \), then \( H \in I[0, A] \) whenever \( F \in I[0, A] \) and \( \varphi \) is nonnegative, Lebesgue integrable on \([0, F(A - 0))\) and satisfies

\[
\int_0^\epsilon \varphi(t) \, dt > 0 \quad \text{for each} \quad \epsilon \in (0, F(A - 0)).
\]

Let \( A \in (0, \infty) \) and \( \psi : R^+_A \to \mathbb{R}_+ \) satisfying

(i) \( \psi(t) < t \) for all \( t \in (0, A) \)

(ii) \( \psi \) is upper semi-continuous.

(iii) \( \psi \) is nondecreasing on \( R^+_A \).

Define \( \Psi[0, A] = \{ \psi : \psi \text{ satisfies (i)-(iii)} \} \).

3. Main results

**Theorem 3.1.** Let \( (X, d) \) be a metric space and \( D = \sup\{d(x, y) : x, y \in X\} \). Set \( A = D \) if \( D = \infty \) and \( A > D \) if \( D < \infty \). Let \( A_1, A_2, S \) and \( T \) be self-mappings of \( (X, d) \) satisfying

\[
A_1(X) \subset T(X) \quad \text{and} \quad A_2(X) \subset S(X),
\]
\[
F(d(A_1x, A_2y)) \leq \psi(F(L(x,y)))
\] (3.1)
for all \(x, y \in X\), where
\[
L(x,y) = \max \left\{ d(Sx, Ty), d(A_1x, Sx), d(Ty, A_2y), \frac{1}{2} [d(Sx, A_2y) + d(A_1x, Ty)] \right\},
\]
\(F \in F[0,A]\) and \(\psi \in \Psi[0,F(A - 0)]\) for all \(A \in (0, \infty)\). Suppose that the pair \((A_1, S)\) is weakly compatible and there exists \(w \in C(A_2, T)\): the set of coincidence points of \(A_2\) and \(T\) such that \(A_2Tw = TA_2w\). If one of \(A_1(X), A_2(X), S(X)\) and \(T(X)\) is a complete subspace of \(X\), then \(A_1, A_2, S\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0\) be arbitrary point in \(X\). Inductively, we can define a sequence \(\{y_n\}\) in \(X\) such that
\[
y_{2n} = A_1x_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Sx_{2n+2} = A_2x_{2n+1}
\]
for all \(n \in \{0, 1, 2, \ldots\}\). As in the proof of [2], \(\{y_n\}\) is a Cauchy sequence in \(X\). Assume that \(S(X)\) is complete. Therefore
\[
\lim_{n \to \infty} A_1x_{2n} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} A_2x_{2n+1} = \lim_{n \to \infty} Sx_{2n+1} = z = Su
\]
for some \(u \in X\). If \(A_1u \neq z\) using (3.1) we obtain
\[
F(d(A_1u, A_2x_{2n+1})) \leq \psi(F(L(u, x_{2n})))
\]
where
\[
L(u, x_{2n}) = \max \left\{ d(Su, Tx_{2n+1}), d(A_1u, Su), d(Tx_{2n+1}, A_2x_{2n+1}), \right.
\]
\[
\left. \quad \frac{1}{2} [d(Su, A_2x_{2n+1}) + d(A_1u, Tx_{2n+1})] \right\}.
\]
Letting \(n \to \infty\), we get
\[
F(d(A_1u, z)) \leq \psi(F(d(A_1u, z))) < F(d(A_1u, z))
\]
which is a contradiction and so \(z = A_1u = Su\). If \(z \neq A_2w\), applying (3.1) we obtain
\[
F(d(A_1u, A_2w)) \leq \psi(F(d(A_1u, A_2w)))
\]
where
\[
L(u, v) = \max \left\{ d(Su, Tw), d(A_1u, Su), d(Tw, A_2w), \frac{1}{2} [d(Su, A_2w) + d(A_1u, Tw)] \right\}.
\]
Hence
\[
F(d(z, A_2w)) \leq \psi(F(d(z, A_2w))) < F(d(z, A_2w))
\]
which is a contradiction and so \(z = A_1u = Su = A_2w = Tw\).
Corollary 3.2. Let \( A \) be a metric space and \( D = \sup\{d(x,y) : x, y \in X\} \). Suppose that the pair \( (A, S) \) is weakly compatible and there exists \( w \in C(A_2, T) \) such that \( A_2Tw = TA_2w \), we have \( Sz = A_1z \) and \( Tz = A_2z \).

If \( A_1z \neq z \) we have by (3.1)
\[
F(d(A_1z, A_2w)) \leq \psi(F(L(z, w)))
\]
where
\[
L(z, w) = \max\{d(Sz, Tw), d(A_1z, Sz), d(Bw, A_2w), \frac{1}{2}[d(Sz, A_2w) + d(A_1z, Tw)]\}.
\]
Therefore
\[
F(d(A_1z, z)) \leq \psi(F(d(A_1z, z))) < F(d(A_1z, z))
\]
and so \( A_1z = Sz = z \). Similarly, we can prove that \( A_2z = Tz = z \).

The proof is similar when \( T(X) \) is assumed to be a complete subspace of \( X \). The case in which \( A_1(X) \) or \( A_2(X) \) is a complete subspace of \( X \) is similar to the case in which \( T(X) \) or \( S(X) \) respectively is complete since \( A_1(X) \subseteq T(X) \) and \( A_2(X) \subseteq S(X) \). The uniqueness of \( z \) follows from (3.1). \( \square \)

Theorem 3.1 generalizes Theorem 2.6 of [3].

Corollary 3.2. Let \((X, d)\) be a metric space and \( D = \sup\{d(x,y) : x, y \in X\} \). Set \( A = D \) if \( D = \infty \) and \( A > D \) if \( D < \infty \). Let \( \{A_i\}, i = 1, 2, \ldots, S \) and \( T \) be self-mappings of \((X, d)\) satisfying
\[
A_i(x) \subseteq T(X)\text{ and } A_i(X) \subseteq S(X), i \geq 2
\]
and
\[
F(d(A_1x, A_iy)) \leq \psi(F(L_i(x, y))), i \geq 2
\]
for all \( x, y \) in \( X \), where
\[
L_i(x, y) = \max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), \frac{1}{2}[d(Sx, A_iy) + d(A_1x, Ty)]\},
\]
\( F \in \mathcal{F}[0, A] \) and \( \psi \in \Psi[0, F(A - 0)] \) for all \( A \in (0, \infty) \). Suppose that the pair \((A_1, S)\) is weakly compatible and there exists \( w \in C(A_1, T)\): the set of coincidence points of \( A_i \) and \( T \) such that \( A_iTw = TA_iw \) for some \( i \geq 2 \). If one of \( A_i(X), S(X) \) and \( T(X) \) is a complete subspace of \( X \). Then \( A_i, S \) and \( T \) have a unique common fixed point in \( X \).

If \( \varphi(t) = 1 \) in Corollary 3.2, we get a generalization of a theorem of [15]. The following example illustrates our corollary 3.2.

Example 3.3. Let \( X = [0, 10] \) be endowed with the metric \( d(x, y) = |x - y| \),

\[
Sx = \begin{cases} 
0, & \text{if } x = 0, \\
x + 8, & \text{if } x \in (0, 2], \\
x - 2, & \text{if } x \in (2, 10],
\end{cases}
\]

\[
T(x) = \begin{cases} 
0, & \text{if } x = 0, \\
x + 5, & \text{if } x \in (0, 2], \\
x - 2, & \text{if } x \in (2, 10],
\end{cases}
\]
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\[
\begin{align*}
A_1x &= \begin{cases} 
3, & \text{if } x \in (0, 2], \\
0, & \text{if } x \in \{0\} \cup (2, 10], 
\end{cases} & A_2x &= \begin{cases} 
0, & \text{if } x \in [0, 2], \\
4, & \text{if } x \in (2, 10], 
\end{cases} \\
A_3x &= \begin{cases} 
0, & \text{if } x \in [0, 2], \\
5, & \text{if } x \in (2, 10], 
\end{cases} & A_4x &= \begin{cases} 
0, & \text{if } x \in [0, 2], \\
6, & \text{if } x \in (2, 10], 
\end{cases} \\
A_ix &= \begin{cases} 
2 + \frac{2}{i}, & \text{if } x \in (0, 2], \\
0, & \text{if } x \in \{0\} \cup (2, 10], 
\end{cases} & \text{for all } i > 4.
\end{align*}
\]

The pair \((A_1, S)\) is weakly compatible, but it is not compatible of type (A), (B), (P) and (C), see [6].

\(A_1 (X) \subset T (X)\) and \(A_i (X) \subset S (X)\).

The pair \((A_i, T)\), \(i > 4\), is weakly compatible because \(A_i\) and \(T\) commute at their coincidence point \(x = 0\), but it is not compatible of type (A), (B), (P) and (C).

Let \(x_n = 2 + \frac{1}{n}\). We have \(Tx_n = \frac{1}{n}\) and \(A_ix_n = 0\), hence

\[
\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} A_ix_n = t = 0.
\]

In the other hand, \(A_iTx_n = A_i(\frac{1}{n}) = 2 + \frac{2}{i}\) and \(TA_ix_n = T0 = 0\) and so

\[
\lim_{n \to \infty} d (A_iTx_n, TA_ix_n) = 2 + \frac{2}{i} \neq 0.
\]

Therefore, the pair \((A_i, T)\) is not compatible.

\[
A_i^2x_n = A_i0 = 0 \quad \text{and} \quad T^2x_n = T \left( \frac{1}{n} \right) = 5 + \frac{1}{n},
\]

so \(\lim_{n \to \infty} |TA_ix_n - A_i^2x_n| = 0\) and

\[
\lim_{n \to \infty} |A_iTx_n - T^2x_n| = \lim_{n \to \infty} (3 + \frac{1}{n} - \frac{2}{i}) \neq 0 \quad \text{for all } i > 3.
\]

Then, \((A_i, T)\) is not compatible of type (A).

\[
\lim_{n \to \infty} |A_iTx_n - T^2x_n| = 3 - \frac{2}{i} > \frac{1}{2} \left[ \lim_{n \to \infty} |A_iTx_n - A_i0| + \lim_{n \to \infty} |A_i0 - A_i^2x_n| \right] = \frac{1}{2} \left( 2 + \frac{2}{i} \right) = \frac{1}{i} + 1,
\]

hence \((A_i, T)\) is not compatible of type (B).

\[
\lim_{n \to \infty} |A_i^2x_n - T^2x_n| = \lim_{n \to \infty} (5 + \frac{1}{n}) = 5 \neq 0.
\]

Therefore, \((A_i, T)\) is not compatible of type (P).

\[
\lim_{n \to \infty} |A_iTx_n - T^2x_n| = 3 - \frac{2}{i} > \frac{1}{3} \left[ \lim_{n \to \infty} |A_iTx_n - A_i0| + \lim_{n \to \infty} |A_i0 - T^2x_n| + \lim_{n \to \infty} |A_i0 - A_i^2x_n| \right] = \frac{1}{3} \left( 7 + \frac{2}{i} \right)
\]

for \(i > 4\). So, the pair \((A_i, T)\) is not compatible of type (C).

It can be verified that the pairs \((A_2, T)\), \((A_3, T)\) and \((A_4, T)\) are not weakly compatible because \(x = 6\) is a coincidence point of \(A_2\) and \(T\), but \(A_2T6 = 4 \neq 0\).
We have the following cases. If $x = 0$ and $y = 0$ we get $R \leq 0$ for all $0 \leq h < 1$. If $x = 0$ and $y \in (0, 2]$, we get
$$R = \ln 4 - h \max \left\{ \ln (x + 9), \ln (x + 6), \frac{1}{2} \ln (x + 9) \right\} \leq 0$$
for $h \geq \frac{\ln 3}{\ln 8}$. Hence, there exists $0 \leq h < 1$. If $x \in (0, 2]$ and $y \in (0, 2]$, we get
$$R = \ln \left( 3 + \frac{2}{i} \right) - h \max \left\{ \ln (y + 6), \ln (y + 4 - \frac{2}{i}) \right\} \leq 0$$
for $h \geq \frac{\ln 3 + \frac{2}{i}}{3 \ln 2}$ and so there exists $0 \leq h < 1$. If $x \in (0, 2]$ and $y \in (2, 10]$, we get
$$R = -h \max \{ \ln (y - 1), \ln (y - 1), \ln (y - 1) \} \leq 0$$
for all $0 \leq h < 1$. If $x \in (0, 2]$ and $y = 0$, we get
$$R = \ln (x + 3) - h \max \left\{ \ln (x + 3), \ln (x + 1), \frac{1}{2} \ln (x + 3) \right\} \leq 0$$
for $h \geq \frac{\ln 3}{\ln 8}$. Hence, there exists $0 \leq h < 1$. If $x \in (0, 2]$, we get
$$R = -h \max \left\{ \ln (y - 1), \ln (y - 1), \frac{1}{2} \ln (x + 9) \right\} \leq 0$$
for all $h \geq 0$. Hence, there exists $0 \leq h < 1$. If $x \in (2, 10]$ and $y \in (0, 2]$, we get
$$R = \ln \left( 3 + \frac{2}{i} \right) - h \max \left\{ \ln (|x - (y + 7)| + 1), \ln (x - 1), \frac{1}{2} \ln (y + 5) + \ln (|x - 4 - \frac{2}{i}| + 1) \right\} \leq 0$$
for $h \geq \frac{\ln 3 + \frac{2}{i}}{\ln 9}$. Hence, there exists $0 \leq h < 1$. If $x, y \in (2, 10]$ we get
$$R = -h \max \left\{ \ln (|x - y| + 1), \ln (x - 1), \ln (y - 1), \frac{1}{2} \ln (y - 1) \right\} \leq 0$$
for all $0 \leq h < 1$.

Now, we verify that $(A_2, T)$ and $(A_3, T)$ satisfy all the conditions of Theorem 4.2. Set

$$R_1 = \int_0^{\| A_1x - A_2y \|} \frac{1}{1 + t} \, dt - h \max \left\{ \int_0^{\| Sx - Ty \|} \frac{1}{1 + t} \, dt, \int_0^{\| A_1x - Sx \|} \frac{1}{1 + t} \, dt, \int_0^{\| A_2y - Ty \|} \frac{1}{1 + t} \, dt \right\}$$

We have the following cases. If $x = 0$ and $y = 0$ we get $R_1 \leq 0$ for all $0 \leq h < 1$. If $x = 0$ and $y \in (0, 2]$, we get

$$R_1 = -h \max \left\{ \ln (y + 6), 0, \ln (y + 6), \frac{1}{2} [y + 6] \right\} \leq 0$$

for all $0 \leq h < 1$. If $x = 0$ and $y \in (2, 10]$, we get

$$R_1 = \ln 5 - h \max \left\{ \ln (y - 1), \ln (|y - 6| + 1), \frac{1}{2} [\ln (y - 1) + \ln 5] \right\} \leq 0$$

for $h \geq \frac{\ln 5}{\ln 9}$. Hence there exists $0 \leq h < 1$. If $x \in (0, 2]$ and $y = 0$, we get

$$R_1 = \ln 4 - h \max \left\{ \ln (x + 9), \ln (x + 6), 0, \frac{1}{2} [\ln 4 + \ln (x + 9)] \right\} \leq 0$$

for all $h \geq \frac{\ln 4}{\ln 11}$. Hence, there exists $0 \leq h < 1$. If $x \in (0, 2]$ and $y \in (0, 2]$, we get

$$R_1 = \ln 4 - h \max \left\{ \ln (4 + x - y), \ln (x + 6), \ln (y + 6), \frac{1}{2} [\ln (y + 3) + \ln (x + 9)] \right\} \leq 0$$

for $h \geq \frac{\ln 4}{\ln 8}$. Hence there exists $0 \leq h < 1$. If $x \in (0, 2]$ and $y \in (2, 10]$, we get

$$R_1 = \ln 2 - h \max \left\{ \ln 11, \ln (x + 6), \ln (|y - 6| + 1), \frac{1}{2} [\ln (5 - y) + \ln (x + 5)] \right\} \leq 0$$

for $h \geq \frac{\ln 2}{\ln 11}$. Hence, there exists $0 \leq h < 1$. If $x \in (2, 10]$ and $y = 0$, we get

$$R_1 = -h \max \left\{ \ln (x - 1), \ln (x - 1), \frac{1}{2} \ln (x - 1) \right\} \leq 0$$

for all $0 \leq h < 1$. In the same manner, if $x \in (2, 10]$ and $y \in (0, 2]$, we get $R_1 \leq 0$ for all $0 \leq h < 1$. If $x \in (2, 10]$ and $y \in (2, 10]$, we get

$$R_1 = \ln 5 - h \max \left\{ \ln (|x - y| + 1), \ln (x - 1), \frac{1}{2} [\ln (y - 1) + \ln (|y - 6| + 1)] \right\} \leq 0$$
for $h \geq \frac{\ln 5}{3}$. Hence, there exists $0 \leq h < 1$. Similarly, we can prove the conditions of Theorem 4.2 if we take the mapping $A_3$ instead of $A_2$. Finally, we remark that all conditions of our theorem are verified and $0$ is the unique common fixed point of $A_i, S$ and $T$.

The following example supports our Theorem 3.1.

**Remark 3.4.** In this example, Theorem 2.6 of [3] is not applicable since the pair $(A_2, T)$ is not weakly compatible, but Theorem 3.1 is applicable. Also, a theorem of [15] for $A_i = A_2$ for all $i \geq 2$ is not applicable since the pairs $(A_1, S)$ and $(A_2, T)$ are not compatible. In the same manner, Theorem 1 of [12] is not applicable.

**Remark 3.5.** In the proof of Lemma 1 of [20] and Theorem 2.1 of [7], the authors applied the inequality

$$a \leq b + c \implies \int_0^a \varphi(t)dt \leq \int_0^b \varphi(t)dt + \int_0^c \varphi(t)dt$$

which is false in general as it is shown by the following example.

**Example 3.6.** Let $\varphi(t) = t$, $a = 1$, $b = \frac{1}{2}$ and $c = \frac{3}{4}$. Then $1 < \frac{1}{2} + \frac{3}{4}$, but

$$\int_0^1 \varphi(t)dt = \frac{1}{2} > \int_0^{\frac{1}{2}} \varphi(t)dt + \int_0^{\frac{3}{4}} \varphi(t)dt$$

$$= \frac{1}{8} + \frac{9}{32} = \frac{13}{32}.$$

To correct these errors, the authors should follow the proof of Theorem 2 of [19].

**Remark 3.7.** In the proof of Theorem 1 of [12], the authors applied the inequality

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \implies \{x_n\} \text{ is a Cauchy sequence}$$

which is false in general. It suffices to take $x_n = \frac{1}{n}$, $n \in \mathbb{N}^*$. Thus, To correct this error, the authors should follow the proof of Theorem 2 of [19].

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A. Aliouche
Department of Mathematics
University of Larbi Ben M’Hidi Oum-El-Bouaghi
04000
Algeria
e-mail: alioumath@yahoo.fr

F. Merghadi
Department of Mathematics
University of Tebessa
12000
Algeria
e-mail: faycel_mr@yahoo.fr