

An almost sure limit theorem for α -mixing random fields

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Abstract

An almost sure limit theorem with logarithmic averages for α -mixing random fields is presented.

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1. Introduction

Let \mathbb{N} be the set of the positive integers, \mathbb{R} the set of real numbers and \mathcal{B} the σ -algebra of Borel sets of \mathbb{R} . Let δ_x be the unit mass at point x , that is $\delta_x: \mathcal{B} \rightarrow \mathbb{R}$, $\delta_x(B) = 1$ if $x \in B$ and $\delta_x(B) = 0$ if $x \notin B$. Denote $\xrightarrow{w} \mu$ the weak convergence to the probability measure μ . In the following all random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Almost sure (a.s.) limit theorems state that

$$\frac{1}{D_n} \sum_{k=1}^n d_k \delta_{\zeta_k(\omega)} \xrightarrow{w} \mu \quad \text{as } n \rightarrow \infty, \quad \text{for almost every } \omega \in \Omega,$$

where ζ_k ($k \in \mathbb{N}$) are random variables. The simplest form of it is the so-called classical a.s. central limit theorem, in which $\zeta_k = (X_1 + \dots + X_k)/\sqrt{k}$, where X_1, X_2, \dots are independent identically distributed (i.i.d.) random variables with expectation 0 and variance 1, moreover $d_k = 1/k$, $D_n = \log n$ and μ is the standard normal distribution $\mathcal{N}(0, 1)$. (See Berkes [1] for an overview.)

Let \mathbb{N}^d be the positive integer d -dimensional lattice points, where d is a fixed positive integer. In this paper $\mathbf{k} = (k_1, \dots, k_d)$, $\mathbf{n} = (n_1, \dots, n_d), \dots \in \mathbb{N}^d$. Relations \leq , $\not\leq$, \min , \max , \rightarrow etc. are defined coordinatewise, i.e. $\mathbf{n} \rightarrow \infty$ means that

$n_i \rightarrow \infty$ for all $i \in \{1, \dots, d\}$. Let $|\mathbf{n}| = \prod_{i=1}^d n_i$ and $|\log \mathbf{n}| = \prod_{i=1}^d \log_+ n_i$, where $\log_+ x = \log x$ if $x \geq e$ and $\log_+ x = 1$ if $x < e$. The general form of the multiindex version of the a.s. limit theorems is

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)} \xrightarrow{w} \mu \quad \text{as } \mathbf{n} \rightarrow \infty, \quad \text{for almost every } \omega \in \Omega,$$

where $\{\zeta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ is a random field (multiindex sequence of random variables). In the multiindex version of the classical a.s. central limit theorem $X_{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d$ i.i.d. random variables with expectation 0 and variance 1, $\zeta_{\mathbf{k}} = \sum_{\mathbf{i} \leq \mathbf{k}} X_{\mathbf{i}} / \sqrt{|\mathbf{k}|}$, $d_{\mathbf{k}} = 1/|\mathbf{k}|$, $D_{\mathbf{n}} = 1/|\log \mathbf{n}|$ and $\mu = \mathcal{N}(0, 1)$. It is well-known that generally the multiindex cases are not direct consequences of the corresponding theorems for ordinary sequences.

Fazekas and Rychlik proved in [5] a general a.s. limit theorem for multiindex sequences of metric space valued random elements. Tómacs proved in [8] an a.s. central limit theorem for m -dependent random fields. In this paper we shall prove an a.s. limit theorem with logarithmic averages for α -mixing random fields (Theorem 2.5). Its onedimension version for $\mu = \mathcal{N}(0, 1)$ is proved by Fazekas and Rychlik (see [4, Proposition 3.2]). In the proof of Theorem 2.5 we shall use a multiindex strong law of large numbers (Theorem 2.1). In the proof of Theorem 2.3 we shall follow ideas of Berkes and Csáki [2].

Throughout the paper we use the following notation. Let \mathbb{R}_+ be the set of the positive real numbers. If $a_1, a_2, \dots \in \mathbb{R}$ then in case $A = \emptyset$ let $\max_{k \in A} a_k = 0$ and $\sum_{k \in A} a_k = 0$. Let $[A]$ be the closure of $A \subset \mathbb{R}$ and $\partial A = [A] \cap [\overline{A}]$.

If ξ is a random variable, then let μ_{ξ} denote the distribution of ξ , $\|\xi\|_{\infty} = \inf\{c \in \mathbb{R} : \mathbb{P}(|\xi| \leq c) = 1\}$ and $\sigma(\xi) = \{\xi^{-1}(B) : B \in \mathcal{B}\}$.

In the following let $\{c_k^{(i)} \in \mathbb{R}_+, k \in \mathbb{N}\}$ be increasing sequences with $c_{k+1}^{(i)}/c_k^{(i)} = O(1)$, $\lim_{n \rightarrow \infty} c_n^{(i)} = \infty$ for each $i = 1, \dots, d$, and the sequences $\{d_k^{(i)} \in \mathbb{R}_+, k \in \mathbb{N}\}$ have the next properties: $d_k^{(i)} \leq \log(c_{k+1}^{(i)}/c_k^{(i)})$ for all $k \in \mathbb{N}$ and $i = 1, \dots, d$, moreover $\sum_{k=1}^{\infty} d_k^{(i)} = \infty$ for each $i = 1, \dots, d$. Let $d_{\mathbf{k}} = \prod_{i=1}^d d_{k_i}^{(i)}$, $D_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}}$ and $D_{n_i}^{(i)} = \sum_{k=1}^{n_i} d_k^{(i)}$.

2. Results

Theorem 2.1. *Let $\{\xi_{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d\}$ be a uniformly bounded random field, namely there exists $c \in \mathbb{R}_+$ such that $|\xi_{\mathbf{i}}| \leq c$ a.s. for all $\mathbf{i} \in \mathbb{N}^d$. Assume that there exist $c_1, c_2, \varepsilon \in \mathbb{R}_+$ and $\alpha_{\mathbf{k}, \mathbf{l}} \in \mathbb{R}$ ($\mathbf{k}, \mathbf{l} \in \mathbb{N}^d$) such that*

$$\sum_{\mathbf{l} \leq \mathbf{n}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \alpha_{\mathbf{k}, \mathbf{l}} \leq c_1 D_{\mathbf{n}}^2 \prod_{i=1}^d \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \tag{2.1}$$

for all enough large $n_i \in \mathbb{N}$, and

$$|\mathbb{E} \xi_{\mathbf{k}} \xi_{\mathbf{l}}| \leq c_2 \left(\prod_{i=1}^d \left(\log_+ \log_+ \frac{c_{m_i}^{(i)}}{c_{h_i}^{(i)}} \right)^{-1-\varepsilon} + \alpha_{\mathbf{k}, \mathbf{l}} \right) \tag{2.2}$$

for each $\mathbf{k}, \mathbf{l} \in \mathbb{N}^d$, where $\mathbf{h} = \min\{\mathbf{k}, \mathbf{l}\}$ and $\mathbf{m} = \max\{\mathbf{k}, \mathbf{l}\}$. Then

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \xi_{\mathbf{k}} \rightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty \quad \text{a.s.}$$

Definition 2.2. The α -mixing coefficient of the random variables ξ and η is

$$\alpha(\xi, \eta) = \alpha(\sigma(\xi), \sigma(\eta)) = \sup_{\substack{A \in \sigma(\xi) \\ B \in \sigma(\eta)}} |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)|.$$

Theorem 2.3. Let $\{\zeta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ be a random field. Assume that there exist random variables $\zeta_{\mathbf{h}, \mathbf{l}}$ ($\mathbf{h} \leq \mathbf{l}$) and $c_1, c_2, c_3, \varepsilon \in \mathbb{R}_+$ such that

$$|\zeta_{\mathbf{k}} - \zeta_{\mathbf{h}, \mathbf{k}}| \geq c_1 \quad \text{a.s.} \quad \forall \mathbf{h}, \mathbf{k} \in \mathbb{N}^d \quad \text{for which } \mathbf{h} \leq \mathbf{k}, \tag{2.3}$$

$$\mathbb{E} \min \{(\zeta_{\mathbf{l}} - \zeta_{\mathbf{h}, \mathbf{l}})^2, 1\} \leq c_2 \prod_{i=1}^d \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{h_i}^{(i)}} \right)^{-2-2\varepsilon} \tag{2.4}$$

for all $\mathbf{h}, \mathbf{l} \in \mathbb{N}^d$ for which $\mathbf{h} \leq \mathbf{l}$, and

$$\sum_{\mathbf{l} \leq \mathbf{n}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \alpha_{\mathbf{k}, \mathbf{l}} \leq c_3 D_{\mathbf{n}}^2 \prod_{i=1}^d \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \tag{2.5}$$

for all enough large $n_i \in \mathbb{N}$, where $\alpha_{\mathbf{k}, \mathbf{l}} = \alpha(\zeta_{\mathbf{k}}, \zeta_{\mathbf{t}, \mathbf{l}})$ with $\mathbf{t} = \min\{\mathbf{k}, \mathbf{l}\}$. Then for any probability distribution μ the following two statements are equivalent:

- (1) $\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)} \xrightarrow{w} \mu$ as $\mathbf{n} \rightarrow \infty$, for almost every $\omega \in \Omega$;
- (2) $\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \mu_{\zeta_{\mathbf{k}}} \xrightarrow{w} \mu$ as $\mathbf{n} \rightarrow \infty$.

Definition 2.4. The α -mixing coefficient of the random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is

$$\alpha(\mathbf{k}) = \sup_{\mathbf{n}} \alpha \left(\bigcup_{i \leq \mathbf{n}} \sigma(X_i), \bigcup_{i \not\leq \mathbf{n} + \mathbf{k}} \sigma(X_i) \right), \quad \mathbf{k} \in \mathbb{N}^d.$$

Theorem 2.5. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an α -mixing random field with mixing coefficient

$$\alpha(\mathbf{k}) \leq \frac{c}{|\log \mathbf{k}|} \quad (2.6)$$

for all $\mathbf{k} \in \mathbb{N}^d$, where $c \in \mathbb{R}_+$ is fixed. Let $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$ and $\sigma_{\mathbf{n}}^2 = \mathbb{E} S_{\mathbf{n}}^2 > 0$. Assume that $\mathbb{E} X_{\mathbf{i}} = 0$ and $\mathbb{E} X_{\mathbf{i}}^2 < \infty$ for all $\mathbf{i} \in \mathbb{N}^d$, moreover there exist $c_1, c_2 \in \mathbb{R}_+$ and $\beta > 2/\log 2$ such that

$$|S_{\mathbf{l}}| \geq c_1 \sigma_{\mathbf{k}} \quad \text{a.s.} \quad \forall \mathbf{l}, \mathbf{k} \in \mathbb{N}^d \quad \text{for which} \quad \mathbf{l} \leq \mathbf{k} \quad (2.7)$$

and

$$\mathbb{E} \min \left\{ \frac{S_{\mathbf{r}}^2}{\sigma_{\mathbf{l}}^2}, 1 \right\} \leq c_2 \left(\frac{|\mathbf{h}|}{|\mathbf{l}|} \right)^{\beta} \quad \forall \mathbf{h}, \mathbf{l} \in \mathbb{N}^d \quad \text{for which} \quad \mathbf{h} \leq \mathbf{l}, \quad (2.8)$$

where $\mathbf{r} = 2\mathbf{h}$ if $2\mathbf{h} < \mathbf{l}$ and $\mathbf{r} = \mathbf{l}$ otherwise. If $\mu_{\zeta_{\mathbf{n}}} \xrightarrow{w} \mu$ as $\mathbf{n} \rightarrow \infty$, where $\zeta_{\mathbf{n}} = S_{\mathbf{n}}/\sigma_{\mathbf{n}}$ and μ is a probability distribution, then

$$\frac{1}{|\log \mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \frac{1}{|\mathbf{k}|} \delta_{\zeta_{\mathbf{k}}(\omega)} \xrightarrow{w} \mu \quad \text{as} \quad \mathbf{n} \rightarrow \infty, \quad \text{for almost every} \quad \omega \in \Omega.$$

3. Lemmas

You can find the proof of the next lemma in [6].

Lemma 3.1 (Covariance inequality). If ξ and η are bounded random variables, then

$$|\text{cov}(\xi, \eta)| \leq 4\alpha(\xi, \eta) \|\xi\|_{\infty} \|\eta\|_{\infty}.$$

The proof of the next lemma follows from that of Theorem 11.3.3 and Corollary 11.3.4 in [3].

Lemma 3.2. Let BL denote the set of all bounded, real-valued Lipschitz function on \mathbb{R} . If μ and μ_n are distributions ($n \in \mathbb{N}$), then there exists a countable set $M \subset BL$ (depending on μ) such that the following are equivalent:

- (1) $\mu_n \xrightarrow{w} \mu$ as $n \rightarrow \infty$;
- (2) $\int g d\mu_n \rightarrow \int g d\mu$ as $n \rightarrow \infty$ for all $g \in M$.

Lemma 3.3 (Theorem 1 of [7], p. 309). If μ and μ_n are distributions ($n \in \mathbb{N}$), then the following are equivalent:

- (1) $\mu_n \xrightarrow{w} \mu$ as $n \rightarrow \infty$;
- (2) $\mu_n(A) \rightarrow \mu(A)$ as $n \rightarrow \infty$ for all $A \in \mathcal{B}$ for which $\mu(\partial A) = 0$.

Lemma 3.4. If μ and $\mu_{\mathbf{n}}$ are distributions ($\mathbf{n} \in \mathbb{N}^d$) and $\mu_{\mathbf{n}} \xrightarrow{w} \mu$ as $\mathbf{n} \rightarrow \infty$, then

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \mu_{\mathbf{k}} \xrightarrow{w} \mu \quad \text{as} \quad \mathbf{n} \rightarrow \infty.$$

Proof. By $\sum_{k_i=1}^{\infty} d_{k_i}^{(i)} = \infty$ we have

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} = \prod_{i=1}^d \frac{\sum_{m_i \leq k_i \leq n_i} d_{k_i}^{(i)}}{\sum_{k_i \leq n_i} d_{k_i}^{(i)}} \rightarrow 1 \quad \text{as } \mathbf{n} \rightarrow \infty \quad \forall \mathbf{m} \in \mathbb{N}^d,$$

which implies, that

$$\frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leq \mathbf{n} \\ \mathbf{k} \not\leq \mathbf{m}}} d_{\mathbf{k}} = 1 - \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \rightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty \quad \forall \mathbf{m} \in \mathbb{N}^d. \quad (3.1)$$

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a bounded and continuous function and $K = \sup_{x \in \mathbb{R}} |f(x)|$. Then

$$\left| \int f d\mu_{\mathbf{n}} - \int f d\mu \right| \leq \int K d\mu_{\mathbf{n}} + \int K d\mu = 2K, \quad (3.2)$$

moreover by $\mu_{\mathbf{n}} \xrightarrow{w} \mu$ and (3.1), for any $\varepsilon > 0$ there exists $\mathbf{n}(\varepsilon) \in \mathbb{N}^d$ such that

$$\left| \int f d\mu_{\mathbf{n}} - \int f d\mu \right| < \frac{\varepsilon}{2} \quad (3.3)$$

and

$$\frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leq \mathbf{n} \\ \mathbf{k} \not\leq \mathbf{n}(\varepsilon)}} d_{\mathbf{k}} < \frac{\varepsilon}{4K} \quad (3.4)$$

for all $\mathbf{n} \geq \mathbf{n}(\varepsilon)$. With notation $\gamma_{\mathbf{n}} = \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \mu_{\mathbf{k}}$ the inequalities (3.2), (3.3) and (3.4) imply, that

$$\begin{aligned} & \left| \int f d\gamma_{\mathbf{n}} - \int f d\mu \right| \leq \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \left| \int f d\mu_{\mathbf{k}} - \int f d\mu \right| \\ &= \frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leq \mathbf{n} \\ \mathbf{k} \not\leq \mathbf{n}(\varepsilon)}} d_{\mathbf{k}} \left| \int f d\mu_{\mathbf{k}} - \int f d\mu \right| + \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{n}(\varepsilon) \leq \mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \left| \int f d\mu_{\mathbf{k}} - \int f d\mu \right| \\ &< \frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leq \mathbf{n} \\ \mathbf{k} \not\leq \mathbf{n}(\varepsilon)}} d_{\mathbf{k}} \cdot 2K + \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{n}(\varepsilon) \leq \mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $\mathbf{n} \geq \mathbf{n}(\varepsilon)$. This fact implies the statement. \square

4. Proof of the theorems

Proof of Theorem 2.1. By (2.2) and (2.1) we have

$$\mathbb{E} \left(\sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \xi_{\mathbf{k}} \right)^2 \leq \sum_{\mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{l} \leq \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} |\mathbb{E} \xi_{\mathbf{k}} \xi_{\mathbf{l}}|$$

$$\begin{aligned}
&\leq c_2 \sum_{\mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{l} \leq \mathbf{n}} \prod_{i=1}^d d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{m_i}^{(i)}}{c_{h_i}^{(i)}} \right)^{-1-\varepsilon} + c_2 \sum_{\mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{l} \leq \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \alpha_{\mathbf{k}, \mathbf{l}} \\
&\leq 2c_2 \prod_{i=1}^d \sum_{k_i \leq l_i \leq n_i} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} \right)^{-1-\varepsilon} + c_2 c_1 D_{\mathbf{n}}^2 \prod_{i=1}^d \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \quad (4.1)
\end{aligned}$$

for all enough large n_i . Now assume that $(k_i, l_i) \in A_{n_i}^{(i)}$, where

$$A_{n_i}^{(i)} = \left\{ (k_i, l_i) : k_i \leq l_i \leq n_i \text{ and } c_{l_i}^{(i)} / c_{k_i}^{(i)} \geq \exp(\sqrt{D_{n_i}^{(i)}}) \right\}.$$

Then $\log_+ \log_+ \left(c_{l_i}^{(i)} / c_{k_i}^{(i)} \right) \geq \frac{1}{2} \log D_{n_i}^{(i)}$, which implies, that

$$\begin{aligned}
&\sum_{(k_i, l_i) \in A_{n_i}^{(i)}} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} \right)^{-1-\varepsilon} \\
&\leq 2^{1+\varepsilon} \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \sum_{(k_i, l_i) \in A_{n_i}^{(i)}} d_{k_i}^{(i)} d_{l_i}^{(i)} \leq 2^{1+\varepsilon} \left(D_{n_i}^{(i)} \right)^2 \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon}. \quad (4.2)
\end{aligned}$$

If $(k_i, l_i) \in B_{n_i}^{(i)}$, where

$$B_{n_i}^{(i)} = \left\{ (k_i, l_i) : k_i \leq l_i \leq n_i \text{ and } c_{l_i}^{(i)} / c_{k_i}^{(i)} < \exp(\sqrt{D_{n_i}^{(i)}}) \right\},$$

then with notation $M_i = \sup_k (c_{k+1}^{(i)} / c_k^{(i)})$, we get

$$\log \frac{c_{l_i+1}^{(i)}}{c_{k_i}^{(i)}} = \log \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} + \log \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} < \log M_i + \sqrt{D_{n_i}^{(i)}}.$$

Thus we have the following inequality, where $B_{n_i, k_i}^{(i)} = \left\{ l_i : (k_i, l_i) \in B_{n_i}^{(i)} \right\}$.

$$\begin{aligned}
&\sum_{(k_i, l_i) \in B_{n_i}^{(i)}} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} \right)^{-1-\varepsilon} \leq \sum_{(k_i, l_i) \in B_{n_i}^{(i)}} d_{k_i}^{(i)} d_{l_i}^{(i)} \\
&\leq \sum_{(k_i, l_i) \in B_{n_i}^{(i)}} d_{k_i}^{(i)} \log \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} = \sum_{k_i=1}^{n_i} \sum_{l_i \in B_{n_i, k_i}^{(i)}} d_{k_i}^{(i)} \log \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} \leq \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} \sum_{l_i=k_i}^{\max B_{n_i, k_i}^{(i)}} \log \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} \\
&= \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} \log \prod_{l_i=k_i}^{\max B_{n_i, k_i}^{(i)}} \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} = \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} \log \frac{c_{\max B_{n_i, k_i}^{(i)}}^{(i)}}{c_{k_i}^{(i)}} \\
&< \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} \left(\log M_i + \sqrt{D_{n_i}^{(i)}} \right) \leq \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} 2\sqrt{D_{n_i}^{(i)}} = 2 \left(D_{n_i}^{(i)} \right)^{3/2}
\end{aligned}$$

for all enough large n_i . It follows from this inequality and (4.2) that

$$\begin{aligned} & \sum_{k_i \leq l_i \leq n_i} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} \right)^{-1-\varepsilon} \\ & \leq 2^{1+\varepsilon} \left(D_{n_i}^{(i)} \right)^2 \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} + 2 \left(D_{n_i}^{(i)} \right)^{3/2} \\ & \leq 2^{1+\varepsilon} \left(D_{n_i}^{(i)} \right)^2 \left(\left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} + \left(D_{n_i}^{(i)} \right)^{-1/2} \right) \\ & \leq 2^{2+\varepsilon} \left(D_{n_i}^{(i)} \right)^2 \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \end{aligned} \tag{4.3}$$

for all enough large n_i . In the last step we use the inequality $\left(D_{n_i}^{(i)} \right)^{-1/2} \leq \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon}$, which follows from $\left(D_{n_i}^{(i)} \right)^{1/2} / \left(\log D_{n_i}^{(i)} \right)^{1+\varepsilon} \rightarrow \infty$ as $n_i \rightarrow \infty$. By (4.1) and (4.3) we get

$$\mathbb{E} \left(\sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \xi_{\mathbf{k}} \right)^2 \leq \text{const.} \prod_{i=1}^d \left(D_{n_i}^{(i)} \right)^2 \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \tag{4.4}$$

for all enough large n_i . Let

$$n_i(t) = \min \left\{ n_i : D_{n_i}^{(i)} \leq \exp \left(t^{\frac{1+\varepsilon/2}{1+\varepsilon}} \right) \right\}$$

and $\mathbf{n}(t) = (n_1(t_1), \dots, n_d(t_d))$. Since $n_i(t_i) \rightarrow \infty$ as $t_i \rightarrow \infty$, thus by (4.4) there exists $\mathbf{T} \in \mathbb{N}^d$, such that

$$\begin{aligned} & \mathbb{E} \sum_{\mathbf{t} \geq \mathbf{T}} \left(\frac{1}{D_{\mathbf{n}(t)}} \sum_{\mathbf{k} \leq \mathbf{n}(t)} d_{\mathbf{k}} \xi_{\mathbf{k}} \right)^2 \leq \sum_{\mathbf{t} \geq \mathbf{T}} \frac{1}{D_{\mathbf{n}(t)}^2} \text{const.} \prod_{i=1}^d \left(D_{n_i(t_i)}^{(i)} \right)^2 \left(\log D_{n_i(t_i)}^{(i)} \right)^{-1-\varepsilon} \\ & \leq \sum_{\mathbf{t} \geq \mathbf{T}} \frac{1}{D_{\mathbf{n}(t)}^2} \text{const.} \prod_{i=1}^d \left(D_{n_i(t_i)}^{(i)} \right)^2 t_i^{-1-\varepsilon/2} = \text{const.} \prod_{i=1}^d \sum_{t_i=T_i}^{\infty} t_i^{-1-\varepsilon/2} < \infty, \end{aligned}$$

which implies

$$\frac{1}{D_{\mathbf{n}(t)}} \sum_{\mathbf{k} \leq \mathbf{n}(t)} d_{\mathbf{k}} \xi_{\mathbf{k}} \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{a.s.} \tag{4.5}$$

For all $\mathbf{n} \in \mathbb{N}^d$ there exists $\mathbf{t} \in \mathbb{N}^d$ such that $\mathbf{n}(t) \leq \mathbf{n} \leq \mathbf{n}(t + \mathbf{1})$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$. Thus the uniformly bounding implies

$$\left| \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \xi_{\mathbf{k}} \right| \leq \left| \frac{1}{D_{\mathbf{n}(t)}} \sum_{\mathbf{k} \leq \mathbf{n}(t)} d_{\mathbf{k}} \xi_{\mathbf{k}} \right| + \frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leq \mathbf{n} \\ \mathbf{k} \not\leq \mathbf{n}(t)}} d_{\mathbf{k}} |\xi_{\mathbf{k}}|$$

$$\begin{aligned}
&\leq \left| \frac{1}{D_{\mathbf{n}(\mathbf{t})}} \sum_{\mathbf{k} \leq \mathbf{n}(\mathbf{t})} d_{\mathbf{k}} \xi_{\mathbf{k}} \right| + \frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leq \mathbf{n} \\ \mathbf{k} \not\leq \mathbf{n}(\mathbf{t})}} d_{\mathbf{k}} \cdot c \\
&\leq \left| \frac{1}{D_{\mathbf{n}(\mathbf{t})}} \sum_{\mathbf{k} \leq \mathbf{n}(\mathbf{t})} d_{\mathbf{k}} \xi_{\mathbf{k}} \right| + c \left(1 - \frac{D_{\mathbf{n}(\mathbf{t})}}{D_{\mathbf{n}(\mathbf{t}+1)}} \right) \quad \text{a.s.}
\end{aligned} \tag{4.6}$$

The reader can easily verify that $D_{\mathbf{n}(\mathbf{t})}/D_{\mathbf{n}(\mathbf{t}+1)} \rightarrow 1$ as $\mathbf{t} \rightarrow \infty$, so by (4.5) and (4.6) imply the statement of Theorem 2.1. \square

Proof of Theorem 2.3. Let $g \in M$, where M is defined in Lemma 3.2. Then there exists $K \geq 1$ such that

$$|g(x)| \leq K \quad \text{and} \quad |g(x) - g(y)| \leq K|x - y| \quad \forall x, y \in \mathbb{R}. \tag{4.7}$$

We shall prove, that with notation $\xi_{\mathbf{k}} = g(\zeta_{\mathbf{k}}) - \mathbb{E}g(\zeta_{\mathbf{k}})$ the conditions of Theorem 2.1 hold true. By (2.5) we get (2.1), moreover by (4.7) we have

$$|\xi_{\mathbf{k}}| \leq |g(\zeta_{\mathbf{k}})| + \mathbb{E}|g(\zeta_{\mathbf{k}})| \leq 2K,$$

thus $\{\xi_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ is a uniformly bounded random field. Now we turn to (2.2). Let $\mathbf{t} = \min\{\mathbf{k}, \mathbf{1}\}$. Lemma 3.1 and (4.7) imply

$$|\mathbb{E} \xi_{\mathbf{k}}(g(\zeta_{\mathbf{t},1}) - \mathbb{E}g(\zeta_{\mathbf{1}}))| = |\text{cov}(g(\zeta_{\mathbf{1}}), g(\zeta_{\mathbf{t},1}))| \leq 4K^2 \alpha_{\mathbf{k},1}. \tag{4.8}$$

On the other hand with notation $\eta_{\mathbf{k},1} = g(\zeta_{\mathbf{1}}) - g(\zeta_{\mathbf{t},1})$

$$|\mathbb{E} \xi_{\mathbf{k}} \eta_{\mathbf{k},1}| = |\text{cov}(g(\zeta_{\mathbf{k}}), \eta_{\mathbf{k},1})| \leq (\mathbb{E}g^2(\zeta_{\mathbf{k}}) \mathbb{E}\eta_{\mathbf{k},1}^2)^{1/2}. \tag{4.9}$$

It is easy to see that $(g(x) - g(y))^2 \leq 4K^2 \min\{(x - y)^2, 1\}$, thus

$$\mathbb{E}\eta_{\mathbf{k},1}^2 \leq 4K^2 \min\{(\zeta_{\mathbf{1}} - \zeta_{\mathbf{t},1})^2, 1\}. \tag{4.10}$$

By (4.7) and (2.3) we have $g^2(\zeta_{\mathbf{k}}) \leq K^2(1 + 1/c_1)^2$ and

$$g^2(\zeta_{\mathbf{k}}) < K^2(c_1 + 1)^2 = K^2 \left(1 + \frac{1}{c_1}\right)^2 \cdot c_1^2 \leq K^2 \left(1 + \frac{1}{c_1}\right)^2 (\zeta_{\mathbf{k}} - \zeta_{\mathbf{t},\mathbf{k}})^2,$$

which imply $g^2(\zeta_{\mathbf{k}}) \leq \text{const.} \min\{(\zeta_{\mathbf{k}} - \zeta_{\mathbf{t},\mathbf{k}})^2, 1\}$ a.s. Using this inequality, (4.10), (4.9) and (2.4) we get the following.

$$\begin{aligned}
|\mathbb{E} \xi_{\mathbf{k}} \eta_{\mathbf{k},1}| &\leq \text{const.} (\mathbb{E} \min\{(\zeta_{\mathbf{k}} - \zeta_{\mathbf{t},\mathbf{k}})^2, 1\} \mathbb{E} \min\{(\zeta_{\mathbf{1}} - \zeta_{\mathbf{t},1})^2, 1\})^{1/2} \\
&\leq \text{const.} \left(\prod_{i=1}^d \log_+ \log_+ \frac{c_{k_i}^{(i)}}{c_{t_i}^{(i)}} \cdot \log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{t_i}^{(i)}} \right)^{-1-\varepsilon} \\
&= \text{const.} \left(\prod_{i=1}^d \log_+ \log_+ \frac{c_{m_i}^{(i)}}{c_{t_i}^{(i)}} \right)^{-1-\varepsilon},
\end{aligned} \tag{4.11}$$

where $\mathbf{m} = \max\{\mathbf{k}, \mathbf{l}\}$. Since $|\mathbb{E} \xi_{\mathbf{k}} \xi_{\mathbf{l}}| \leq |\mathbb{E} \xi_{\mathbf{k}} \eta_{\mathbf{k}, \mathbf{l}}| + |\mathbb{E} \xi_{\mathbf{k}} (g(\zeta_{\mathbf{l}, 1}) - \mathbb{E} g(\zeta_{\mathbf{l}}))|$, using (4.11) and (4.8) we have (2.2). Now applying Theorem 2.1 we get

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \xi_{\mathbf{k}} \rightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty \quad \text{a.s.} \tag{4.12}$$

Let $\mu_{\mathbf{n}} = \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \mu_{\zeta_{\mathbf{k}}}$ and $\mu_{\mathbf{n}, \omega} = \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)}$ ($\omega \in \Omega$).

First assume that (2) is true, that is $\mu_{\mathbf{n}} \xrightarrow{w} \mu$ as $\mathbf{n} \rightarrow \infty$. Then Lemma 3.2 implies

$$\int g \, d\mu_{\mathbf{n}} \rightarrow \int g \, d\mu \quad \text{as } \mathbf{n} \rightarrow \infty, \tag{4.13}$$

and (4.12) implies

$$\int g \, d\mu_{\mathbf{n}, \omega} - \int g \, d\mu_{\mathbf{n}} = \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \zeta_{\mathbf{k}}(\omega) \rightarrow 0 \tag{4.14}$$

as $\mathbf{n} \rightarrow \infty$, for almost every $\omega \in \Omega$. By (4.13) and (4.14) we get $\int g \, d\mu_{\mathbf{n}, \omega} \rightarrow \int g \, d\mu$ as $\mathbf{n} \rightarrow \infty$, for almost every $\omega \in \Omega$, thus by Lemma 3.2 we get (1).

Finally assume that (1) is true, that is $\mu_{\mathbf{n}, \omega} \xrightarrow{w} \mu$ as $\mathbf{n} \rightarrow \infty$, for almost every $\omega \in \Omega$. Let $A \in \mathcal{B}$ and $\mu(\partial A) = 0$. Then by Lemma 3.3 $\mu_{\mathbf{n}, \omega}(A) \rightarrow \mu(A)$ as $\mathbf{n} \rightarrow \infty$, for almost every $\omega \in \Omega$. It follows that $\mu_{\mathbf{n}}(A) = \int \mu_{\mathbf{n}, \omega}(A) \, d\mathbb{P}(\omega) \rightarrow \mu(A)$ as $\mathbf{n} \rightarrow \infty$. Thus using Lemma 3.3 we get (2). This completes the proof of Theorem 2.3. \square

Proof of Theorem 2.5. Let $d_k^{(i)} = 1/k$, $c_k^{(i)} = k^{1/\log 2}$, $\varepsilon = (\beta \log 2 - 2)/2$, $\zeta_{\mathbf{k}, \mathbf{l}} = \zeta_{\mathbf{l}} - S_{2\mathbf{k}}/\sigma_1$ if $2\mathbf{k} < \mathbf{l}$ and $\zeta_{\mathbf{k}, \mathbf{l}} = 0$ if $\mathbf{k} \leq \mathbf{l}$ and $2\mathbf{k} \not< \mathbf{l}$. We shall prove that conditions of Theorem 2.3 hold. It is easy to see that $\alpha_{\mathbf{k}, \mathbf{l}} \leq \alpha(\mathbf{k})$ for all $\mathbf{k}, \mathbf{l} \in \mathbb{N}^d$, where $\alpha_{\mathbf{k}, \mathbf{l}}$ is defined in Theorem 2.3. Therefore by (2.6) we have

$$\begin{aligned} \sum_{\mathbf{l} \leq \mathbf{n}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \alpha_{\mathbf{k}, \mathbf{l}} &\leq \sum_{\mathbf{l} \leq \mathbf{n}} \sum_{\mathbf{k} \leq \mathbf{n}} \frac{c}{|\mathbf{k}| \cdot |\mathbf{l}| \cdot |\log \mathbf{k}|} \\ &= c \prod_{i=1}^d \left(\sum_{k=1}^{n_i} \frac{1}{k \log_+ k} \right) \left(\sum_{l=1}^{n_i} \frac{1}{l} \right). \end{aligned} \tag{4.15}$$

It is well-known that $\sum_{k=1}^n \frac{1}{k} \sim \log n$ and $\sum_{k=1}^n \frac{1}{k \log_+ k} \sim \log \log n$, where $a_n \sim b_n$ iff $\lim_{n \rightarrow \infty} a_n/b_n = 1$. So by (4.15) we have

$$\begin{aligned} \sum_{\mathbf{l} \leq \mathbf{n}} \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \alpha_{\mathbf{k}, \mathbf{l}} &\leq \text{const.} \prod_{i=1}^d \log \log n_i \cdot \log n_i \leq \text{const.} \prod_{i=1}^d (\log n_i)^2 (\log \log n_i)^{-1-\varepsilon} \\ &\leq \text{const.} \prod_{i=1}^d (\log n_i)^2 (\log D_{n_i}^{(i)})^{-1-\varepsilon} \leq \text{const.} D_{\mathbf{n}}^2 \prod_{i=1}^d (\log D_{n_i}^{(i)})^{-1-\varepsilon} \end{aligned}$$

for all enough large n_i , which implies (2.5). Using (2.8)

$$\mathbb{E} \min \{ (\zeta_{\mathbf{l}} - \zeta_{\mathbf{h}, \mathbf{l}})^2, 1 \} = \mathbb{E} \min \{ S_{\mathbf{r}}^2 / \sigma_{\mathbf{l}}^2, 1 \} \leq \text{const.} \prod_{i=1}^d \left(\log_+ \log_+ \frac{c_{h_i}^{(i)}}{c_{h_i}^{(\mathbf{l})}} \right)^{-2-2\varepsilon}$$

for all $\mathbf{h}, \mathbf{l} \in \mathbb{N}^d$ for which $\mathbf{h} \leq \mathbf{l}$, where $\mathbf{r} = 2\mathbf{h}$ if $2\mathbf{h} < \mathbf{l}$ and $\mathbf{r} = \mathbf{l}$ if $\mathbf{h} \leq \mathbf{l}$ and $2\mathbf{h} \not< \mathbf{l}$, so we get (2.4). The reader can readily verify that (2.3) is hold as well. Now applying Lemma 3.4 and Theorem 2.3, we have

$$\frac{1}{\sum_{\mathbf{k} \leq \mathbf{n}} \frac{1}{|\mathbf{k}|}} \sum_{\mathbf{k} \leq \mathbf{n}} \frac{1}{|\mathbf{k}|} \delta_{\zeta_{\mathbf{k}}(\omega)} \xrightarrow{w} \mu \quad \text{as } \mathbf{n} \rightarrow \infty, \quad \text{for almost every } \omega \in \Omega.$$

Since $\sum_{\mathbf{k} \leq \mathbf{n}} \frac{1}{|\mathbf{k}|} \sim |\log \mathbf{n}|$, we get the statement. \square

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