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On the resolution of simultaneous Pell equations*

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Abstract

We descibe an alternative procedure for solving automatically simultaneous Pell equations with relatively small coefficients. The word "automatically" means to indicate that the algorithm can be implemented in Magma. Numerous famous examples are verified and a new theorem is proved by running simply the corresponding Magma procedure requires only the six coefficients of the system

$$\begin{aligned} a_1x^2 + b_1y^2 &= c_1, \\ a_2x^2 + b_2z^2 &= c_2. \end{aligned}$$

Keywords: Simultaneous Pell equations, to compute all solutions, Thue equations

MSC: 11D09, 11D25, 11Y50

1. Introduction

In this paper an alternative method is presented for solving the simultaneous Pell equations

$$a_1x^2 + b_1y^2 = c_1, \tag{1.1}$$
$$a_2x^2 + b_2z^2 = c_2, \tag{1.2}$$

in non-negative integers x, y and z, where the coefficients are given integers satisfying the natural conditions

$$a_1b_1 < 0, \quad a_2b_2 < 0, \quad c_1c_2 \neq 0, \quad a_1c_2 - a_2c_1 \neq 0.$$

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The algorithm depends on the combination of (1.1) and (1.2), which leads to Thue equations of degree four can be solved, for example, by the computer package MAGMA. Further, if (1.2) is replaced by

$$a_2x^2 + b_2z^2 = c_2y^2, \tag{1.3}$$

then the method still works. Unfortunately, the number and the coefficients of the Thue equations need to be solved may increase if a_i, b_i, c_i $(i = 1, 2)$ are getting larger. Nevertheless, applying the new idea, the classical examples have been handled before by different methods were verified in a short time (see Appendix). One of the examples gives a new result by showing that there is no Lucas balancing number.

The first paper concerning simultaneous Pell equations is due to Boutin and Teilhet [8]. In 1904, they proved the unsolvability (in positive integers α, β, γ) of the system $6\beta^2 + 1 = \alpha^2$, $\gamma^2 - 3\beta^2 = 1$. In Appendix there are given some more papers from the early period. Ljunggren [20] has a remarkable result from the first part of the twentieth century. Using the properties of the units of quadratic fields, he showed that the equations $x^2 - Dy^2 = 1$ and $y^2 - D_1z^2 = 1$ with fixed D and D_1 have only finitely many solutions, and he was able to solve the case $D = 2$, $D_1 = 3$. Generally, the finiteness of the number of solutions of (1.1), (1.2) (or (1.1), (1.3) or (1.4), (1.5)) follows from the works of Thue [27] or Siegel [26].

In 1969, Baker and Davenport discovered that the theory of linear forms in logarithm can be also applied to solve simultaneous Pell equations. Their famous paper [3] provided the number 120 as a unique extension of the Diophantine triple $\{1, 3, 8\}$ to quadruple. A set of positive integers is called Diophantine m-tuple if the product of any two elements increased by one is a perfect square. Following them, many authors applied the Baker-Davenport method to investigate similar problems (see Appendix). Taking t_{12}, t_{13}, $t_{23} \in \mathbb{Z}$ the set $S = \{a_1, a_2, a_3\}$ is called Diophantine triple with t_{12}, t_{13}, t_{23} if each $a_ia_j + t_{ij}$ equals a perfect square. Can S be extended to Diophantine quadruple by some integer $x = a_4$ with the new integers t_{14}, t_{24}, t_{34}? This question leads to the equations

$$\begin{aligned} a_1x + t_{14} &= x_1^2, \\ a_2x + t_{24} &= x_2^2, \\ a_3x + t_{34} &= x_3^2, \end{aligned}$$

or, equivalently, to an (1.1), (1.2)-type system of the form

$$\begin{aligned} a_2x_1^2 - a_1x_2^2 &= a_2t_{14} - a_1t_{24}, \\ a_3x_1^2 - a_1x_3^2 &= a_3t_{14} - a_1t_{34}. \end{aligned}$$

Clearly, starting from an Diophantine quadruple with fixed six integers t_{ij} $(1 \leqslant i < j \leqslant 4)$, one can make efforts to solve the problem of Diophantine quintuple with the new integers t_{i5} $(i = 1, \dots, 4)$.

Pinch [23] generalized the procedure of Baker and Davenport, and his approach was applied by Gaál, Pethő and Pohst [13]. They reduced the resolution of index form equations to the resolution of certain simultaneous Pell equations.

Although Kedlaya [18] described an elementary method to solve the generalization

$$\begin{aligned} x^2 - ay^2 &= b, \qquad (1.4)\\ P(x,y) &= z^2 \qquad (1.5)\end{aligned}$$

of (1.1), (1.3), where $P(x,y)$ is a polynomial with integer coefficients, it is fact, that in his examples P is univariate with degree at most two.

Tzanakis [28] suggests the elliptic logarithm method. The procedure provides a corresponding elliptic curve and then determines all rational points on it. But his algorithm requires an initial non-trivial rational solution, and it may cause difficulties. This idea has been partially described by Katayama [16] as well.

An other direction is to study the number of solutions of simultaneous Pell equations. In [5] Bennett proved that if a and b are distinct nonzero integers then the simultaneous equations $x^2 - az^2 = 1 = y^2 - bz^2$ possess at most three solutions in positive integers (x,y,z). Further, he also gave an upper bound for the cardinality of positive triplets (x,y,z) satisfying $x^2 - az^2 = u$, $y^2 - bz^2 = v$.

In the end of this section we quote two preliminary result required by our method. First we recall a criterion due to Legendre for the existence of a nonzero integer solution (x,y,z) to the diophantine equation

$$ax^2 + by^2 + cz^2 = 0, \qquad (1.6)$$

where a, b and c are nonzero integers. (See, for example, in [7].)

Theorem 1.1. *Let a, b, c be three squarefree integers, $a > 0$, $b < 0$, $c < 0$ which are pairwise coprime. Then there exists a nonzero integer solution (x,y,z) to the diophantine equation* (1.6) *if and only if all three congruences*

$$t^2 \equiv -ab \pmod{c} \qquad t^2 \equiv -ac \pmod{b} \qquad t^2 \equiv -bc \pmod{a}$$

are solvable. Furthermore, if a nonzero solution exists, then there exists a nonzero solution (x_0,y_0,z_0) of equation (1.6) *satisfying the inequality*

$$\max\{x_0,y_0,z_0\} \leqslant \sqrt{abc}.$$

By applying the next statement (see [21]), if (1.6) has a non-zero solution, one can determine all (x,y,z) satisfying (1.6).

Theorem 1.2. *Assume that (x_0,y_0,z_0) is an integer solution of equation* (1.6) *with $z_0 \neq 0$. Then, all integer solutions (x,y,z) with $z \neq 0$ of equation* (1.6) *are of the form*

$$\begin{aligned} x &= \pm\frac{D}{d}\left(-ax_0s^2 - 2by_0rs + bx_0r^2\right),\\ y &= \pm\frac{D}{d}\left(ay_0s^2 - 2ax_0rs - by_0r^2\right),\end{aligned}$$

$$z = \pm\frac{D}{d}\left(az_0s^2 + bz_0r^2\right),$$

where r and $s > 0$ are coprime integers, D is a nonzero integer, and $d \mid 2a^2bcz_0^3$ is a positive integer.

2. The algorithm

Consider the aforesaid system of two diophantine equations

$$a_1x^2 + b_1y^2 = c_1, \tag{2.1}$$
$$a_2x^2 + b_2z^2 = c_2, \tag{2.2}$$

in non-negative integers x, y and z, where the coefficients are given integers satisfying the conditions $a_1b_1 < 0$, $a_2b_2 < 0$, $c_1c_2 \neq 0$ and $a_1c_2 - a_2c_1 \neq 0$.

After multiplying (2.1) by c_2 and (2.2) by c_1 and subtracting the second equation from the first, we obtain

$$(a_1c_2 - a_2c_1)x^2 + b_1c_2y^2 - b_2c_1z^2 = 0. \tag{2.3}$$

Note that none of the coefficients in (2.3) is zero. We should achieve that the conditions of Legendre's theorem be fulfilled. Therefore we divide (2.3) by $\gcd(a_1c_2 - a_2c_1, b_1c_2, b_2c_1)$ and we get $a_3x^2 + b_3y^2 + c_3z^2 = 0$, further if $a_3b_3c_3 < 0$ then even multiply $a_3x^2 + b_3y^2 + c_3z^2 = 0$ by (-1). Moreover, let this new equation be multiplied by $\gcd(a_3, b_3) \cdot \gcd(a_3, c_3) \cdot \gcd(b_3, c_3)$ and let assimilate the squarefull part of the coefficients into the corresponding variables, relabelling them, and we have

$$aX^2 + bY^2 + cZ^2 = 0, \tag{2.4}$$

where X, Y, Z is a permutation of c_xx, c_yy, c_zz with some suitable positive integers c_x, c_y and c_z, moreover $a > 0$, $b < 0$ and $c < 0$ are pairwise coprime, squarefree integers. Clearly, the choice of X is unique, but the role of Y and Z can be switched. By the theorem of Legendre, we need a basic solution (X_0, Y_0, Z_0).

If (2.4) is not solvable then the system (2.1), (2.2) has no solution. Otherwise, let (X_0, Y_0, Z_0) with $Z_0 \neq 0$ satisfy (2.4), and possibly $d(2a^2bcZ_0^3) \leqslant d(2a^2bcY_0^3)$, where $d(\ )$ denotes the number of divisors function. Such a triplet can easily be found by a simply search in the intervals $0 \leqslant X_0, Y_0, Z_0 \leqslant \sqrt{abc}$.

Now, applying Theorem 1.2, X, Y and Z can be expressed by

$$X = \pm\frac{D}{d}(\alpha_1s^2 + \beta_1sr + \gamma_1r^2),$$
$$Y = \pm\frac{D}{d}(\alpha_2s^2 + \beta_2sr + \gamma_2r^2),$$
$$Z = \pm\frac{D}{d}(\alpha_3s^2 + \beta_3sr + \gamma_3r^2),$$

where $s > 0$ and r are coprime, D is an arbitrary integer, $d \mid h_d = 2a^2bcZ_0^3$ is a positive integer and $\beta_3 = 0$. Consequently,

$$\begin{aligned} x &= \pm\frac{D}{c_x d}(\alpha_{i_1}s^2 + \beta_{i_1}sr + \gamma_{i_1}r^2), \\ y &= \pm\frac{D}{c_y d}(\alpha_{i_2}s^2 + \beta_{i_2}sr + \gamma_{i_2}r^2), \\ z &= \pm\frac{D}{c_z d}(\alpha_{i_3}s^2 + \beta_{i_3}sr + \gamma_{i_3}r^2), \end{aligned}$$

where i_1, i_2, i_3 is a permutation of the subscripts 1, 2, 3 of α, β and γ.

These results can be applied to return with x, y and z, for instance, to (2.1), and we obtain

$$a_1\left(\frac{D}{c_x d}(\alpha_{i_1}s^2 + \beta_{i_1}sr + \gamma_{i_1}r^2)\right)^2 + b_1\left(\frac{D}{c_y d}(\alpha_{i_2}s^2 + \beta_{i_2}sr + \gamma_{i_2}r^2)\right)^2 = c_1,$$

which implies

$$a_1c_y^2\left(\alpha_{i_1}s^2 + \beta_{i_1}sr + \gamma_{i_1}r^2\right)^2 + b_1c_x^2\left(\alpha_{i_2}s^2 + \beta_{i_2}sr + \gamma_{i_2}r^2\right)^2 = c_1c_x^2c_y^2\left(\frac{d}{D}\right)^2.$$

Note that the left hand side is a homogenous form of degree 4 in s and r, denote it by $T_1(s, r)$. Simplify the latest equation by the greatest common divisor of $c_1c_x^2c_y^2$ and the coefficients of T_1. Hence we obtain $T(s, r) = c_4(d/D)^2$. On the right hand side, let c_0 be the squarefree part of c_4. Thus there exist a positive integer c_6 such that $c_4 = c_0c_6^2$. Then the above equation is equivalent to

$$T(s, r) = c_0\left(\frac{c_6 d}{D}\right)^2. \tag{2.5}$$

(2.5) means finitely many Thue equations of order 4, because $T(s, r)$ is given, $0 < d$ is a divisor of $h_d = 2a^2bcZ_0^3$ and $j = \frac{c_6 d}{D}$ must be integer. To determine all solutions of equations (2.5) we use MAGMA system. Suppose that (s_j, r_j) is a solution of $T(s, r) = c_0j^2$ for some eligible j. We reject (s_j, r_j) if $s_j \leqslant 0$ or $\gcd(s_j, r_j) > 1$, otherwise we get

$$\begin{aligned} x &= \pm\frac{c_6}{c_x j}(\alpha_{i_1}s_j^2 + \beta_{i_1}s_jr_j + \gamma_{i_1}r_j^2), \\ y &= \pm\frac{c_6}{c_y j}(\alpha_{i_2}s_j^2 + \beta_{i_2}s_jr_j + \gamma_{i_2}r_j^2), \\ z &= \pm\frac{c_6}{c_z j}(\alpha_{i_3}s_j^2 + \beta_{i_3}s_jr_j + \gamma_{i_3}r_j^2). \end{aligned}$$

If all x, y and z are non-negative integers then a solution of the system (2.1), (2.2) is found.

3. Examples

Example 3.1. A positive integer y is called balancing number with balancer $r \in \mathbb{N}^+$ if

$$1 + 2 + \cdots + (y-1) = (y+1) + \cdots + (y+r). \tag{3.1}$$

The problem of determining balancing numbers leads to the solutions of the Pell equation $z^2 - 8y^2 = 1$, where y can be described by the recurrence $y_n = 6y_{n-1} - y_{n-2}$, $y_0 = 1$, $y_1 = 6$ (see Behera and Panda, [4]). Note that $y = y_0 = 1$ is not a balancing number in the sense of equation (3.1).

In [19], Liptai showed that there are no Fibonacci balancing numbers, i.e. neither of balancing numbers y is a term of the Fibonacci sequence $\{F\}$ defined by the initial values $F_0 = 0$, $F_1 = 1$ and by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, ($n \geqslant 2$). Liptai used the Baker-Davenport method to have the solution of the simultaneous Pell equation $x^2 - 5y^2 = \pm 4$, $z^2 - 8y^2 = 1$.

Now we show that no Lucas balancing number exists. Lucas sequence is defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$, ($n \geqslant 2$) and $L_0 = 2$, $L_1 = 1$. It is well known that the terms of Lucas and Fibonacci sequences satisfy $L_n^2 - 5F_n^2 = \pm 4$.

Theorem 3.2. *There is no Lucas balancing number.*

Proof. We are showing that the system

$$x^2 - 5y^2 = \pm 4, \tag{3.2}$$
$$z^2 - 8x^2 = 1. \tag{3.3}$$

has only the positive integer solution $(x, y, z) = (1, 1, 3)$, consequently there exists no Lucas balancing number x.

Taking the case $+4$, with the notation $X := x$, $Y := y$ and $Z := 2z$, we have

$$33X^2 - 5Y^2 - Z^2 = 0.$$

By Theorem 1.1, it has no nonzero solution, because $t^2 \not\equiv 33 \pmod{(-5)}$.

The case of -4 with $X := 2z$, $Y := y$, $Z := x$ provides

$$X^2 - 5Y^2 - 31Z^2 = 0.$$

The coefficients suggest the solution $(X_0, Y_0, Z_0) = (6, 1, 1)$. Applying Theorem 1.2, it follows that

$$\begin{aligned} x &= Z = \pm\frac{D}{d}(s^2 - 5r^2), \\ y &= Y = \pm\frac{D}{d}(s^2 - 12sr + 5r^2), \\ z &= \frac{X}{2} = \pm\frac{D}{2d}(-6s^2 + 10sr - 30r^2) = \pm\frac{D}{d}(-3s^2 + 5sr - 15r^2). \end{aligned}$$

Substitute x and z to (3.3) to have

$$(-3s^2 + 5sr - 15r^2)^2 - 8(s^2 - 5r^2)^2 = \left(\frac{d}{D}\right)^2,$$

where, by Theorem 1.2 again, $0 < d \mid 310$. Obviously, $\frac{d}{D}$ is integer, therefore we have to solve the Thue equations

$$s^4 - 30s^3r + 195s^2r^2 - 150sr^3 + 25r^4 = \left(\frac{d}{D}\right)^2 \tag{3.4}$$

for some positive integers $j = d/D \mid 310$. There are only three values of j when the solution (s_j, r_j) satisfies the condition $s_j > 0$ and $\gcd(s_j, r_j) = 1$, these are $j = 1$, 31 and 155. All the three triplets $(j, s_j, r_j) = (1, 1, 0)$, $(31, 6, 1)$, $(155, 5, 6)$ provide the same solution $(x, y, z) = (1, 1, 3)$. Hence, we conclude that there are no Lucas balancing number. □

Example 3.3 (Brown [9]). The system

$$x^2 - 8y^2 = 1 \tag{3.5}$$
$$z^2 - 5y^2 = 1 \tag{3.6}$$

leads to the equation $X^2 - 3Y^2 - Z^2 = 0$, where $X := x$, $Y := y$, $Z := z$. The coefficients of the Legendre equation and the basic solution $(X_0, Y_0, Z_0) = (1, 0, 1)$ imply $d \leqslant 6$. Theorem 2 gives $x = X = \pm\frac{D}{d}(-s^2 - 3r^2)$, $y = Y = \pm\frac{D}{d}(-2sr)$, $z = Z = \pm\frac{D}{d}(s^2 - 3r^2)$, which together with the first equation of the system leads to

$$s^4 - 26s^2r^2 + 9r^4 = \left(\frac{d}{D}\right)^2. \tag{3.7}$$

Since $d \mid 6$, we have to solve only four Thue equations. Only one of them have solution satisfying the conditions, namely if $j = (d/D) = 1$ then $(s_j, r_j) = (1, 0)$. It gives $(x, y, z) = (1, 0, 1)$.

Example 3.4. To determine all the non-negative solutions of the system

$$3x^2 - 10y^2 = -13, \tag{3.8}$$
$$x^2 - 3y^2 = z^2, \tag{3.9}$$

first we consider (3.9), which has already been solved in the previous example. Applying $x = X = \pm\frac{D}{d}(-s^2 - 3r^2)$, $y = Y = \pm\frac{D}{d}(-2sr)$, $d \mid 6$ and (3.8), we obtain

$$3s^4 - 22s^2r^2 + 27r^4 = -13\left(\frac{d}{D}\right)^2. \tag{3.10}$$

These Thue equations has eight solutions in coprime $s_j > 0$ and r_j providing $(x, y, z) = (7, 4, 1)$ and $(73, 40, 23)$.

In the next section we enumerate chronologically several systems of Pell equations in order to illustrate experiences and statistical data regarding the MAGMA program on my average home computer. The last coloumn shows the running time of the algorithm. Then we notify four more examples.

4. Appendix

Year	Cit.	Author(s)	System(s)	(x, y, z)	h_d	$d(h_d)$	Time
1904	[8]	Boutin, Teilhet	$x^2 - 6y^2 = 1$ $z^2 - 3y^2 = 1$	$(1, 0, 1)$	6	4	1 sec
1918	[25]	Rignaux	$x^2 - 2z^2 = 1$ $y^2 - 3z^2 = 1$	$(1, 1, 0)$	2	2	1 sec
1922	[2]	Arwin	$x^2 - 2y^2 = 1$ $y^2 - 3z^2 = 1$	$(3, 2, 1)$	6	4	2 sec
1941	[20]	Ljunggren (see Arwin)	$x^2 - 2y^2 = 1$ $y^2 - 3z^2 = 1$	$(3, 2, 1)$	6	4	2 sec
1949	[12]	Gloden (see Rignaux)	$2x^2 + 1 = y^2$ $3x^2 + 1 = z^2$	$(0, 1, 1)$	2	2	1 sec
1969	[3]	Baker, Davenport (details below)	$3x^2 - y^2 = 2$ $8x^2 - z^2 = 7$	$(1, 1, 1)$ $(11, 19, 31)$	980	18	20 sec
1975	[15]	Kanagasabapathy Ponnudurai	$y^2 - 3x^2 = -2$ $z^2 - 8x^2 = -7$	$(1, 1, 1)$ $(11, 19, 31)$	980	18	20 sec
1978	[14]	Grinstead	$x^2 - 8y^2 = 1$ $3z^2 - 2y^2 = 1$	$(3, 1, 1)$	36	9	5 sec
1980	[29]	Vellupilai	$z^2 - 3y^2 = -2$ $z^2 - 6x^2 = -5$	$(1, 1, 1)$ $(29, 41, 71)$	50	6	2 sec
1984	[22]	Mohanty Ramasamy	$x^2 - 5y^2 = -20$ $z^2 - 2y^2 = 1$	$(0, 2, 3)$	10	4	3 sec
1985	[9]	Brown	$x^2 - 8y^2 = 1$ $z^2 - 5y^2 = 1$	$(1, 0, 1)$	6	4	1 sec
1987	[30]	Zheng	$y^2 - 2x^2 = 1$ $z^2 - 5x^2 = 4$	$(0, 1, 2)$	6	4	2 sec
1987	[30]	Zheng	$y^2 - 5x^2 = 4$ $z^2 - 10x^2 = 9$	$(0, 2, 3)$	80	10	3 sec
1988	[23]	Pinch (example)	$x^2 - 2y^2 = -1$ $x^2 - 10z^2 = -9$	$(1, 1, 1)$ $(41, 29, 13)$	10	4	1 sec
1995	[13]	Gaál, Pethő Pohst (example)	$2x^2 - y^2 = \pm 1$ $5x^2 - z^2 = \pm 4$	$(0, 1, 2)$ $(1, 1, 1)$ $(5, 7, 11)$	6, 2704 2704, 6	4, 15	Σ : 16 sec
1996	[24]	Riele (details below)	$2x^2 - y^2 = 1$ $y^2 - 3z^2 = 1$	$(1, 1, 0)$ $(5, 7, 4)$	96	12	1 sec
1996	[10]	Chen (details below)	$5x^2 - 3y^2 = 2$ $16y^2 - 5z^2 = 11$	$(1, 1, 1)$	7436	18	13.5 min
1996	[1]	Anglin (example)	$x^2 - 11y^2 = 1$ $z^2 - 56y^2 = 1$	$(1, 0, 1)$ $(199, 60, 449)$	10	4	26 sec
1997	[11]	Chen	$x^2 - 7y^2 = 2$ $z^2 - 32y^2 = -23$	$(3, 1, 3)$ $(717, 271, 1533)$	92	6	40 sec
1998	[18]	Kedlaya (example)	$x^2 - 2y^2 = -1$ $3z^2 - 4y^2 = -1$	$(1, 1, 1)$	36	9	2 sec
1995	[17]	Katayama, Levesque Nakahara (example)	$x^2 - 3y^2 = 1$ $y^2 - 2z^2 = -1$	$(1, 1, 1)$	8	4	2 sec
2004	[6]	Bennett (example)	$x^2 - 2y^2 = 1$ $9z^2 - 3y^2 = -3$	$(3, 2, 1)$	54	8	1 sec
2004	[19]	Liptai (details below)	$x^2 - 5y^2 = \pm 4$ $z^2 - 8y^2 = 1$	$(2, 0, 1)$ $(3, 1, 3)$ $(1, 1, 3)$	6, 2738	4, 6	Σ : 40 sec
2005		Szalay	$x^2 - 5y^2 = \pm 4$ $z^2 - 8x^2 = 1$	$(1, 1, 3)$	–, 310	8	4 sec
2005		Szalay	$3x^2 - 10y^2 = -13$ $x^2 - 3y^2 = z^2$	$(7, 4, 1)$ $(73, 40, 23)$	6	4	—

1. BAKER and DAVENPORT, [3].
$3x^2 - y^2 = 2$, $8x^2 - z^2 = 7 \Longrightarrow 7X^2 - 5Y^2 - 2Z^2 = 0$,
$X := y$, $Y := x$, $Z := z$, $(X_0, Y_0, Z_0) = (1, 1, 1)$,
$d \mid 980$,
$49s^4 - 224s^3r + 314s^2r^2 - 160sr^3 + 25r^4 = (d/D)^2$.

2. RIELE. [24]
$2x^2 - y^2 = 1$, $y^2 - 3z^2 = 1 \Longrightarrow X^2 - Y^2 - 6Z^2 = 0$,
$X := 2y$, $Y := 2x$, $Z := z$, $(X_0, Y_0, Z_0) = (5, 1, 2)$,
$d \mid 96$,
$23s^4 + 20s^3r - 150s^2r^2 + 20sr^3 + 23r^4 = -(2d/D)^2$.

3. CHEN, [10].
$5x^2 - 3y^2 = 2$, $16y^2 - 5z^2 = 11 \Longrightarrow 13X^2 - 2Y^2 - 11Z^2 = 0$,
$X := y$, $Y := x$, $Z := z$, $(X_0, Y_0, Z_0) = (1, 1, 1)$,
$d \mid 7436$,
$169s^4 - 1534s^3r + 1718s^2r^2 - 236sr^3 + 4r^4 = (d/D)^2$.

4. LIPTAI, [19].
$x^2 - 5y^2 = \pm 4$, $z^2 - 8z^2 = 1 \Longrightarrow X^2 - 3Y^2 - Z^2 = 0$ and $37X^2 - Y^2 - Z^2$,
$X := 2z$, $Y := 3y$, $Z := x$, $(X_0, Y_0, Z_0) = (1, 0, 1)$ and
$X := y$, $Y := x$, $Z := 2z$, $(X_0, Y_0, Z_0) = (1, 6, 1)$
$d \mid 6$ and $d \mid 2738$
$9s^4 - 74s^2r^2 + 81r^4 = (6d/D)^2$ and
$42439s^4 - 28416s^3r + 7050s^2r^2 - 768sr^3 + 31r^4 = -(2d/D)^2$.

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