*Annales Mathematicae et Informaticae*
***34*** *(2007) pp. 29–31*
http://www.ektf.hu/tanszek/matematika/ami

# On a sum involving powers of reciprocals of an arithmetical progression

**Hacène Belbachir, Abdelkader Khelladi**

USTHB/ Faculté de Mathématiques, Alger
e-mail: hbelbachir@usthb.dz, hacenebelbachir@gmail.com, akhelladi@usthb.dz, akhelladi@wissal.dz

*Submitted 25 September 2007; Accepted 9 December 2007*

**Abstract**

Our purpose is to establish the following result: Let $a$ and $d$ be coprime integers and $a, a+d, a+2d, \dots, a+(k-1)\,d$ $(k \geqslant 2)$ be an arithmetical progression. Then for all integers $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ the rational number $1/a^{\alpha_0} + 1/(a+d)^{\alpha_1} + \dots + 1/(a+(k-1)\,d)^{\alpha_{k-1}}$ is never an integer. This result extends theorems of Taeisinger (1915) and Kürschák (1918), and also generalizes a result of Erdős (1932).

*Keywords:* Harmonic sums, arithmetical progression, greatest prime factor.

In 1915, Taeisinger proved that the harmonic number $H_n := 1 + \frac{1}{2} + \dots + \frac{1}{n}$ is never an integer except for $H_1$. The more general result that the sum of reciprocals of consecutive terms, not necessarily starting with 1, is never an integer was proved by Kürschák in 1918 [3, p.157]. In 1932, Erdős proved that the sum of reciprocals of any integers in arithmetical progression is never a reciprocal and then an integer [2]. Our purpose is to give some extensions of the cited results.

Let $n$ be a positive integer and $p$ be a prime number. We define the $p$-valuation of $n$ as the unique positive integer $v_p(n)$ satisfying $n = u \cdot p^{v_p(n)}$ with $\gcd(u,p) = 1$.

Our idea relies on the fundamental inequality about the valuation of a sum of two positive integers. Let $n$ and $m$ be integers. It is well known that $v_p(n+m) \geqslant \min\{v_p(n), v_p(m)\}$, with a remarkable implication that if $v_p(n) > v_p(m)$ then $v_p(n+m) = v_p(m)$.

The following Theorem is the key assertion behind all the results of this paper.

**Theorem 1.1.** *Let $n_1, n_2, \dots, n_k$ be positive integers. Assume that there exists a prime $P$ such that $v_P(n_{j_P})$ is maximal (non zero) for a unique $j_P \in \{1, 2, \dots, k\}$. Then*

$$\frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k}$$

*is never an integer.*

In fact this result is well-known and simple consequence of elementary properties of valuations (see [1]). However, for the convenience of the reader we give the proof of this statement.

**Proof.** Let us suppose that $N := \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$ is an integer. By setting $R := n_1 n_2 \cdots n_k / P^v$, where $v = 1 + \sum_{j \neq j_P} v_P(n_j)$, one has

$$RN - \sum_{j \neq j_P} \frac{R}{n_j} = \frac{R}{n_{j_P}}.$$

Each term of the left hand side is an integer, while the right hand side is not. It is contradiction, so the statement is proved. □

We get the following as a simple and immediate consequence.

**Corollary 1.2.** *Let $n_1, n_2, \ldots, n_k$ be positive integers. Assume that there exists a prime $P$ such that $P \mid n_i$ for some $i$, and $P \nmid n_j$ when $j \neq i$. Then*

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

*is never an integer.*

The first main result of our paper is an extension of Taeisinger's Theorem.

**Theorem 1.3.** *Let $n$ be an integer $\geqslant 2$ and $\alpha_2, \ldots, \alpha_n$ be positive integers. Then*

$$1 + \frac{1}{2^{\alpha_2}} + \cdots + \frac{1}{n^{\alpha_n}}$$

*is never an integer.*

**Proof.** Let $P$ be the greatest prime number $\leqslant n$. By Bertrand's postulate we have $n < 2P$. Thus $P$ is coprime to all $k \in \{1, 2, \ldots, n\} \setminus \{P\}$. The theorem follows then from Corollary 1.2. □

To study the case of an arithmetical progression, we give the following result which is an immediate consequence of a theorem of Shorey and Tijdeman [4].

**Theorem 1.4.** *Let $a, d$ and $k$ be positive integers, satisfying* $\gcd(a, d) = 1$, $k \geqslant 2$. *By setting $\Delta = \prod_{j=1}^{k} (a + (j-1) d)$ and $P := \max_{p \mid \Delta} p$, the greatest prime factor of $\Delta$, then for $d > 1$, we have $P \geqslant k$.*

Now we are able to establish an extension of Erdős theorem, then of Kürschák's Theorem.

**Theorem 1.5.** *Let $a, d$ and $k$ be positive integers satisfying $k \geqslant 2$, and $a, a+d, a+2d, \ldots, a+(k-1)\,d$ be an arithmetical progression. Then for all positive integers $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ the rational number*

$$\frac{1}{a^{\alpha_0}} + \frac{1}{(a+d)^{\alpha_1}} + \cdots + \frac{1}{(a+(k-1)\,d)^{\alpha_{k-1}}}$$

*is never an integer.*

**Proof.** Let $\delta := \gcd(a, d)$. Consider the arithmetical progression $(a' + jd')$, $j = 0, \ldots, k-1$, where $a' = a/\delta$ and $d' = d/\delta$. For this progression, let $P$ the prime given by Theorem 1.4. If $P \nmid \delta$, we conclude by using Corollary 1.2. Otherwise, we have

$$\frac{1}{a^{\alpha_0}} + \frac{1}{(a+d)^{\alpha_1}} + \cdots + \frac{1}{(a+(k-1)\,d)^{\alpha_{k-1}}} < \frac{k}{P} \leqslant 1.$$

□

**Acknowledgement.** The authors are grateful to the referee and would like to thank him/her for comments and several suggestions which improved the quality of this paper.

# References

[1] Bachman, G., Introduction to $p$-adic numbers and valuation theory, *Academic Press, N.Y.* (1964).

[2] Erdős, P., Verallgemeinerung eines elementar-zahlentheoretischen Satzes von Kürschák (Generalization of an elementary number-theoretic theorem of Kürschák.), *Mat. Fiz. Lapok* 39 (1932), 17–24.

[3] Hoffman, P., The man who loved only numbers: The story of Paul Erdős and the search for mathematical truth, *N.Y. Hyperion* (1998).

[4] Shorey, T.N. and Tijdeman, R., On the greatest prime factor of an arithmetical progression, in "A tribute of Paul Erdős", Edited by A. Baker, B. Bollobas, and A. Hainal. *Cambridge University Press* (1990), 385–389.

**Hacène Belbachir**
**Abdelkader Khelladi**
USTHB/ Faculté de Mathématiques
BP 32, El Alia, 16111 Bab Ezzouar, Alger
Algeria