

On the shape parameter and constrained modification of GB-spline curves*

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Abstract

GB-spline curves can be considered as the generalization of B-spline curve incorporating a shape parameter into the polynomial basis functions. The geometric effect of the alteration of the shape parameter is discussed in this paper, including constrained shape control of the curve.

Keywords: GB-spline curves, shape parameter, paths, shape control, constrained modification

MSC: 68U05

1. Introduction

Although B-spline curve still plays central role in computer aided geometric design, the recently developed generalizations of this curve are also in the forefront of research. The well-known result of this attempt is the NURBS curve (c.f. [9]), but this curve has rational coefficient functions, yielding computational stability problems. Some recently developed methods tried to incorporate shape parameters into the original, polynomial basis functions. One of the earliest methods in this way is β -spline curve with two global parameters ([1, 2]). Further methods have been provided by direct generalization of B-spline curves as α B-splines in [8] and [10] and recently as GB-splines in [3]. Some alternative spline curves with shape parameters can be found in [4, 5, 6].

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In this paper we examine the GB-spline curves. At first we study the effect of the shape parameter on the points of the curve, extending the method we applied for trigonometric CB-spline curves in [7]. In Section 2 we study the paths obtained by altering the shape parameter of the curve, and prove that points of the curve move along straight line segments. Applying this fact in Section 5 linear blending is used for constrained shape control, where the shape parameter is modified in a way that the new GB-spline curve passes through a given point.

2. GB-spline curve and its λ -paths

In [3] the GB-spline curve as a generalization of the classical uniform cubic B-spline curve with shape parameter has been introduced. The definition of an arc of a GB-spline curve with shape parameter λ is as follows.

Definition 2.1. Given a sequence of control points $P_i, (i = 0, \dots, 3)$ the arc of the GB-spline curve is

$$C(\lambda, t) = \sum_{i=0}^3 P_i b_i(\lambda, t), \quad \lambda \in [-8, \infty), \quad t \in [0, 1], \quad (2.1)$$

where the GB-spline basic functions are

$$\begin{aligned} b_0(\lambda, t) &= \frac{2}{12 + \lambda} (1 - t)^3 \\ b_1(\lambda, t) &= \frac{1}{12 + \lambda} (2(3 + \lambda)t^3 - 3(4 + \lambda)t^2 + 8 + \lambda) \\ b_2(\lambda, t) &= \frac{1}{12 + \lambda} (-2(3 + \lambda)t^3 + 3(2 + \lambda)t^2 + 6t + 2) \\ b_3(\lambda, t) &= \frac{2}{12 + \lambda} t^3. \end{aligned} \quad (2.2)$$

This arc can simply be extended to a multi-arc non-uniform cubic GB-spline curve in a usual way, using four consecutive control points and applying the substitution

$$t = \frac{u - u_i}{u_{i+1} - u_i}$$

at each arc, where $u \in [u_i, u_{i+1})$. Since the shape parameter has the same effect on each arc, we will focus on the single arc (2.1) in this paper.

Now we consider the paths $P(\lambda, t_0)$ of the point $C(t_0)$ of the curve as the parameter λ has been changed. Note, that in these paths λ is the running parameter and t is the family parameter. Throughout this paper these paths are called λ -paths.

Theorem 2.2. *The limit points of the λ -paths $P(\lambda, t_0)$ at $\lambda \rightarrow \infty$ are fixed points of the control leg P_1P_2 and have symmetrical positions for the midpoint of the leg.*

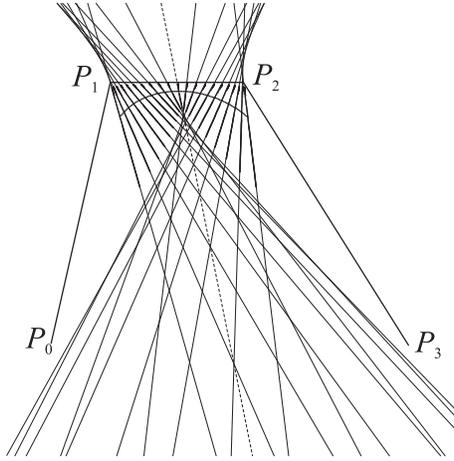


Figure 1: GB-spline curve and its λ -paths

Proof. By simple calculation

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} b_0(\lambda, t) &= \lim_{\lambda \rightarrow \infty} b_3(\lambda, t) = 0 \\ \lim_{\lambda \rightarrow \infty} b_1(\lambda, t) &= 2t^3 - 3t^2 + 1 \\ \lim_{\lambda \rightarrow \infty} b_2(\lambda, t) &= -2t^3 + 3t^2. \end{aligned}$$

Denoting the latter limits by $b_1(\infty, t)$ and $b_2(\infty, t)$, and observing, that $b_2(\infty, t) = 1 - b_1(\infty, t)$ it is obvious, that the limit points of the paths are

$$\lim_{\lambda \rightarrow \infty} P(\lambda, t) = L(t) = b_1(\infty, t)P_1 + (1 - b_1(\infty, t))P_2 \quad (2.3)$$

while at $t = 0.5$ we obtain $0.5P_1 + 0.5P_2$ and this completes the proof. \square

Theorem 2.3. *The λ -paths are straight line segments.*

Proof. We prove that for any fixed $t \in [0, 1]$ the points of the path $P(\lambda, t)$ can be described as barycentric combination of the two endpoints $P(-8, t)$ and $P(\infty, t) = \lim_{\lambda \rightarrow \infty} P(\lambda, t)$. The blending functions at $P(-8, t)$ are

$$\begin{aligned} b_0(-8, t) &= \frac{1}{2}(1 - t)^3 \\ b_1(-8, t) &= \frac{-5}{2}t^3 + 3t^2 \\ b_2(-8, t) &= \frac{1}{2}(5t^3 - 9t^2 + 3t + 1) \end{aligned}$$

$$b_3(-8, t) = \frac{1}{2}t^3.$$

We can observe that

$$\frac{b_0(\lambda, t)}{b_0(-8, t)} = \frac{b_3(\lambda, t)}{b_3(-8, t)} = \frac{4}{12 + \lambda} \quad (2.4)$$

and denoting this quotiens by $q(\lambda)$, after some calculation we obtain that

$$\begin{aligned} b_1(\lambda, t) &= q(\lambda)b_1(-8, t) + (1 - q(\lambda))b_1(\infty, t) \\ b_2(\lambda, t) &= q(\lambda)b_2(-8, t) + (1 - q(\lambda))b_2(\infty, t), \end{aligned}$$

thus finally for any point of the path we get

$$P(\lambda, t) = q(\lambda)P(-8, t) + (1 - q(\lambda))P(\infty, t) \quad (2.5)$$

and this was to be proved. \square

Theorem 2.4. *Considering the symmetric λ -paths $P(\lambda, t_0)$ and $P(\lambda, 1 - t_0)$, these lines may intersect each other. These intersection points are on the path of the point associated to the parameter value $t = 1/2$, that is at the line $P(\lambda, 1/2)$, if the lines P_0P_3 and P_1P_2 are parallel (see Fig. 2).*

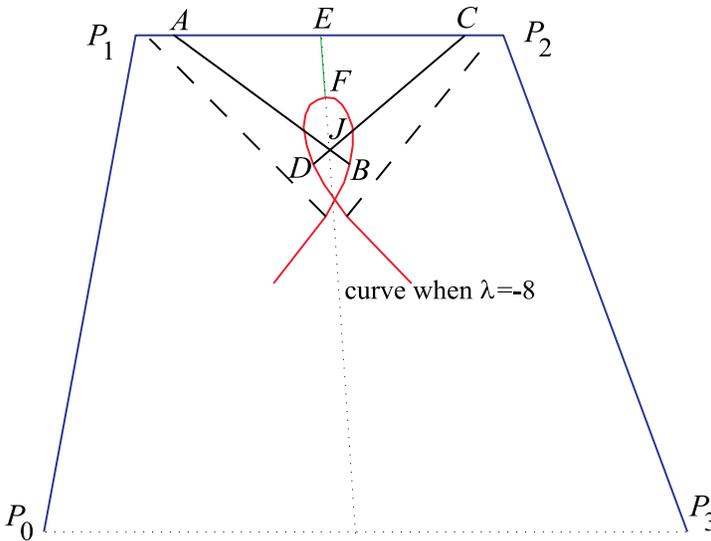


Figure 2: Symmetric paths intersect each other at a path associated to $t = 1/2$

Proof. It is easy to prove that the shape of a GB-spline curve is independent of the choice of coordinates, i.e. (2.2) satisfies the following two equations:

$$C(\lambda, t, P_0 * T + r, P_1 * T + r, P_2 * T + r, P_3 * T + r) \equiv C(\lambda, t, P_0, P_1, P_2, P_3) * T + r \quad (2.6)$$

where r is an arbitrary vector, and T is an arbitrary 3×3 matrix. From above we know that the GB-spline curve, the symmetric lines and the midpoint of the segments are all preserved by an affine transformation, so we can prove the result in a special case using the coordinate system given in Fig. 2.

For arbitrary parameter t , let the symmetric paths $P(\lambda, t_0)$ and $P(\lambda, 1 - t_0)$ intersect the control leg P_1P_2 and the curve $C(-8, t)$ at the point A, B, C, D respectively, and the middle path corresponding to $t = 0.5$ is on the line segment EF with E, F are midpoints of P_0P_3 and P_1P_2 respectively. Then using the definition of GB-spline curve, the coordinates of these points can be computed as follows:

$$\begin{aligned} A &= ((3t^2 - 2t^3)a, 1) \\ B &= 1/2((5t^3 - 9t^2 + 3t + 1)a + t^3, 1 + 3t - 3t^2) \\ C &= ((1 - 3t^2 + 2t^3)a, 1) \\ D &= 1/2((6t^2 - 5t^3)a + (1 - t)^3, 1 + 3t - 3t^2) \\ E &= (a/2, 1) \\ F &= (7/16a + 1/16, 7/8) \end{aligned}$$

Thus we obtain the coordinates of intersection point J of the line AB and CD :

$$J = \left(\frac{\frac{(3t^4 - 6t^3 - 2t^2 + 3t + 1)a^2 + (-3t^4 + 6t^3 - 3t^2)a}{(9t^2 - 9t - 3)a + t^2 + 1}}{\frac{(6t^4 - 12t^3 - 4t^2 + 10t + 2)a + (-t^2 + t - 1)}{(9t^2 - 9t - 3)a + t^2 + 1}} \right). \quad (2.7)$$

To prove that the symmetric paths intersect each other at the path of the point associated to the parameter value $t = 0.5$, we can prove that the point J is on the line segment EF . The reciprocal of slope of EF is

$$\frac{1}{k_{EF}} = \frac{a - 1}{2} \quad (2.8)$$

Connecting the points EJ , the reciprocal of slope of EJ is $\frac{a-1}{2}$ too. So the point J is located on the line EF (or located on its extending part). That completes the proof. \square

3. Passing through a given point

For practical applications, we would like to find a GB-spline curve passing through a given point among the family curves with the same control polygon. Of

course, the given point should be in a constrained region filled by the family of curves with running parameter λ . For $\lambda \geq 0$ this region is bounded by the B-spline curve, the control leg P_1P_2 and the paths when $\lambda = 0, 1$ (See Fig. 3.a). If we let $\lambda > -8$ then the shape of the constrained region is a bit more complex. For a convex polygon, the region includes two parts in general. There is only one curve passing through a given point in one region, while there are two curves passing through a given point in another. Definitely, for every point in this region, we can find no less than one curve passing through it. By the property of the given convex control polygon $P_i, i = 0, \dots, 3$, we can give the shape of the constrained region:

1) If two legs P_0P_1, P_2P_3 contend outside, the region $H \oplus G$ is circled by leg P_1P_2 , paths when $\lambda = 0, 1$ and the curve when $\lambda = -8$ as shown in Fig. 3. b.

2) When control polygon is a parallelogram, this region $H \oplus G$ is a triangle (See Fig. 3. c).

3) Otherwise this region is circled by leg P_1P_2 , paths when $\lambda = 0, 1$ and the curve when $\lambda = -8$ as shown in Fig. 3. d.

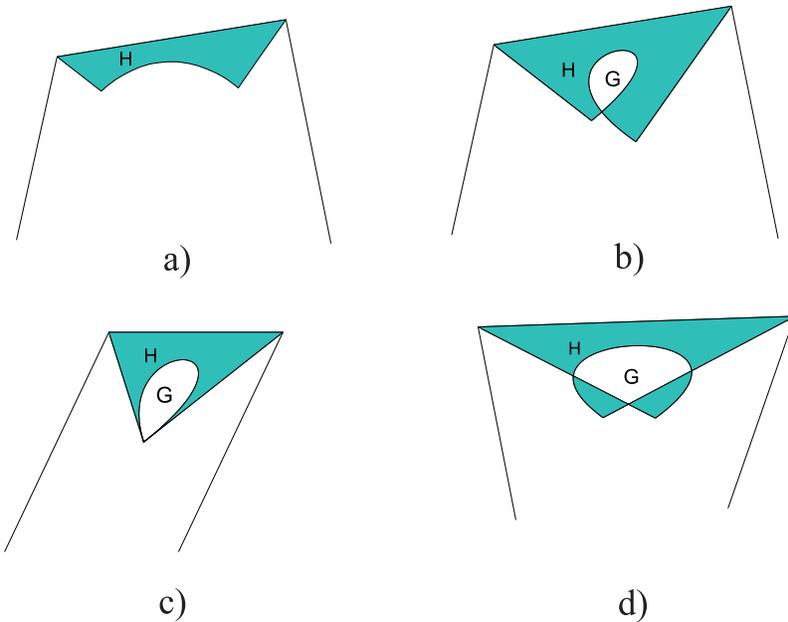


Figure 3: Different cases of constrained region for shape control. At each point in region H exactly one curve passes through, while in region G there are two solutions for each point.

Then for every point P in this region, we should find two parameter values λ_0 and t_0 for which $C(\lambda_0, t_0) = P$. As we have mentioned before, when $\lambda < 0$, the GB-spline curve is “below” the standard B-spline curve and in this case the variation diminishing property does not necessarily fulfilled. Thus in the following

we restrict ourselves for the case $\lambda \geq 0$, however the described method works for $\lambda < 0$ as well.

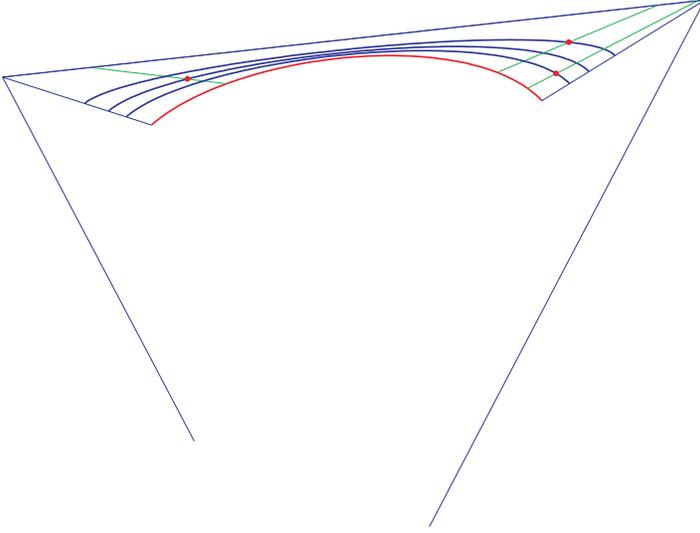


Figure 4: Given three points in the constrained region ($\lambda \geq 0$) the shape parameter is modified in a way that the curves pass through at the given points

We know that GB-spline paths are all lines, so we can find the value of t_0 by the following dichotomy method.

Let $first = 0, last = 1$:

a) Let $t^* = (first + last)/2$ and compute two endpoints $C(0, t^*)$ and $C(\infty, t^*)$ of path line $C(\lambda, t^*)$.

b) If P is just on the path line $C(\infty, t^*)$ within an allowed error, we get $t_0 = t^*$. The algorithm ends.

c) Otherwise we let $last = t^*$ (when P and b_0 are on the same side of path line) or $first = t^*$ (when P and b_3 are on the same side of path line). Then we return to step a).

After obtaining the value of t_0 , we can get the value of λ_0 by the following calculation. From (2.5) one can get

$$P(\lambda_0, t_0) = q(\lambda_0)P(-8, t_0) + (1 - q(\lambda_0))P(\infty, t_0)$$

which yields

$$q(\lambda_0) = \frac{P - P(\infty, t_0)}{P(-8, t_0) - P(\infty, t_0)}$$

for each coordinates of the points $P, P(\infty, t_0)$ and $P(-8, t_0)$. Choosing for example

the x coordinates of these points, one can find

$$\lambda_0 = \frac{4(P(-8, t_0) - P_x(\infty, t_0))}{P_x - P_x(\infty, t_0)} - 12.$$

By the above algorithm the curve $C(\lambda_0, t)$ passes through the given point P at the parameter value t_0 (Fig. 4).

4. Conclusion and further research

GB-spline curves has been studied in the paper with special emphasis on the numerical and geometrical effects of the alteration of its shape parameter λ . The curve has also been described in a linear blending way, where a cubic blending function was used to combine the classical B-spline curve and its control polygon leg. This approach may worth for further examination to study other curves with shape parameters as linear blending curves to give an overall view and comparison of these curve types.

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