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Distribution of terms of a logarithmic sequence*

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Abstract

The number $L(a,b) = \frac{a-b}{\ln a - \ln b}$ for $a \neq b$ and $L(a,a) = a$, is said to be the logarithmic mean of the positive numbers a, b. We shall say that a sequence $(a_n)_{n=1}^{\infty}$ with positive terms is a logarithmic sequence if $a_n = L(a_{n-1}, a_{n+1})$. In the present paper some basic estimations of the terms of logarithmic sequences are investigated.

Keywords: logarithmic mean, power mean, logarithmic sequence.

MSC: Primary 11K31, Secondary 26E60.

1. Introduction

Let a, b be positive real numbers. The logarithmic mean of a, b is defined as follows:

$$L(a,b) = \frac{a-b}{\ln a - \ln b} \quad \text{if} \quad a \neq b \quad \text{and} \quad L(a,a) = a$$

(see [5]).

The logarithmic sequence is defined in paper [2] by means of logarithmic mean in the following way:

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Definition 1.1. A sequence $(a_n)_{n=1}^{\infty}$ of positive real numbers is called logarithmic if

$$a_n = L(a_{n-1}, a_{n+1}) \text{ for each } n \geqslant 2.$$

Moreover, in [2] the existence of logarithmic sequence is proved and even it is shown that if a sequence $(a_n)_{n=1}^{\infty}$ is logarithmic and $a_1 < a_2$ then $a_1 < a_2 < \cdots < a_n < \cdots$. On the other hand, if $a_1 > a_2$ then $a_1 > a_2 > \cdots > a_n > \cdots$ (see [2], Theorem 2.1). Thus we see that the logarithmic sequence is either increasing or decreasing if $a_1 \neq a_2$. In the case $a_1 = a_2$ the logarithmic sequence $(a_n)_{n=1}^{\infty}$ is stationary and $a_n = a_1$ $(n = 1, 2, \dots)$. In the present paper we will consider only the logarithmic sequences $(a_n)_{n=1}^{\infty}$ for which $a_1 \neq a_2$.

The following theorem holds for logarithmic sequences.

Theorem 1.2. ([2; Th. 2.2., Th. 2.3.]) *Let the sequence $(a_n)_{n=1}^{\infty}$ be logarithmic and $a_1 \neq a_2$. Then the following implications hold.*

(i) *If $a_1 < a_2$ then*

$$\lim_{n\to\infty} a_n = \infty.$$

(ii) *If $a_1 > a_2$ then the series*

$$\sum_{n=1}^{\infty} a_n$$

converges.

Now we introduce the power mean of degree $\alpha \in \mathbb{R}$ of two positive numbers a, b as follows:

$$M_\alpha(a,b) = \left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}} \text{ if } \alpha \neq 0 \text{ and } M_0(a,b) = \lim_{\alpha\to 0} M_\alpha(a,b).$$

It is well known that $M_0(a,b) = \sqrt{a.b}$ and $M_\alpha(a,b)$ is increasing with respect to α (see [6]).

In paper [3] the following relation between $L(a,b)$ and $M_\alpha(a,b)$ is proved for arbitrary positive numbers a, b:

$$M_0(a,b) \leqq L(a,b) \leqq M_{\frac{1}{3}}(a,b), \tag{1.1}$$

and the equality occurs if and only if $a = b$.

As $M_\alpha(a,b)$ is increasing with respect to α, from (1.1) we have

$$M_0(a,b) \leqq L(a,b) \leqq M_\alpha(a,b) \tag{1.2}$$

for all $a, b > 0$ and $\alpha \geqslant \frac{1}{3}$.

Thus, if the sequence $(a_n)_{n=1}^{\infty}$ is logarithmic then (1.2) implies that for all $n \geqslant 2$ and $\alpha \geqslant \frac{1}{3}$ the inequality

$$\sqrt{a_{n-1}a_{n+1}} \leqslant a_n \leqslant \left(\frac{a_{n-1}^\alpha + a_{n+1}^\alpha}{2}\right)^{\frac{1}{\alpha}}$$

holds. Consequently we have for all $n \geqslant 2$ and $\alpha \geqslant \frac{1}{3}$

$$\frac{a_{n+1}}{a_n} \leqslant \frac{a_n}{a_{n-1}} \quad \text{and} \quad a_n^\alpha - a_{n-1}^\alpha \leqslant a_{n+1}^\alpha - a_n^\alpha. \tag{1.3}$$

From (1.3) we obtain that in the case of increasing logarithmic sequence $(a_n)_{n=1}^\infty$ for each $n \geqslant 2$ the inequalities

$$1 < \frac{a_{n+1}}{a_n} < \frac{a_n}{a_{n-1}} \quad \text{and} \quad 0 < a_n - a_{n-1} < a_{n+1} - a_n \tag{1.4}$$

hold.

A natural question arises. What can be said about the asymptotic behaviour of differences $a_{n+1} - a_n$ and fractions $\frac{a_{n+1}}{a_n}$ if $(a_n)_{n=1}^\infty$ is an increasing logarithmic sequence? More precisely, does it hold

$$\lim_{n\to\infty} (a_{n+1} - a_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1\ ? \tag{1.5}$$

In the first part of the present paper, among others, we give the answer to the previous question. We will determine the lower bounds for terms a_n, differences $a_{n+1} - a_n$ and fractions $\frac{a_{n+1}}{a_n}$ if $(a_n)_{n=1}^\infty$ is a logarithmic sequence.

2. Estimates for differences and quotients of consecutive terms of a logarithmic sequence

Theorem 2.1. *Let $(a_n)_{n=1}^\infty$ be a logarithmic sequence. Then the following implications hold.*

(i) *If $(a_n)_{n=1}^\infty$ is increasing then*

$$a_n > \left(\frac{{a_2}^\alpha - {a_1}^\alpha}{2}\right)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}} \tag{2.1}$$

for every $\alpha \geqslant \frac{1}{3}$ and $n \in \mathbb{N}$.

(ii) *If $(a_n)_{n=1}^\infty$ is decreasing then*

$$a_n < \left(\frac{{a_2}^\beta - {a_1}^\beta}{2}\right)^{\frac{1}{\beta}} n^{\frac{1}{\beta}} \tag{2.2}$$

for every $\beta < 0$ and $n \in \mathbb{N}$.

Proof. (i) Let $(a_n)_{n=1}^\infty$ be an increasing logarithmic sequence. Then (1.3) implies for $\alpha \geqslant \frac{1}{3}$

$$a_n^\alpha - a_{n-1}^\alpha < a_{n+1}^\alpha - a_n^\alpha \quad \text{for} \quad n \geqslant 2.$$

Consequently, for every $n \geqslant 2$ we have

$$a_2^\alpha - a_1^\alpha < a_{n+1}^\alpha - a_n^\alpha, \text{ i.e.}$$

$$(a_n^\alpha + a_2^\alpha - a_1^\alpha)^{\frac{1}{\alpha}} < a_{n+1}. \tag{2.3}$$

Now we will show by induction the inequality

$$((n-1)a_2^\alpha - (n-2)a_1^\alpha)^{\frac{1}{\alpha}} \leqslant a_n \tag{2.4}$$

for every $n \geqslant 2$. For $n = 2$ evidently the equality takes place in (2.4). Suppose that (2.4) holds for some $n = k \geqslant 2$. The we obtain

$$(ka_2^\alpha - (k-1)a_1^\alpha)^{\frac{1}{\alpha}} = ((k-1)a_2^\alpha - (k-2)a_1^\alpha + a_2^\alpha - a_1^\alpha)^{\frac{1}{\alpha}} \leqslant \\ \leqslant (a_k^\alpha + a_2^\alpha - a_1^\alpha)^{\frac{1}{\alpha}}.$$

Consequently, using (2.3) we obtain

$$(ka_2^\alpha - (k-1)a_1^\alpha)^{\frac{1}{\alpha}} \leqslant a_{k+1}$$

proving (2.4) for every $n \geqslant 2$. Finally, for $n \geqslant 2$ we obtain

$$a_n \geqslant ((n-1)(a_2^\alpha - a_1^\alpha) + a_1^\alpha)^{\frac{1}{\alpha}} > (n-1)^{\frac{1}{\alpha}} (a_2^\alpha - a_1^\alpha)^{\frac{1}{\alpha}} \geqslant n^{\frac{1}{\alpha}} \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)^{\frac{1}{\alpha}}.$$

(ii) Let $(a_n)_{n=1}^\infty$ be a decreasing logarithmic sequence. Then (1.2) and the fact that $M_\alpha(a,b)$ is increasing with respect to α imply the inequality

$$\left(\frac{a_{n-1}^\beta + a_{n+1}^\beta}{2}\right)^{\frac{1}{\beta}} < a_n = L(a_{n-1}, a_{n+1})$$

holding for every real $\beta < 0$. Consequently

$$a_n^\beta - a_{n-1}^\beta < a_{n+1}^\beta - a_n^\beta$$

holds for every $n \geqslant 2$. Especially,

$$a_{n+1}^\beta - a_n^\beta > a_2^\beta - a_1^\beta, \quad \text{i.e.}$$

$$a_{n+1} < \left(a_n^\beta + a_2^\beta - a_1^\beta\right)^{\frac{1}{\beta}} \tag{2.5}$$

holds for every $n \geqslant 2$. Now we will show by induction the inequality

$$a_n \leqq \left((n-1)a_2^\beta - (n-2)a_1^\beta\right)^{\frac{1}{\beta}} \tag{2.6}$$

for every $n \geqslant 2$. In the case $n = 2$ the equality takes place in (2.6). Suppose that (2.6) holds for some $n = k \geqslant 2$. The we obtain

$$\left(ka_2^\beta - (k-1)a_1^\beta\right)^{\frac{1}{\beta}} = \left((k-1)a_2^\beta - (k-2)a_1^\beta + a_2^\beta - a_1^\beta\right)^{\frac{1}{\beta}} \geqslant$$

$$\geqslant \left(a_k^\beta + a_2^\beta - a_1^\beta\right)^{\frac{1}{\beta}}.$$

Applying (2.5) we obtain

$$\left(ka_2^\beta - (k-1)a_1^\beta\right)^{\frac{1}{\beta}} \geqslant a_{k+1}$$

proving (2.6) for every integer $n \geqslant 2$. Finally, for every $n \geqslant 2$ we have

$$a_n \leqq \left((n-1)(a_2^\beta - a_1^\beta) + a_1^\beta\right)^{\frac{1}{\beta}} <$$

$$< \left(a_2^\beta - a_1^\beta\right)^{\frac{1}{\beta}} \frac{1}{(n-1)^{-\frac{1}{\beta}}} \leqq \left(\frac{a_2^\beta - a_1^\beta}{2}\right)^{\frac{1}{\beta}} \frac{1}{n^{-\frac{1}{\beta}}}.$$

□

Corollary 2.2. *Let $(a_n)_{n=1}^\infty$ be an increasing logarithmic sequence. Then for every $n \geqslant 2$ the inequality*

$$a_n > \left(\frac{\sqrt[3]{a_2} - \sqrt[3]{a_1}}{2}\right)^3 n^3$$

holds.

Proof. Follows directly from Theorem 2.1 *(i)* for $\alpha = \frac{1}{3}$. □

Corollary 2.3. *If $(a_n)_{n=1}^\infty$ is an increasing logarithmic sequence then the series*

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

converges.

Proof. By Corollary 2.2 we have for every $n \geqslant 2$

$$a_n > c.n^3 \quad \text{where} \quad c = \left(\frac{\sqrt[3]{a_2} - \sqrt[3]{a_1}}{2}\right)^3.$$

Evidently the series $\sum\limits_{n=2}^{\infty} \frac{1}{cn^3}$ majorises the series $\sum\limits_{n=2}^{\infty} \frac{1}{a_n}$. Consequently the series $\sum\limits_{n=1}^{\infty} \frac{1}{a_n}$ converges. □

Corollary 2.4. *Let $(a_n)_{n=1}^{\infty}$ be a decreasing logarithmic sequence and let $l > 0$ be a real number. Then the inequality*

$$a_n < c_1 \frac{1}{n^{\frac{1}{l}}}, \quad \text{where} \quad c_1 = \left(\frac{a_2^{-l} - a_1^{-l}}{2}\right)^{-\frac{1}{l}}$$

holds for every $n \geqslant 2$.

Proof. Follows from Theorem 2.1 (ii) for $\beta = -l,\ l > 0$. □

Corollary 2.5. *If $(a_n)_{n=1}^{\infty}$ is a decreasing logarithmic sequence then the series $\sum_{n=1}^{\infty} a_n$ converges.*

Theorem 2.6. *Let $(a_n)_{n=1}^{\infty}$ be an increasing logarithmic sequence. Then the inequality*

$$a_{n+1} - a_n > \left(\sqrt{a_2} - \sqrt{a_1}\right)^2 (n+1) \tag{2.7}$$

holds for every $n \geqslant 2$.

Proof. We will proceed by induction. From (1.3) for $\alpha = \frac{1}{2}$ follows the inequality

$$\sqrt{a_n} - \sqrt{a_{n-1}} < \sqrt{a_{n+1}} - \sqrt{a_n}. \tag{2.8}$$

For $n = 2$ we obtain from (2.8)

$$\sqrt{a_3} - \sqrt{a_2} > \sqrt{a_2} - \sqrt{a_1}$$

and

$$a_3 - a_2 > (\sqrt{a_2} - \sqrt{a_1})(\sqrt{a_3} + \sqrt{a_2}) > 3(\sqrt{a_2} - \sqrt{a_1})(\sqrt{a_2} - \sqrt{a_1}).$$

Suppose that (2.7) holds for some $n = k \geqslant 2$. Then from (2.8) for $n = k+1$ we obtain

$$\sqrt{a_{k+2}} - \sqrt{a_{k+1}} > \sqrt{a_{k+1}} - \sqrt{a_k}.$$

Moreover

$$a_{k+2} - a_{k+1} > (a_{k+1} - a_k)\frac{\sqrt{a_{k+2}} + \sqrt{a_{k+1}}}{\sqrt{a_{k+1}} + \sqrt{a_k}} =$$

$$= (a_{k+1} - a_k) + (a_{k+1} - a_k)\frac{\sqrt{a_{k+2}} - \sqrt{a_k}}{\sqrt{a_{k+1}} + \sqrt{a_k}} =$$

$$= (a_{k+1} - a_k) + (\sqrt{a_{k+1}} - \sqrt{a_k})(\sqrt{a_{k+2}} - \sqrt{a_k}) >$$

$$> (a_{k+1} - a_k) + \left(\sqrt{a_{k+1}} - \sqrt{a_k}\right)^2.$$

As (2.8) implies

$$\sqrt{a_{k+1}} - \sqrt{a_k} > \sqrt{a_2} - \sqrt{a_1}$$

we have

$$a_{k+2} - a_{k+1} > a_{k+1} - a_k + (\sqrt{a_2} - \sqrt{a_1})^2.$$

Finally

$$a_{k+2} - a_{k+1} > (k+2)(\sqrt{a_2} - \sqrt{a_1})^2.$$

□

Theorem 2.7. *Let $(a_n)_{n=1}^{\infty}$ be a logarithmic sequence. Then $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and the following implications hold.*

1. *If $(a_n)_{n=1}^{\infty}$ is increasing then*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

2. *If $(a_n)_{n=1}^{\infty}$ is decreasing then*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

Proof. The sequence $(a_n)_{n=1}^{\infty}$ is logarithmic, thus

$$a_n = \frac{a_{n+1} - a_{n-1}}{\ln a_{n+1} - \ln a_{n-1}} \quad \text{for } n \geqslant 2.$$

Consequently

$$\frac{a_n}{a_{n-1}} = \frac{\frac{a_{n+1}}{a_{n-1}} - 1}{\ln \frac{a_{n+1}}{a_{n-1}}}$$

which is equivalent with

$$\frac{a_n}{a_{n-1}} \ln \frac{a_{n+1}}{a_n}\frac{a_n}{a_{n-1}} = \frac{a_{n+1}}{a_n}\frac{a_n}{a_{n-1}} - 1. \tag{2.9}$$

The first relation in (1.3) implies that the sequence $\left(\frac{a_{n+1}}{a_n}\right)_{n=1}^{\infty}$ is decreasing and bounded from below. Consequently the limit $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and it is finite. Denote $x = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$.

If the sequence $(a_n)_{n=1}^{\infty}$ is increasing then obviously $x \geqslant 1$. Taking limit in (2.9) for $n \to \infty$ we obtain

$$x \ln x^2 = x^2 - 1 \quad \text{i.e.} \quad 2x \ln x = x^2 - 1.$$

The above inequality can not hold for $x > 1$ since for all real $x \in (0,1) \cup (1,\infty)$ the inequality $2x \ln x < x^2 - 1$ holds. Thus $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

If the sequence $(a_n)_{n=1}^{\infty}$ is decreasing then obviously $0 \leqslant x < 1$. In the case $0 < x < 1$ again we obtain $2x \ln x = x^2 - 1$ what is impossible. Thus we have $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$ in the case of a decreasing sequence $(a_n)_{n=1}^{\infty}$. □

Corollary 2.8. *Let $(a_n)_{n=1}^{\infty}$ be a logarithmic sequence. Then*

$$\lim_{n\to\infty} \frac{a_n}{q^n} = 0$$

1. *for every real $q > 1$ if $(a_n)_{n=1}^{\infty}$ is increasing,*
2. *for every real $q > 0$ if $(a_n)_{n=1}^{\infty}$ is decreasing.*

Proof. 1. Consider the power series

$$\sum_{n=1}^{\infty} a_n x^n .$$

Then Theorem 2.7 implies that the radius of its convergence is $R = 1$. Thus for every $0 < x < 1$ the series $\sum\limits_{n=1}^{\infty} a_n x^n$ converges. Consequently

$$\lim_{n\to\infty} a_n x^n = 0.$$

Denoting $q = \frac{1}{x}$ we have $q > 1$ arbitrary and $\frac{a_n}{q^n} \to 0 \quad (n \to \infty)$ holds.
2. If $(a_n)_{n=1}^{\infty}$ is decreasing then Theorem 2.7 implies that the radius of convergence R of the considered power series is infinity. Thus for every real $x > 0$ we have $\lim\limits_{n\to\infty} a_n x^n = 0$. □

Corollary 2.9. *If $(a_n)_{n=1}^{\infty}$ is an increasing logarithmic sequence then the set*

$$\left\{ \frac{a_m}{a_n} : \ m, n = 1, 2, \dots \right\}$$

is dense in $(0, \infty)$.

Proof. The proof follows from Theorem 2.7 and the following theorem: If for an unbounded sequence $(a_n)_{n=1}^{\infty}$ of positive real numbers

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

holds then the set $\left\{ \frac{a_m}{a_n} : \ m, n = 1, 2, \dots \right\}$ is dense in $(0, \infty)$ (see Theorem 1.1 of [1]). □

3. Comparison of terms of logarithmic sequence with terms of other sequences

First we will show that the function $L(x, b)$ is increasing in $x > 0$ with fixed $b > 0$. This property of the function $L(x, b)$ will be later used in the proof of Theorem 3.3.

Theorem 3.1. *Let $a, b, c, \in \mathbb{R}^+$. Then*

$$L(c, b) \leqslant L(a, b) \quad \Leftrightarrow \quad c \leqslant a.$$

Proof. For $0 < x \neq b$ we have

$$L(x, b) = b\frac{\frac{x}{b} - 1}{\ln \frac{x}{b}}.$$

Thus $L(x, b)$ is increasing with respect to x if and only if the function

$$f(y) = b\frac{y - 1}{\ln y}$$

is increasing with respect to y ($y \neq 1$), i.e. $\frac{df}{dy} \geqslant 0$ for $y > 0$, $y \neq 1$. This is equivalent to

$$g(y) = \frac{1}{y} + \ln y - 1 \geqslant 0$$

for each $y > 0$. Since $\frac{dg}{dy} = \frac{1}{y} - \frac{1}{y^2} = \frac{y-1}{y^2}$ we obviously have $\frac{dg}{dy} \leqslant 0$ for $0 < y < 1$ and $\frac{dg}{dy} \geqslant 0$ for $y \geqslant 1$. Thus $g(y)$ attains its minimum at $y = 1$, i.e. $g(y) \geqslant g(1) = 0$ for each $y > 0$. □

First we are going to compare the terms of a given logarithmic sequence with terms of another logarithmic sequence.

Theorem 3.2. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be such logarithmic sequences that $a_1 = b_1$ and $a_2 \geqslant b_2$. Then*

$$a_n \geqslant b_n \quad \textit{and} \quad \frac{a_n}{a_{n-1}} \geqslant \frac{b_n}{b_{n-1}}$$

hold for every $n \geqslant 2$.

Proof. We will proceed by induction. For $n = 2$ the statement obviously holds. Assume that it holds for some $n = k \geqslant 2$, i.e.

$$a_k \geqslant b_k \quad \text{and} \quad \frac{a_k}{a_{k-1}} \geqslant \frac{b_k}{b_{k-1}}. \tag{3.1}$$

Let us consider the terms a_{k+1}, b_{k+1}. Since both $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are logarithmic sequences, we have

$$a_k = L(a_{k-1}, a_{k+1}) \quad \text{and} \quad b_k = L(b_{k-1}, b_{k+1}).$$

Consequently

$$\frac{a_k}{a_{k-1}} = L\left(\frac{a_{k+1}}{a_{k-1}}, 1\right) \quad \text{and} \quad \frac{b_k}{b_{k-1}} = L\left(\frac{b_{k+1}}{b_{k-1}}, 1\right). \tag{3.2}$$

We will use the notation

$$\alpha_1=\frac{a_k}{a_{k-1}},\ \alpha_2=\frac{a_{k+1}}{a_{k-1}},\ \beta_1=\frac{b_k}{b_{k-1}}\quad\text{and}\quad\beta_2=\frac{b_{k+1}}{b_{k-1}}$$

in the rest of the proof. Then (3.2) implies

$$\frac{\alpha_1}{\beta_1}=\frac{L(\alpha_2,1)}{L(\beta_2,1)}=\frac{\alpha_2-1}{\beta_2-1}\cdot\frac{\ln\beta_2}{\ln\alpha_2}=\frac{\alpha_2^{\frac12}+1}{\beta_2^{\frac12}+1}\cdot\frac{\alpha_2^{\frac12}-1}{\beta_2^{\frac12}-1}\cdot\frac{\ln\beta_2^{\frac12}}{\ln\alpha_2^{\frac12}}=$$

$$=\frac{\alpha_2^{\frac12}+1}{\beta_2^{\frac12}+1}\cdot\frac{\alpha_2^{\frac14}+1}{\beta_2^{\frac14}+1}\cdot\frac{\alpha_2^{\frac14}-1}{\beta_2^{\frac14}-1}\cdot\frac{\ln\beta_2^{\frac14}}{\ln\alpha_2^{\frac14}}=\dots=\left(\prod_{k=1}^{n}\frac{\alpha_2^{\frac{1}{2^k}}+1}{\beta_2^{\frac{1}{2^k}}+1}\right)\cdot\frac{\alpha_2^{\frac{1}{2^n}}-1}{\beta_2^{\frac{1}{2^n}}-1}\cdot\frac{\ln\beta_2^{\frac{1}{2^n}}}{\ln\alpha_2^{\frac{1}{2^n}}}.$$

Taking into account that $\frac{a+1}{b+1}\leqslant\frac{a}{b}$ holds in the case when $a\geqslant b>0$, we obtain:

$$\frac{\alpha_1}{\beta_1}\leqslant\left(\frac{\alpha_2}{\beta_2}\right)^{\sum\limits_{k=1}^{n}\frac{1}{2^k}}\cdot\frac{L\left(\alpha_2^{\frac{1}{2^n}},1\right)}{L\left(\beta_2^{\frac{1}{2^n}},1\right)}.$$

taking limit for $n\to\infty$ we obtain $\frac{\alpha_1}{\beta_1}\leqslant\frac{\alpha_2}{\beta_2}$ as

$$\lim_{n\to\infty}L\left(a^{\frac{1}{2^n}},1\right)=1\quad\text{where}\quad a>0.$$

The inequality $\frac{\alpha_1}{\beta_1}\leqslant\frac{\alpha_2}{\beta_2}$ is equivalent with the inequality $\frac{a_{k+1}}{a_k}\geqslant\frac{b_{k+1}}{b_k}$. Since $a_k\geqslant b_k$ using the induction assumption (3.1) we obtain $a_{k+1}\geqslant b_{k+1}$ which completes the proof.

□

The next theorem generalizes the previous one.

Theorem 3.3. *Let $(a_n)_{n=1}^{\infty}$ be a logarithmic sequence and let a sequence $(b_n)_{n=1}^{\infty}$ fulfils the following conditions*

$$b_1=a_1,\ b_2\leqslant a_2\quad\textit{and}\quad b_n\geqslant L(b_{n-1},b_{n+1})\quad\textit{for}\quad n\geqslant 2\tag{3.3}$$

Then for every positive integer n the inequality

$$a_n\geqslant b_n$$

holds.

Proof. Let $k\geqslant 0$ be a given integer. Define the sequence $(a_{k,n})_{n=1}^{\infty}$ as follows:

$$a_{k,1}=b_{k+1},\ a_{k,2}=b_{k+2}\quad\text{and}\quad a_{k,n}=L(a_{k,n-1},a_{k,n+1})\quad\text{for}\quad n\geqslant 2.\tag{3.4}$$

Thus the sequence $(a_{k,n})_{n=1}^{\infty}$ is logarithmic for every $k\geqslant 0$.
We will show that

$$a_{k,n}\leqslant a_{k+n}\quad\text{and}\quad b_{k+3}\leqslant a_{k,3}\tag{3.5}$$

holds for every integer $k \geqslant 0$ and positive integer n. We will proceed by induction with respect to k.
For $k = 0$ from (3.3), (3.4) we have

$$a_{0,1} = b_1 = a_1, \quad a_{0,2} = b_2 \leqslant a_2.$$

The assumption that both sequences $(a_n)_{n=1}^{\infty}$ and $(a_{0,n})_{n=1}^{\infty}$ are logarithmic and Theorem 3.2 imply that for every $n \in \mathbb{N}$ the inequality

$$a_{0,n} \leqslant a_n$$

holds. On the other hand, (3.3) and (3.4) imply

$$L(b_3, b_1) \leqslant b_2 = a_{0,2} = L(a_{0,3}, a_{0,1}) = L(a_{0,3}, b_1),$$

and consequently, using Theorem 3.1, we obtain

$$b_3 \leqslant a_{0,3}.$$

Suppose that for some $k = l \geqslant 0$ inequalities (3.5) hold. In the case $k = l + 1$ we obtain

$$a_{l+1,1} = b_{l+2} = a_{l,2} \quad \text{and} \quad a_{l+1,2} = b_{l+3} \leqslant a_{l,3}.$$

By use of Theorem 3.2 and induction assumption we obtain

$$a_{l+1,n} \leqslant a_{l,n+1} \leqslant a_{l+1+n}$$

for every $n \in \mathbb{N}$. On the other hand, (3.3) and (3.4) imply

$$L(b_{l+4}, b_{l+2}) \leqslant b_{l+3} = a_{l+1,2} = L(a_{l+1,3}, a_{l+1,1}).$$

As $b_{l+2} = a_{l+1,1}$, Theorem 3.1 implies

$$b_{l+4} \leqslant a_{l+1,3}.$$

Thus we proved (3.5) by induction. Finally, from (3.5) we obtain

$$b_k \leqslant a_{k-3,3} \leqslant a_k$$

for every $k \geqslant 3$. □

The proof of the following theorem is an application of the previous one.

Theorem 3.4. *Let $(a_n)_{n=1}^{\infty}$ be such a logarithmic sequence that $a_1 < a_2$. Then the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges and*

$$\sum_{n=1}^{\infty} \frac{1}{a_n} < \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{(\sqrt{a_2} - \sqrt{a_1})^2} \frac{\pi^2}{6}$$

holds.

Proof. Define the sequence $(b_n)_{n=1}^{\infty}$ by:

$$b_1 = a_1,\ b_2 = a_2 \quad \text{and} \quad b_n = M_{\frac{1}{2}}(b_{n-1}, b_{n+1}) \quad \text{for} \quad n \geqslant 2.$$

As

$$M_{\frac{1}{2}}(b_{n-1}, b_{n+1}) \geqslant L(b_{n-1}, b_{n+1}),$$

we have $b_n \geqslant L(b_{n-1}, b_{n+1})$. Thus the sequence $(b_n)_{n=1}^{\infty}$ fulfils the assumptions of Theorem 3.2.
Consequently $b_n \leqslant a_n$ for every $n \in \mathbb{N}$. Using ([2] Th.1.1) we have

$$b_n = \Big((n-1)\sqrt{b_2} - (n-2)\sqrt{b_1}\Big)^2,$$

i.e. for every $n > 2$

$$b_n = \Big((n-2)(\sqrt{b_2} - \sqrt{b_1}) + \sqrt{b_2}\Big)^2 > (n-2)^2 \Big(\sqrt{b_2} - \sqrt{b_1}\Big)^2 =$$

$$= (n-2)^2 \left(\sqrt{a_2} - \sqrt{a_1}\right)^2$$

holds. Finally we obtain

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \leqslant \sum_{n=1}^{\infty} \frac{1}{b_n} < \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{(\sqrt{a_2} - \sqrt{a_1})^2} \frac{\pi^2}{6}.$$

□

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