Annales Mathematicae et Informaticae
33 *(2006) pp. 57–68*
http://www.ektf.hu/tanszek/matematika/ami

Solving ordinary differential equation systems by approximation in a graphical way

Gábor Geda[a], **Anikó Vágner**[b]

[a]Department of Computer Science
Eszterházy Károly College
e-mail: gedag@aries.ektf.hu

[b]Department of Information Technology
Eszterházy Károly College
e-mail: vagnera@aries.ektf.hu

Submitted 11 November 2006; Accepted 18 December 2006

Abstract

Our aim was to find a graphic numeric solution method for higher-order differential equations and differential equation systems. To understand this method the basic mathematical knowledge taught in the secondary school must be enough, we have to complete it with geometric meaning of differential quotient and generalization of knowledge about two-dimensional vector space. We considered it important to make this method easy to algorithm. Such method and its practical experience are shown in this paper.

MSC: 65L05, 65L06, 53A04,97D99

1. Introduction

It is well-known how important the differential equation models are in the mathematical description of different processes and systems.

Our aim is to find approximate methods which are based on the demonstration and there is no need for higher mathematical knowledge to understand and apply them. Moreover, it is easy to algorithmise them even in the possession of the secondary school material.

The problem concerning this topic such as mechanical oscillations can be given as an ordinary n-order differential equation:

$$y^{(n)}(t) = g(t, y(t), y^{(1)}(t), y^{(2)}(t), \ldots, y^{(n-1)}(t))$$

This can be transformed to explicit ordinary differential equation system (abbreviated as ODES in the followings):

$$\dot{x}_i(t) = f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) \quad (i = 1, \ldots, n). \tag{1.1}$$

The solution for these equations, if it exist at all, can be given with $x_i(t)$ $(i = 1, \ldots, n)$ functions. In most cases to produce such solutions is a difficult task which needs the knowledge of serious mathematical devices.

We consider the solutions $n+1$-dimensional space curve

$$\mathbf{x}(t) = (t, x_1(t), x_2(t), \ldots, x_n(t)).$$

So (1.1) corresponds a vector to any point in $n+1$-dimensional vector place, and the vector is parallel with the tangent line at a given P point of the solution of ODES. The only problem is that we do not know which points should be considered belonging to the same curve among the points close to one another.

In certain cases there is no need to present all the possible solutions, that is all curves, only the $\mathbf{x}(t)$ curve is necessary of which a given

$$P_0(p_0; p_1; p_2; \ldots; p_n)$$

fits, and on the coordinates of which

$$x_i(p_0) = p_i \quad (i = 1, \ldots, n)$$

is realized. In this case we can say that we solve an initial value problem. By expressing an initial value problem we choose one of the curves which are solutions for ODES. Other times we have to be contented with the approximate solution of the problem.

In a geometrical point of view the solution for an initial value problem by approximation is giving a $P_0, P_1, \ldots, P_k$ point serial the elements of which fit to the chosen curve by desired accuracy. The serial of points $(0 \leqslant i \leqslant k)$ determines a broken line the points of which approximate well the points of the curve.

The accuracy of the approximation is influenced by several factors. The most important ones among them are the approximate algorithm and ODES itself.

This way, when we select the successive elements of the point serial we should take the changes of the curve of the function into consideration.

2. Demonstration of an approximate method

Let (1.1),

$$P_0(p_0; p_1; p_2; \ldots; p_n)$$

point on coordinates of which

$$x_i(p_0) = p_i \quad (i = 1, \ldots, n)$$

is realized and a suitable minor d distance. We would like to determine the broken line running through P_0 point and approaching the

$$\mathbf{x}(t) = (t, x_1(t), x_2(t), \ldots, x_n(t))$$

function curve meaning the solution in the surroundings of P_0 given point.

Let $\mathbf{m^p}$ vector be parallel with tangent line to curves at P_0 point. The coordinates of $\mathbf{m^p}$ are:

$$m_0^p = 1$$

$$m_i^p = \dot{x}_i(p_0) \quad (i = 1, \ldots, n).$$

Define $\mathbf{p}$ vector, where $\mathbf{p}$ is parallel with $\mathbf{m^p}$ vector and $\|\mathbf{p}\|$=d, that is

$$\mathbf{p} = \frac{\mathbf{m^p}}{\|\mathbf{m^p}\|} d. \tag{2.1}$$

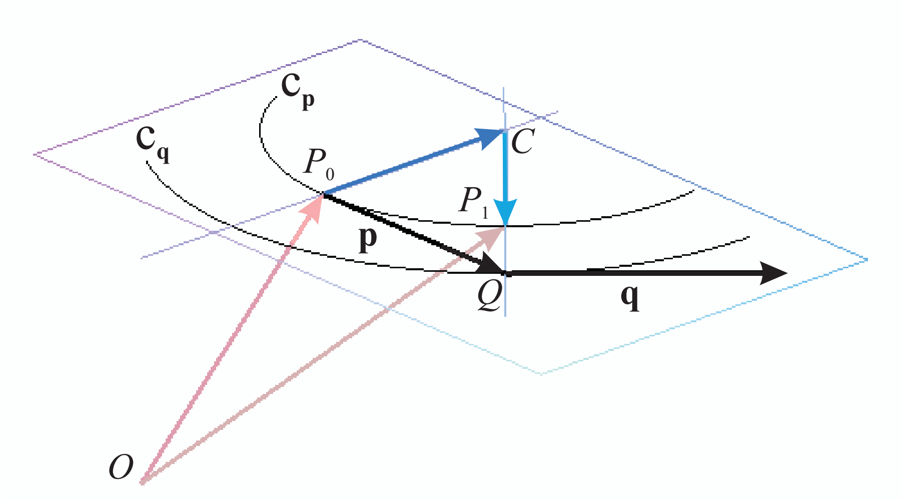

Figure 1: The c_p and the c_q osculation circles in case $n = 2$ (in 3 dimension)

Define Q point, where $\overrightarrow{P_0Q} = \mathbf{p}$. Then coordinates of Q point can be calculated (see figure 1). Let coordinates of Q point be $Q(q_0; q_1; \ldots; q_n)$.

Coordinates of $\mathbf{m^q}$ vector, which is parallel with tangent line to curves at Q point is:

$$m_0^q = 1$$

$$m_i^q = \dot{x}_i(q_0) \quad (i = 1, \ldots, n).$$

If d is minor enough, then Q is close enough to the curve which is the solution for the initial value problem. This way, $\mathbf{m^q}$ well approximates the steepness of the curve in one of its points near to Q.

Define $\mathbf{q}$ vector, where $\mathbf{q}$ is parallel $\mathbf{m^q}$ vector and $\|\mathbf{q}\| = d$, in other words

$$\mathbf{q} = \frac{\mathbf{m^q}}{\|\mathbf{m^q}\|} d. \tag{2.2}$$

If $\mathbf{q}$ vector is parallel with $\mathbf{p}$ vector, then we accept Q point as the next element of serial of points, and we continue the approaching from this point.

Otherwise in the narrow surroundings of P_0 the curve can be well approximated in the plane, which $\mathbf{p}$ and $\mathbf{q}$ vectors define with a proper arc (c_p), which is the osculating circle of the curve in P_0. Similarly, we can fit an arch (c_q) in $(\mathbf{p}, \mathbf{q})$ plane in the narrow surroundings of Q to the curve on which Q fits (see figure 2.a). The lines which are perpendicular tangent lines in P_0 and Q points intersect at point C. This point can be considered to be the common central point of the two circles (c_p and c_q) if the d is minor enough.

Define $\mathbf{a}$ and $\mathbf{b}$ vectors for coordinates of C point: $\mathbf{a} = \mathbf{p} + \lambda\mathbf{q}$, and let $\mathbf{a}$ be perpendicular to $\mathbf{p}$ vector, and $\mathbf{b} = \mathbf{q} + \omega\mathbf{p}$ and let $\mathbf{b}$ be perpendicular to $\mathbf{q}$ vector (see figure 2.b; 2.c).

As $\mathbf{p}$ and $\mathbf{a}$ are perpendicular to each other, their scalar product is null, from which λ can be calculated:

$$\lambda = -\frac{\sum_{i=0}^n p_i^2}{\sum_{i=0}^n p_i q_i}.$$

Similar way, can we get value of ω from scalar product of $\mathbf{q}$ and $\mathbf{b}$:

$$\omega = -\frac{\sum_{i=0}^n q_i^2}{\sum_{i=0}^n p_i q_i}.$$

On the one hand, $\overrightarrow{OC}$ local vector can be written with $\overrightarrow{OP_0}$ local vector and $\mathbf{a}$ vector multiplies by a constant, on the other hand, with $\overrightarrow{OQ}$ local vector and $\mathbf{b}$ vector multiplied by an other constant. That is:

$$\overrightarrow{OC} = \overrightarrow{OP_0} + \phi\mathbf{a} = \overrightarrow{OP_0} + \phi\mathbf{p} + \phi\lambda\mathbf{q}, \tag{2.3}$$

$$\overrightarrow{OC} = \overrightarrow{OQ} + \psi\mathbf{b} = \overrightarrow{OQ} + \psi\mathbf{q} + \psi\omega\mathbf{p}. \tag{2.4}$$

We know, that

$$\overrightarrow{OQ} - \overrightarrow{OP_0} = \mathbf{p}.$$

Then in the $(\mathbf{p}, \mathbf{q})$ base are the value of ϕ and ψ can be calculated:

$$\phi = \frac{1}{\lambda\omega - 1}; \quad \psi = \frac{\lambda}{\lambda\omega - 1}. \tag{2.5}$$

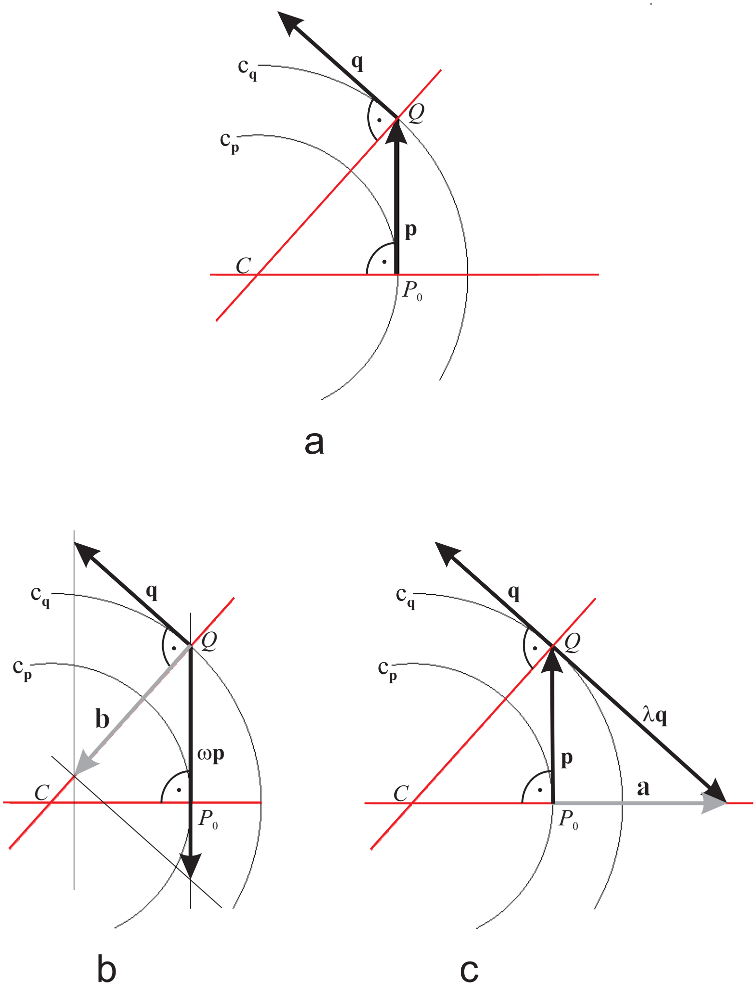

Figure 2: The c_p and c_q osculation circles in the plane of $\mathbf{p}$ and $\mathbf{q}$

Then coordinates of C point can also be calculated in both $(\mathbf{p}, \mathbf{q})$ base and $n+1$-dimensional vector place.

Knowing coordinates of C point we can define P_1 point, as a point of the line defined by C and Q points and of c_p arc, namely $\overrightarrow{CP_1}$ vector is parallel with $\overrightarrow{CQ}$ vector, $\left\|\overrightarrow{CP_1}\right\| = \left\|\overrightarrow{CP}\right\|$ and P_1 point is on the $\overline{CQ}$ half-line. So

$$\overrightarrow{OP_1} = \overrightarrow{OP_0} + \overrightarrow{P_0C} + \overrightarrow{CP_1},$$

where $\overrightarrow{OP_0}$ and $\overrightarrow{OP_1}$ are local vectors (see figure 1).

To determine the following approximate point, the starting point will be P_1 as it was P_0 earlier.

The promptness of the approximation depends on the selection of the value d. If it is too big, C will not be a good approximation of the common centre of the two osculating circles (c_p and c_q).

At the same time, if we find an appropriate C point then the distance of C and P_0 approximate the radius of the circle of curvature at P_1. This can be used to get a better defining of the value of d. If we can choose the value of d according to the characteristics of the curve we can approximate the function more precisely, and the algorithm will be faster.

To understand the operation of this method we only need the knowledge of graphic meaning of the differential quotient as the exact definition is not used in this case.

If we regard an ODES as a function which orders vector to the point of $n+1$-dimensional place, where the vector is parallel with the tangent line at the point then the point serial giving the solution can be written by the use of vector operation based on the method mentioned above (in the followings OCM–osculating circle method) which approximates the solution of initial value problem.

To give the algorithm we need the knowledge of the equation system and vector operations (such as scalar product, vector addition).

3. Look at the problem in case n = 2

Let the next initial value problem be given

$$\dot{x}_1(t) = f_1(t, x_1(t), x_2(t)),$$

$$\dot{x}_2(t) = f_i(t, x_1(t), x_2(t)),$$

$$P_0(p_0; x_1(p_0); x_2(p_0)).$$

The solution of initial value problem is the $\mathbf{x}(t) = (t, x_1(t), x_2(t))$ curve, which can be approximated with $P_0, P_1, \dots, P_k$ serial of points. The $\mathbf{m^p}(1, \dot{x}_1(p_0), \dot{x}_2(p_0))$ vector is parallel with tangent line of the curve at P_0 point. Then coordinates of $\mathbf{p}$ vector can be calculated on grounds (2.1).

If coordinates of Q point are $Q(q_0; q_1; \dots; q_n)$ then $\mathbf{m^q}(1, \dot{x}_1(q_0), \dot{x}_2(q_0))$ is the vector belonging to Q. From $\mathbf{m^q}$ coordinates of $\mathbf{q}$ vector are calculatable ground of (2.2). Value of λ and ω can be defined:

$$\lambda = -\frac{p_0^2 + p_1^2 + p_2^2}{p_1 q_1 + p_1 q_1 + p_2 q_2},$$

$$\omega = -\frac{q_0^2 + q_1^2 + q_2^2}{p_1 q_1 + p_1 q_1 + p_2 q_2}.$$

Ground of these considering (2.3), (2.4) and (2.5) coordinates of C point can be calculatable, from which coordinates of P_1 are also calculatable grounding of $\overrightarrow{CP_0}$, $\overrightarrow{CQ}$ vectors and

$$\overrightarrow{OP_1} = \overrightarrow{OP_0} + \overrightarrow{P_0C} + \overrightarrow{CP_1}$$

vector.

4. Examples

To illustrate the usefulness of algorithm we show two examples. Data for the figures of the examples were provided by a program, which was made on the base of above demonstrated algorithm. The initial values and parameters can be chosen randomly, the values in the examples provide the demonstration of working of algorithm. The aim was not to show the mathematical model.

4.1. Equation of damped oscillation

Generally:

$$x^{(2)}(t) + \frac{c}{m}x^{(1)}(t) + \omega^2 x(t) = 0,$$

where c, m and ω are constants characteristic of the system. We get the next equation system after transforming:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -\frac{c}{m}x_2(t) - \omega^2 x_1(t).\end{aligned}$$

Choose $c = 1$, $m = 2$ and $\omega = \pi$ in the example. Then

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -\frac{1}{2}x_2(t) - \pi^2 x_1(t).\end{aligned} \tag{4.1}$$

Define the approximated solution of equation system where the initial conditions are

$$\begin{aligned}x_1(0) &= 0.28\\ x_2(0) &= 0.28\end{aligned}$$

values by using OCM algorithm.

Compare our solution to numeric solution produced by Runge-Kutta4 method of Maple program. In both methods we have approximated the solution ($h = 350$).

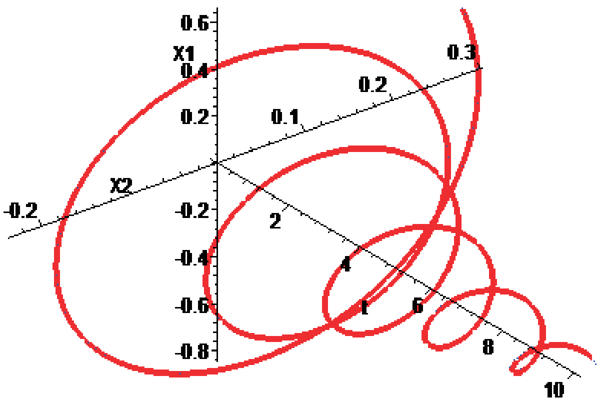

Figure 3: Curve of meaning the solution of (4.1) equation system.

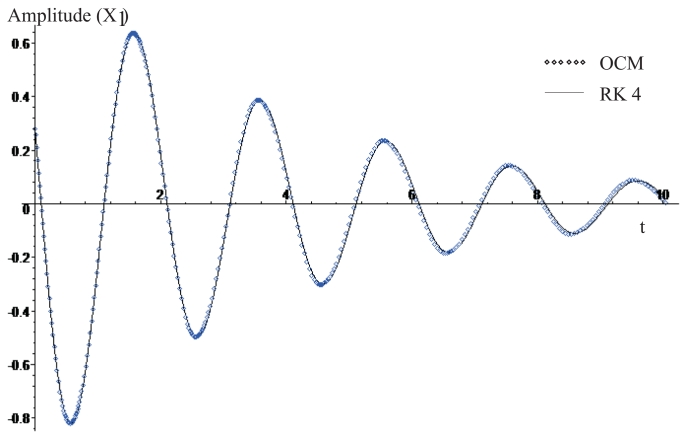

Figure 4: Deflection-time function.

4.2. Lotka-Volterra model

Lotka-Volterra equations are suitable for modelling various occurrences, systems for examples ecological systems, chemical processes. The model can be defined with next equations:

$$\dot{x}_1(t) = -ax_1(t) + bx_1(t)x_2(t) - mx_1^2(t),$$
$$\dot{x}_2(t) = cx_1(t) - dx_1(t)x_2(t) - lx_2^2(t).$$

The actual entity number of predatory is X_1, prey is X_2. a, b, c, d, m, l are constant characteristic of the system. In our example we examine the system $a = 2$; $b = 0.015$; $c = 1$; $d = 0.03$; $m = 0$; $l = 0.0005$:

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + 0.015x_1(t)x_2(t), \\ \dot{x}_2(t) &= x_1(t) - 0.03x_1(t)x_2(t) - 0.0005x_2^2(t). \end{aligned} \tag{4.2}$$

Initial condition is:

$$\begin{aligned} x_1(0) &= 150, \\ x_2(0) &= 50. \end{aligned}$$

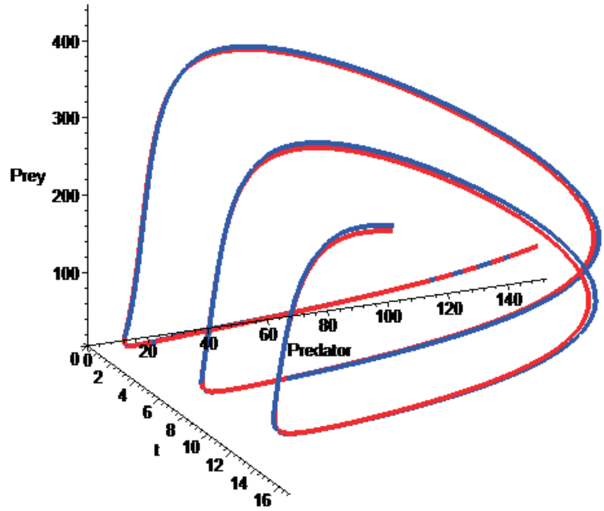

Figure 5: Curve meaning the solution of (4.2) equation system.

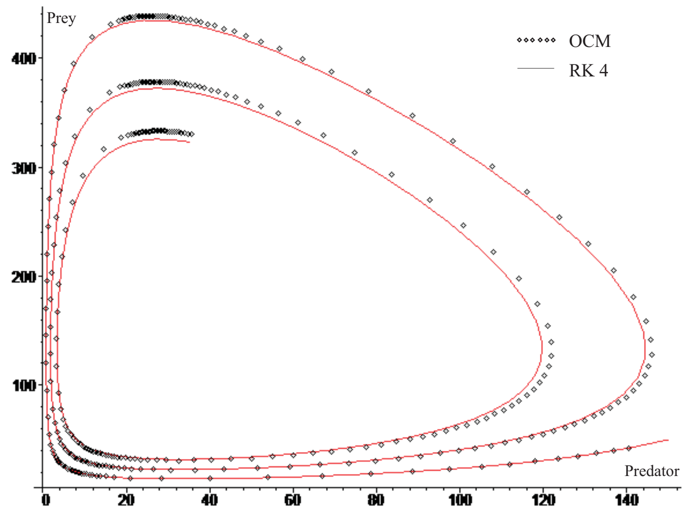

Figure 6: Trajectory of (4.2).

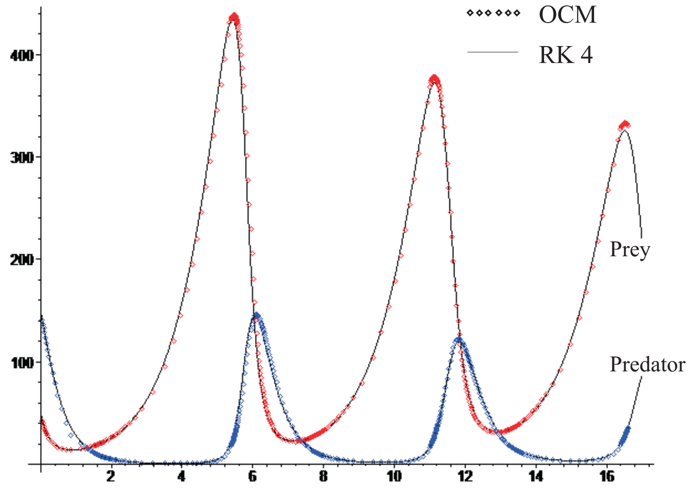

Figure 7: Change of entity number of predators and preys in time.

5. Conclusion

In case of various approximating method the solution can be approximated more precisely by increasing of the applied basis points. But this makes the method more difficult and needs more counting. Despite OCM, considers two basis points, the method provides comparatively great precision. It can be explained by approximating the curve on short periods with arcs. The curve piece, which is between any definite two basis points, can be approximated with an arc with suitable radius, in other words, any point on the curve piece between P_i, P_{i+1} points can be approximated with a suitable point of arch. Giving the solution by vector equation, despite we give in $n + 1$ dimensional vector space, can be easy. As we reduce the solution to a 2-dimensional case, we can avoid the solution of equation system, which has more equation than two.

References

[1] Geda, G.,Solving initial value problem by different numerical methods: Practical investigation, *Annales Mathematicae et Informaticae* (2005).

[2] Geda, G., Solving Initial Value Problem by Approximation in Different Graphic Ways, 5^{th} *Intrnational Conference of PhD Students, Miskolc* (2005).

[3] Arató, M. A Famous Nonlinear Stocshastic Equation (Lotka-Volterra Model with Diffusion), *Mathematical and Computer Modelling*, (2003).

[4] Geda, G. Investigation of Stability of Nonlinear Differential Equations with Stochastic Methods, *XXVI. Seminar on Stability Problems for Stochastic Models*, (2005).

[5] Stoyan, G., Tako, G., Numerikus módszerek, *ELTE-TypoTEX, Budapest*, (1993).

[6] Hatvani, L., Pintér, L., Differenciálegyenletes modellek a középiskolában, *POLYGON, Szeged*, (1997).

[7] Rontó, M., Raisz, Péterné., Differenciálegyenletek műszakiaknak, *Miskolci Egyetemi Kiadó, Miskolc*, (2004).

[8] Szőkefalvi-Nagy Gy., Gehér L., Nagy P., Differenciálgeometria, *Műszaki Könyvkiadó, Budapest,*, (1979).

[9] Geda, G., Kezdetiérték-probléma közelítő megoldásának egy geometriai szemléltetése, *Tavaszi Szél, Debrecen*, (2005).

[10] Pólya, Gy., Matematikai módszerek a természettudományban, *Gondolat, Budapest*, (1984).

Gábor Geda
Department of Computer Science
Eszterházy Károly College
Leányka str. 6.
H-3300 Eger, Hungary

Anikó Vágner
Department of Information Technology
Eszterházy Károly College
Leányka str. 6.
H-3300 Eger, Hungary