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Construction of ECT-B-splines, a survey*

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Abstract

s-dimensional *generalized polynomials* are linear combinations of functions forming an ECT-system on a compact interval with coefficients from $\mathbb{R}^s$. *ECT-spline curves in* $\mathbb{R}^s$ are constructed by glueing together at interval endpoints generalized polynomials generated from different local ECT-systems via connection matrices. If they are nonsingular, lower triangular and totally positive there is a basis of the space of 1-dimensional ECT-splines consisting of functions having minimal compact supports normalized to form a nonnegative partition of unity. Its functions are called *ECT-B-splines*. One way (which is semiconstructional) to prove existence of such a basis is based upon zero bounds for ECT-splines. A constructional proof is based upon a definition of ECT-B-splines by generalized divided differences extending Schoenberg's classical construction of ordinary polynomial B-splines. This fact eplains why ECT-B-splines share many properties with ordinary polynomial B-splines. In this paper we survey such constructional aspects of ECT-splines which in particular situations reduce to classical results.

Key Words: ECT-systems, ECT-B-splines, ECT-spline curves, de-Boor algorithm

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1. ECT-systems and their duals, rET- and lET-systems

Let J be a nontrivial compact subinterval of the real line $\mathbb{R}$. A system of functions $U = (u_1, \ldots, u_n)$ in $C^{n-1}(J;\mathbb{R})$ is called an *extended Tchebycheff system*

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(*ET-system*, for short) *of order* n *on* J provided for all $T = (t_1, \ldots, t_n)$, $t_1 \leq \ldots \leq t_n$, $t_j \in J$,

$$V \left| \begin{matrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{matrix} \right|_r := \det \ (D_r^{\nu_j} u_i(t_j)) \,|_{i,j=1,\ldots,n} > 0$$

with

$$\nu_j := \ \max\{l : t_j = t_{j-1} = \ldots = t_{j-l} \geq t_1\}, \quad j = 1, \ldots, n, \tag{1.1}$$

where $Df(x) := \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ denotes the operator of differentiation appropriately one sided at an endpoint of J. Then span U will be called an *ET-space of dimension* n *on* J. An ET-system $U = (u_1, \ldots, u_n)$ is called *complete* or an *ECT-system* provided $(u_1, \ldots, u_k)$ is an ET-system of order k on J for $k = 1, \ldots, n$.

The following characterization of ECT-systems is well known [8] p. 376f, [30] p. 364:

Theorem 1.1. *Let* $u_1, \ldots, u_n$ *be of class* $C^{n-1}(J; \mathbb{R})$. *Then the following assertions are equivalent:*

(i) $(u_1, \ldots, u_n)$ *is an ECT-system of order* n *on* J.

(ii) All Wronskian determinants

$$W(u_1, \ldots, u_k)(x) = det \ \left(D^{j-1} u_i(x)\right)_{i=1,\ldots,k}^{j=1,\ldots,k} > 0 \quad k = 1, \ldots, n; \ x \in J$$

are positive on J.

(iii) There exist positive weight functions $w_j \in C^{n-j}(J; \mathbb{R})$, $j = 1, \ldots, n$, *and for every* $c \in J$ *coefficients* $c_{j,i} \in \mathbb{R}$ *such that*

$$\begin{aligned} u_j(x) = w_1(x) \cdot \int_c^x w_2(t_2) \int_c^{t_2} w_3(t_3) \int_c^{t_3} \cdots \int_c^{t_{j-1}} w_j(t_j)\, dt_j \ldots dt_2 \\ + \sum_{i=1}^{j-1} c_{j,i} \cdot u_i(x), \quad j = 1, \ldots, n; \quad x \in J. \end{aligned} \tag{1.2}$$

Clearly, the functions $s_j(x, c) := u_j(x) - \sum_{i=1}^{j-1} c_{j,i} \cdot u_i(x) \quad j = 1, \ldots, n$ satisfy

$$s_j(x, c) = w_1(x) \cdot h_{j-1}(x, c; w_2, \ldots, w_j) \quad j = 1, \ldots, n \tag{1.3}$$

where $h_0(x, c) := 1$ and for $1 \leq m \leq n$

$$h_m(x, c; w_1, \ldots, w_m) := \int_c^x w_1(t) \cdot h_{m-1}(t, c; w_2, \ldots, w_m) dt.$$

The system (1.3) $(s_1, \ldots, s_n)$ forms a special basis of span$\{u_1, \ldots, u_n)$ which we call an ECT-system in *canonical form with respect to c*.

Example 1.2. If $w_j = \mathbf{1}$ for $j = 1, \dots, n$ where $\mathbf{1}$ denotes the constant function equal to one then

$$\begin{aligned} h_m(x, c; \mathbf{1}, \dots, \mathbf{1}) &= \frac{(x-c)^m}{m!} && m = 0, \dots, n \quad \text{and} \\ s_j(x, c) &= \frac{(x-c)^{j-1}}{(j-1)!} && j = 1, \dots, n \end{aligned}$$

and $\operatorname{span}\{s_1, \dots, s_n\} = \pi_{n-1}$, the space of ordinary polynomials of degree $n-1$ or of order n at most.

Example 1.3. (cf. also [3], [31]) If $n \geq 3$ and $w_j = \mathbf{1}$ for $j = 1, \dots, n-2$,

$$w_{n-1}(x) = \frac{(n-2)!}{(x-a+\varepsilon)^{n-1}}, \quad w_n(x) = \frac{(n-1)(b-a+2\varepsilon)(x-a+\varepsilon)^{n-2}}{(b+\varepsilon-x)^n}$$

with $\varepsilon > 0$ a parameter, then for any $c \in [a, b]$

$$s_j(x, c) = \frac{(x-c)^{j-1}}{(j-1)!}, \qquad j = 1, \dots, n-2 \tag{1.4}$$

$$s_{n-1}(x, c) = \frac{(x-c)^{n-2}}{(x-a+\varepsilon)(c-a+\varepsilon)^{n-2}} \tag{1.5}$$

$$s_n(x, c) = \frac{(x-c)^{n-1}(b-a+2\varepsilon)}{(x-a+\varepsilon)(b+\varepsilon-x)(b+\varepsilon-c)^{n-1}} \tag{1.6}$$

is a *Cauchy-Vandermonde-system* in canonical form with respet to c whose first $n-2$ functions are polynomials and the last two are proper rational functions, s_{n-1} having a pole of order 1 at $x = a - \varepsilon$ and s_n having poles of order 1 at $x = a - \varepsilon$ and at $x = b + \varepsilon$.

Associated with an ECT-system (1.2) or (1.3) are the linear differential operators

$$\begin{aligned} &D_0 u = u, \quad D_j u = D\left(\frac{u}{w_j}\right) && j = 1, \dots, n \\ &\hat{L}_j u = D_j \cdots D_0 u && j = 0, \dots, n \\ &L_j u = \frac{1}{w_{j+1}} \hat{L}_j u && j = 0, \dots, n-1. \end{aligned}$$

For $\mu \in \mathbb{N}$ by $\boldsymbol{L}[f](t) := (L_0 f(t), \dots, L_{\mu-1} f(t))^T$ we denote *the ECT-derivative vector of dimension* μ of a sufficiently smooth function f. Also, we will use the limits

$$\boldsymbol{L}^\mu[f](t-) := \lim_{\tau \to t-0} \boldsymbol{L}^\mu[f](\tau), \qquad \boldsymbol{L}^\mu[f](t+) := \lim_{\tau \to t+0} \boldsymbol{L}^\mu[f](\tau).$$

Obviously, $\ker \hat{L}_j = \operatorname{span}\{u_1, \dots, u_j\}, \quad j = 1, \dots, n, \quad$ and

$$L_j s_{j+1}(x, c) = 1 \qquad j = 0, \dots, n-1,$$

$$L_j s_{l+1}(c,c) = \delta_{j,l} \qquad j,l = 0,\dots,n-1.$$

There is a Taylor's Theorem with respect to ECT-systems. The initial value problem

$$\begin{aligned} \hat{L}_n u(x) &= f(x), && x \in J \\ L_j u(c) &= c_j, && j = 0,\dots,n-1, \end{aligned}$$

with $f \in C(J;\mathbb{R})$ and $c_j \in \mathbb{R}$ given, has the solution

$$u(x) = \sum_{j=0}^{n-1} c_j s_{j+1}(x,c) + \int_c^x f(t) s_n(x,t) dt. \tag{1.7}$$

Associated with any ECT-system $U = (s_j)_{j=1}^n$ of order n on J in canonical form with respect to $c \in J$ with weights $w_1,\dots,w_n$ its *dual canonical system* $U^* = (s_i^*)_{i=1}^n$ *with respect to* $c \in J$ is defined by

$$s_{j,n}^*(x,c) := h_{j-1}(x,c;w_n,\dots,w_{n+2-j}) \qquad j = 1,\dots,n. \tag{1.8}$$

It is again an ECT-system of order n on J with weights $(w_1^*,\dots,w_n^*) = (\mathbf{1}, w_n,\dots,w_2)$ provided

$$w_j \in C^{\max\{n-j,j-2\}}(J;\mathbb{R}), \qquad j = 2,\dots,n. \tag{1.9}$$

Assuming this, with the dual canonical ECT-system with respect to c associated are the linear differential operators

$$\begin{aligned} &D_0^* f = f, \quad D_1^* f = Df, \quad D_j^* f = D\left(\frac{f}{w_{n+2-j}}\right), && j = 2,\dots,n \\ &\hat{L}_j^* f = D_j^* \cdots D_0^* f, && j = 0,\dots,n, \\ &L_0^* f = f, \quad L_j^* f = \frac{1}{w_{n+1-j}} \hat{L}_j^* f, && j = 1,\dots,n. \end{aligned}$$

The function

$$g(x,y) := \begin{cases} w_1(x) h_{n-1}(x,y;w_2,\dots,w_n) & x \geq y \\ 0 & \text{otherwise} \end{cases}$$

has the characteristic behaviour of a Green's function for the differential operator L_{n-1} acting on the varable x, i.e.

$$\begin{aligned} &L_{n-1} g_n(x,y)|_{x=y-} = 0 \\ &L_{n-1} g_n(x,y)|_{x=y+} = L_{n-1} s_n(x,y)|_{x=y} = 1. \end{aligned}$$

In particular, for $x,y,c \in J$

$$h(x,y) := s_n(x,y) = w_1(x) h_{n-1}(x,y;w_2,\dots,w_n)$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} (-1)^{n-k} s_k(x,c) s^*_{n+1-k,n}(y,c) \qquad (1.10)\\
&= (-1)^{n-1} w_1(x) h_{n-1}(y,x;w_n,\dots,w_2)\\
&= (-1)^{n-1} w_1(x) s^*_{n,n}(y,x)
\end{aligned}
$$

where the right hand side of (1.10) is independent of c [10].

Example 1.1. (continued) If $w_1 = \ldots = w_n = \mathbf{1}$, then $s^*_j(x,c) = \frac{(x-c)^{j-1}}{(j-1)!}$, $j = 1,\dots,n$, and (1.10) reduces to the Binomial Theorem

$$
s_n(x,y) = h(x,y) = \frac{(x-y)^{n-1}}{(n-1)!} = \sum_{k=1}^{n} (-1)^{n-k} \frac{(x-c)^{k-1}}{(k-1)!} \cdot \frac{(y-c)^{n-k}}{(n-k)!}.
$$

Example 1.2. (continued, cf. [31]) If $n \geq 3$ and the weight functions are taken as in Example 1.2 then $(w^*_1,\dots,w^*_n) = (\mathbf{1}, w_n, w_{n-1},\dots,w_2)$, and if for any $c \in [a,b]$

$$
\begin{aligned}
\gamma(k,n,c) &:= \frac{(n-2)!}{(k-3)!}(c-a+\varepsilon)^{k-1-n} \sum_{\kappa=0}^{k-3} \binom{k-3}{\kappa} \frac{(-1)^{k-3-\kappa}}{n-2-\kappa}\\
\delta(k,n) &:= (b-a+2\varepsilon)\frac{(n-1)!}{(k-3)!}\\
\lambda(\nu,n) &:= (-1)^{n-1-\nu}\binom{n-1}{\nu}(b-a+2\varepsilon)^{\nu}, \qquad 1 \leq \nu \leq n-1\\
\mu(k,n,\nu,c) &:= \frac{1}{\nu}\binom{k-3}{n-\nu-1} \sum_{i=0}^{\nu+k-n-2} (-1)^{k-i} \binom{\nu+k-n-2}{i}\cdot\\
&\quad \cdot (b-a+2\varepsilon)^i \frac{(c-a+\varepsilon)^{\nu+k-n-2-i}}{\nu-1-i}, \qquad n-k-2 \leq \nu \leq n-1\\
\psi_\nu &:= \psi_\nu(x,b,c,\varepsilon) := \frac{1}{(b+\varepsilon-x)^{\nu}} - \frac{1}{(b+\varepsilon-c)^{\nu}}, \quad 1 \leq \nu \leq n-1
\end{aligned}
$$

then

$$
\begin{aligned}
s^*_{1,n}(x,c) &= \mathbf{1}\\
s^*_{2,n}(x,c) &= \sum_{\nu=1}^{n-1} \psi_\nu \cdot \alpha(2,n,\nu,c), \qquad \alpha(2,n,\nu,c) = \lambda(\nu,n) \qquad (1.11)
\end{aligned}
$$

and for $3 \leq k \leq n$

$$
s^*_{k,n}(x,c) = \sum_{\nu=1}^{n-1} \psi_\nu \cdot \alpha(k,n,\nu,c) \qquad (1.12)
$$

where

$$\alpha(k,n,\nu,c) = \begin{cases} \gamma(k,n,c)\lambda(\nu,n) & 1 \le \nu \le n-k+1 \\ \gamma(k,n,c)\lambda(\nu,n) + \delta(k,n)\mu(k,n,\nu,c) & n-k+2 \le \nu \le n-1. \end{cases}$$

The representations (1.11) and (1.12) are proved by calculating the integrals according to the definition of the dual system in its canonical form with respect to c. In example 1.2 according to (1.10)

$$h(x,y) = (-1)^{n-1} \cdot s^*_{n,n}(y,x).$$

Let J be a subinterval of the real line $\mathbb{R}$ that is open to the right. For $n \in \mathbb{N}_0$ let

$$C_r^n(J;\mathbb{R}) := \{ f \in C(J;\mathbb{R}) \ : \ \text{for every } x \in J \text{ and for } \nu = 1,\dots,n \text{ there exists the right derivative of } f \text{ of order } \nu \text{ at } x \text{ and } J \ni x \mapsto D_r^\nu f(x) \text{ is right continuous}\}.$$

A system of functions $U = (u_1,\dots,u_n)$ in $C_r^{n-1}(J;\mathbb{R})$ is called a *right-sided extended Tchebycheff system* (*rET-system*, for short) *of order* n *on* J provided for all $T = (t_1,\dots,t_n)$, $t_1 \le \dots \le t_n$, $t_j \in J$,

$$V \left| \begin{matrix} u_1,\dots,u_n \\ t_1,\dots,t_n \end{matrix} \right|_r := \det \ (D_r^{\nu_j} u_i(t_j)) \,|_{i,j=1,\dots,n} > 0$$

with ν_j defined by (1.1) where $D_r f(x) := \lim_{h\to 0+} \frac{f(x+h)-f(x)}{h}$ denotes the operator of ordinary right differentiation. Then span U will be called an *rET-space of dimension* n *on* J.

If $q \in$ span U where U is an rET-system of order n on J, a point $x_0 \in J$ is called a *zero of* q *of right multiplicity* ν_0 iff $q(x_0) = 0, D_r^1 q(x_0) = 0, \dots, D_r^{\nu_0 - 1} q(x_0) = 0, D_r^{\nu_0} q(x_0) \neq 0$.

The following characterization of rET-spaces is an immediate consequence of the Alternative Theorem of Linear Algebra, as is the corresponding well known characterization for ET-spaces (cf. [8], p. 376).

Theorem 1.4. *(i)* $(u_1,\dots,u_{n-1},u_n)$ *or* $(u_1,\dots,u_{n-1},-u_n)$ *is an rET-sytem of order* n *on* J.

(ii) Every nontrivial element of span$(u_1,\dots,u_n)$ *has at most* $n-1$ *zeros in* J *counting right multiplicities.*

(iii) Every problem of right sided Hermite interpolation

$$H(U,T+,f): \quad \begin{cases} \text{given points } t_1 \le \dots \le t_n \text{ in } J, \\ \text{given } f \in C_r^{n-1}(J;\mathbb{R}), \\ \text{find } q \in \text{ span } U \text{ such that} \\ D_r^{\nu_j} q(t_j) = D_r^{\nu_j} f(t_j) \qquad j = 1,\dots,n \end{cases} \tag{1.13}$$

has a unique solution.

Analogously, *left sided ET-systems* and *lET-spaces* and related concepts as the problem of *left sided Hermite interpolation* $H(U,T-,f)$ are defined. In the analysis of dual functionals to ECT-B-splines naturally certain rET-and lET-spaces arise (see (5.1) and (5.2) below) that are no ET-spaces.

If $U=(u_1,\ldots,u_n)$ is an rET-system on J then the leading coefficient (that before u_n) of the unique $q\in$ span U that solves $H(U,T+,f)$ is called the *right sided generalized divided difference of* f with respect to $u_1,\ldots,u_n$ and with nodes $t_1,\ldots,t_n$. By Cramer's rule it is

$$\begin{bmatrix} u_1,\ldots,u_n \\ t_1,\ldots,t_n \end{bmatrix}_r f = \frac{V\begin{vmatrix} u_1,\ldots,u_{n-1},f \\ t_1,\ldots,t_{n-1},t_n \end{vmatrix}_r}{V\begin{vmatrix} u_1,\ldots,u_{n-1},u_n \\ t_1,\ldots,t_{n-1},t_n \end{vmatrix}_r}.$$

Developing the numerator determinant along its last column one sees

$$\begin{bmatrix} u_1,\ldots,u_n \\ t_1,\ldots,t_n \end{bmatrix}_r f = \sum_{j=1}^{n} c_j\cdot D_r^{\nu_j} f(t_j),\qquad c_n = \frac{V\begin{vmatrix} u_1,\ldots,u_{n-1} \\ t_1,\ldots,t_{n-1} \end{vmatrix}_r}{V\begin{vmatrix} u_1,\ldots,u_{n-1},u_n \\ t_1,\ldots,t_{n-1},t_n \end{vmatrix}_r} \tag{1.14}$$

with coefficients c_j that do not depend on f.

For lET- or ET-systems we use similar notations with the suffix r replaced by l or omitted, respectively.

It is known [17] that if $(u_1,\ldots,u_{n+1})$, $(u_1,\ldots,u_n)$ are ECT-systems, and, if $n\geq 2$, also $(u_1,\ldots,u_{n-1})$ is an ECT-system, then if $t_1\neq t_{n+1}$

$$\begin{bmatrix} u_1,\ldots,u_{n+1} \\ t_1,\ldots,t_{n+1} \end{bmatrix} f = \frac{\begin{bmatrix} u_1,\ldots,u_n \\ t_2,\ldots,t_{n+1} \end{bmatrix} f - \begin{bmatrix} u_1,\ldots,u_n \\ t_1,\ldots,t_n \end{bmatrix} f}{\begin{bmatrix} u_1,\ldots,u_n \\ t_2,\ldots,t_{n+1} \end{bmatrix} u_{n+1} - \begin{bmatrix} u_1,\ldots,u_n \\ t_1,\ldots,t_n \end{bmatrix} u_{n+1}}.$$

This formula holds for the right or left sided generalized divided differences as well [25].

2. rECT-splines; the spaces $\mathcal{S}_n(\mathcal{U},\mathcal{A}^+,M,X)$ and $\mathcal{S}_n(\mathcal{U},\mathcal{A}^+,\boldsymbol{\xi}_{\mathbf{ext}})$

Assume that x is a real number and that in nontrivial closed intervals $J_0=[a,x]$ and $J_1=[x,b]$ left and right to x there are given two ECT-systems of order n

$$U^{[0]} := U_n^{[0]} := (u_1^{[0]},\ldots,u_n^{[0]}),\quad U^{[1]} := U_n^{[1]} := (u_1^{[1]},\ldots,u_n^{[1]}),$$

with weights $w_j^{[i]}$ $(j=1,\ldots,n;\ i=0,1)$ and associated linear differential operators $L_j^{[i]}$ $(j=0,\ldots,n-1;\ i=0,1)$, correspondingly. Suppose that μ is an integer, $0\leq\mu\leq n$, and that A is a square $(n-\mu)-$dimensional real matrix which is nonsingular. A function $s:[a,b]\mapsto\mathbb{R}$ such that $s|_{[a,x)}\in$ span $U^{[0]}$ and $s|_{[x,b]}\in$ span $U^{[1]}$ and

$$\boldsymbol{L}^{[1]n-\mu}[s](x+) = A\cdot\boldsymbol{L}^{[0]n-\mu}[s](x-) \tag{2.1}$$

where $\boldsymbol{L}^{[i]n-\mu}[s](t)$ $(i=0,1)$ denote the ECT-derivative vectors of s at t of dimension $n-\mu$ is called $(U^{[0]},U^{[1]},A)-$*smooth of order* $n-\mu$ *at* x. The equations (2.1) are called the *connection equations of* s *at the knot* x and A is called a *connection matrix* at x. We allow $0\le\mu\le n$ where in case $\mu=n$ there is no condition on s at x. In case $\mu=0$ the knot x is a knot with no freedom. If $1\le\mu\le n$ at x, given s on $[a,x)$, there are μ degrees of freedom in extending s to $[x,b]$ as a function belonging to span $U^{[1]}$ such that $s\in C_r^{n-1}([a,b];\mathbb{R})$. Symmetrically, if $1\le\mu\le n$ at x, given s on $(x,b]$, there are μ degrees of freedom in extending s to $[a,x]$ as a function belonging to span $U^{[0]}$ such that $s\in C_l^{n-1}([a,b];\mathbb{R})$.

It should be observed that $(U^{[0]},U^{[1]},A)-$smoothness in general does not imply smoothness in the ordinary sense. But it is not hard to give conditions that a function being $(U^{[0]},U^{[1]},A)-$smooth at x of order $n-\mu$ is smooth at x of order m in the usual sense [31].

Let $[a,b]\subset\mathbb{R}$ be either a nontrivial compact interval or the real line. By X we denote a finite or a bi-infinite partition of $[a,b]$ respectively, i.e.

$$X=\{x_0,\ldots,x_{k+1}\}\quad\text{with}\quad a=x_0<x_1<\ldots<x_{k+1}=b\quad\text{or}$$
$$X=(x_i)_{i\in\mathbb{Z}}\ \text{with}\ \ldots<x_{-1}<x_0<x_1<\ldots\ \text{and}\ \lim_{i\to\pm\infty}x_i=\pm\infty.$$

The points of X which are not endpoints are called *inner knots* and endpoints are called *auxiliary knots*. The index sets for inner knots are

$$K_X:=\begin{cases}\{1,\ldots,k\} & \text{if } X=\{x_0,\ldots,x_{k+1}\}\\ \mathbb{Z} & \text{if } X=(x_i)_{i\in\mathbb{Z}}.\end{cases}$$

In any case by $\Delta=(J_i)$, $J_i:=[x_i,x_{i+1})$ and $\check\Delta=(\check J_i)$, $\check J_i:=(x_i,x_{i+1}]$ for all i except the last resp. first we denote the corresponding partition of $[a,b]$ into subintervals called r- *resp.* l-*knot intervals* where in case of a finite partition of a compact interval the last $r-$ resp. first $l-$knot interval is $J_k:=[x_k,x_{k+1}]$ resp. $\check J_0=[x_0,x_1]$.

Assume that on each closed interval $\bar J_i=[x_i,x_{i+1}]$ the system

$$U_n^{[i]}=\left(u_1^{[i]},\ldots,u_n^{[i]}\right)\tag{2.2}$$

is an ECT-system of order n with associated weight functions

$$w_j^{[i]}\in C^{n-j}(\bar J_i;(0,\infty)),\quad j=1,\ldots,n\tag{2.3}$$

and associated linear differential operators $L_j^{[i]}$ and ECT-derivative vectors $\boldsymbol{L}^{[i]\mu}[f](t)=\left(L_0^{[i]}f(t),\ldots,L_{\mu-1}^{[i]}f(t)\right)^T$ of dimension μ.

By $\mathcal{U}=\mathcal{U}_n=\left(U^{[i]}\right)_i$ we denote the sequence of ECT-systems. Assume that corresponding to the inner knots we are given a sequence of integers $M=(\mu_i)$, $0\le\mu_i\le n$, and a sequence of nonsingular matrices

$$\mathcal{A}=\mathcal{A}_n=(A^{[i]}),\qquad A^{[i]}\in\mathbb{R}^{(n-\mu_i)\times(n-\mu_i)}.$$

A function $s : [a,b] \mapsto \mathbb{R}$ is called an *rECT-* resp. *lECT-spline function on* $[a,b]$ *with respect to the generating sequences* $\mathcal{U}, \mathcal{A}, M, X$ provided

$$\begin{aligned} &s|_{J_i} \in \text{ span } U^{[i]} \text{ resp. } s|_{\check{J}_i} \in \text{ span } U^{[i]} \quad \text{for all } i \text{ and} \\ &s \text{ is } (U^{[i-1]}, U^{[i]}, A^{[i]})\text{–smooth at } x_i \text{ for all inner knots.} \end{aligned} \tag{2.4}$$

The sets of all such functions will be denoted by $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ and $\check{\mathcal{S}}_n(\mathcal{U}, \mathcal{A}, M, X)$, respectively.

Clearly, every rECT-spline function is right continous everywhere and jumps may occur only at the knots. If all ECT-systems $U^{[i]}$ have the first weight function

$$w_1^{[i]}(x) = \mathbf{1} \qquad x \in \bar{J}_i, \quad \text{for all } i \tag{2.5}$$

and all connection matrices $A^{[i]}$ have the form

$$A^{[i]} = \text{ diag } (1, \bar{A}^{[i]}) \tag{2.6}$$

where $\mu_i \leq n-1$ and $\bar{A}^{[i]} \in \mathbb{R}^{(n-1-\mu_i)\times(n-1-\mu_i)}$ is nonsingular for all i then $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ and $\check{\mathcal{S}}_n(\mathcal{U}, \mathcal{A}, M, X) \subset C([a,b];\mathbb{R})$ and both spaces contain the constant functions.

In the sequel we shall treat rECT-spaces only. Clearly, every result for rECT-splines has an analogue for lECT-splines.

Under the assumptions (2.5), (2.6) and that $\mathcal{A} = \mathcal{A}^+ := (A^{[i]})_i$ where for every i the connection matrix

$$A^{[i]} \text{ is nonsingular, lower triangular, totally positive} \tag{2.7}$$

it is possible to construct for the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ a local support basis (N_j) that is normalized to form a nonnegative partition of unity. In order to give the definitions the following notation is usefull. For any partition $X = (x_i)$ of $[a,b]$, finite or biinfinite, with corresponding sequence of multiplicities of inner knots $M = (\mu_i)$ such that $1 \leq \mu_i \leq n$ for all i, we denote by $\boldsymbol{\xi}$ resp. by $\boldsymbol{\xi}_{\text{ext}}$ the weakly increasing sequence of inner resp. of all knots where auxiliary knots by definition have multiplicity n, each repeated according to its multiplicity, the enumeration being fixed by the convention $\xi_1 = \xi_2 = \ldots = \xi_{\mu_1} = x_1$. In this case we will also use the notation $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) = \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{\text{ext}})$.

By $\varphi : j \mapsto i_j$ we denote the mapping which assigns to each ξ_j the unique knot x_{i_j} such that $\xi_j = x_{i_j}$. Then

$$X = \varphi(\boldsymbol{\xi}_{\text{ext}}),\ M_{\text{ext}} := (\mu_i) \text{ with } \mu_i = \text{card } \varphi^{-1}(\{x_i\}).$$

It will be convenient to use the index set

$$J_\varphi = J_\varphi^n := \begin{cases} \{-n+1, \ldots, \mu\} & \text{if } [a,b] \text{ is compact} \\ \mathbb{Z} & \text{if } [a,b] = \mathbb{R}. \end{cases}$$

Observe that the sequences $\boldsymbol{\xi}$ or $\boldsymbol{\xi}_{\text{ext}}$ are well defined as nonvoid sequences of $\mu := \sum \mu_i$ terms also in case $0 \le \mu_i \le n$ for all i provided $1 \le \mu_1 \le n$. Only in case that all inner knots have multiplicities zero, $M = (0)_i$, we have $\boldsymbol{\xi} = (\ )$, a void sequence.

Remark 2.1. The space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ was introduced by Barry [1], p. 396. Barry has constructed de Boor-Fix functionals first and used them to derive existence of a local support basis for this space. ECT-splines are studied from a blossom point of view by Mazure [13],[14],[16] and Pottmann [15],[27] and more recently by Prautzsch [28], and from a constructive point of view by Mühlbach [23],[24]. Cardinal ECT-splines with simple knots are discussed in [31].

Remark 2.2. If $U^{[i]} = U_n|_{\bar{J}_i}$ where U_n is a fixed global ECT-system of order n on $[a,b]$ and $A^{[i]}$ is the $(n-\mu_i)-$dimensional identity matrix then $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ is the space of *Tchebycheff splines* of order n on $[a,b]$ with knots $x_1, \ldots, x_k$ of multiplicities $\mu_1, \ldots, \mu_k$, respectively.

Remark 2.3. If $U^{[i]} = (1, x, \ldots, x^{n-1})|_{\bar{J}_i}$ for $i = 0, \ldots, k$ then $\mathcal{S}_n = \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) = \mathcal{S}_n(x_1, \ldots, x_k | A^{[1]}, \ldots, A^{[k]})$ is the space of piecewise ordinary polynomials of order n generated by connection matrices $A^{[i]}$ considered by Dyn and Micchelli [4], p. 321, and by Barry et al [2]. If moreover each $A^{[i]}$ is an identity matrix then $\mathcal{S}_n$ is the well known Schoenberg space of ordinary polynomial spline functions of order n with knots x_i of multilicity μ_i, $i = 1, \ldots, k$.

According to the definitions given an rECT-spline $s \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ may be represented by

$$s = \sum_i \sum_{j=1}^{n} c_j^{[i]} \cdot u_j^{[i]}$$

meaning that

$$s|_{J_i} = \sum_{j=1}^{n} c_j^{[i]} \cdot u_j^{[i]}, \qquad \text{all } i$$

with coefficients $c_j^{[i]}$ $(j = 1, \ldots, n-\mu_i)$ that are related by the connection equations (2.4). There remain μ_i degrees of freedom for s right to x_i.

From this it is easily seen that with the usual pointwise defined algebraic operations $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ is a linear space over the reals whose dimension is

$$d = \dim \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) = n + \mu, \qquad \mu = \sum \mu_i \tag{2.8}$$

where the sum is extended over all inner knots. A basis generalizing the *truncated powers* is constructed as follows. It will be suficient to consider the case of a compact interval $[a,b]$. For $i = 0$ let $b_j(x)|_{J_0} := s_j^{[0]}(x,c)$, $j = 1, \ldots, n$, where

$s_1^{[0]}, \dots, s_n^{[0]}$ is the prescribed ECT-system on $\bar{J}_0$ in its canonical form with respect to a fixed point c with $x_0 \le c \le x_1$. Then extend b_j to J_1 such that the extension satisfies the connection equations (2.4) at x_1. Since $A^{[1]}$ is nonsingular there is a μ_1-parameter family of such extensions. Actually, in extending the basic functions b_j to the right for every knot x_i we choose in the connection equations (2.4) the connection matrix of the form

$$C^{[i]} := \text{diag }(A^{[i]}, I_{\mu_i}) \in \mathbb{R}^{n\times n} \tag{2.9}$$

where I_ν denotes the identity matrix of dimension ν requiring

$$L_l^{[i]} b_j(x_i+) = L_l^{[i-1]} b_j(x_i-), \quad l = n-\mu_i, \dots, n-1,\ i = 1, \dots, k.$$

If $1 \le i \le k$ and $j = n + \sum_{l=1}^{i-1} \mu_l + m$, $m = 1, \dots, \mu_i$, take $b_j(x)|_{J_i} = s_{n-\mu_i+m}^{[i]}(x, x_i)$ where $s_1^{[i]}, \dots, s_n^{[i]}$ is the ECT-system on $\bar{J}_i$ in its canonical form with respect to $c = x_i$, extend b_j to the left by zero and to the right across each knot x_p, $i+1 \le p \le k$, via the connection equations (2.4) with the connection matrices (2.9). By construction, the functions $b_1, \dots, b_d$ belong to $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ and they are linearly independent on $[a, b]$. Since their cardinality equals the dimension of $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ we have constructed a basis of this space.

3. A zero bound for splines in $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$

We use the zero counting convention due to Goodman [6]). In this section and in the rest of the paper we make the basic assumptions (2.5),(2.6) and (2.7) which ensure, in particular, $\mathbf{1} \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \subset C([a, b]; \mathbb{R})$.

Definition 3.1. Let $f \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ and $t \in (a, b)$. We set

$$f(t)^+ := \begin{cases} 1 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ is positive on } (t, t+\varepsilon) \\ 0 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ vanishes identically on } (t, t+\varepsilon) \\ -1 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ is negative on } (t, t+\varepsilon) \end{cases}$$

$$f(t)^- := \begin{cases} 1 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ is positive on } (t-\varepsilon, t) \\ 0 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ vanishes identically on } (t-\varepsilon, t) \\ -1 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ is negative on } (t-\varepsilon, t). \end{cases}$$

If f is not identically zero in some neighborhood of t then $f(t)^+ f(t)^- \ne 0$. In this case there exist nonnegative integers $l, r \le n-1$ such that

$$f(t-) = f'(t-) = \ldots = f^{(l-1)}(t-) = f(t+) = f'(t+) = \ldots = f^{(r-1)}(t+) = 0$$

and $f^{(l)}(t-) f^{(r)}(t+) \ne 0$. Let $q^* :=$ max (l, r). We say that f has a *point zero of multiplicity m at the point t* where

$$m = \begin{cases} q^* & \text{if } f(t)^- f(t)^+ (-1)^{q^*} > 0 \\ q^* + 1 & \text{if } f(t)^- f(t)^+ (-1)^{q^*} < 0 \end{cases}$$

As a consequence, $f(t)^+f(t)^- = (-1)^m$. If $x_0 \le \alpha < \beta \le x_{k+1}$ we set $k(\alpha,\beta) := \sum_{\alpha<x_l<\beta} \mu_l$.

Definition 3.2. Let $f \in \mathcal{S}_n(\mathcal{U},\mathcal{A}^+,M,X)$ and $a \le \alpha < \beta \le b$.

(i) If $f(\alpha)^-f(\beta)^+ \neq 0$ and $f(x) = 0$ for $\alpha < x < \beta$, then α and β are knots, $\alpha = x_p$ and $\beta = x_q$ with $0 < p < q < k+1$, and we say that f *has an interval zero* $[\alpha,\beta]$ *of multiplicity*

$$Z(f|[\alpha,\beta]) = n + 1 + k(\alpha,\beta).$$

(ii) If $f(x) = 0$ for all $a \le x < \beta$ while $f(\beta)^+ \neq 0$, then β is a knot, $\beta = x_q$ with $0 < q < k+1$, and we say that f *has an interval zero* $[a,\beta]$ *of multiplicity*

$$Z(f|[a,\beta]) = n + k(a,\beta).$$

(iii) If $f(x) = 0$ for all $\alpha < x \le b$ while $f(\alpha)^- \neq 0$, then α is a knot, $\alpha = x_p$ with $0 < p < k+1$, and we say that f *has an interval zero* $[\alpha,b]$ *of multiplicity*

$$Z(f|[\alpha,b]) = n + k(\alpha,b).$$

The total number of zeros of f in an interval J will be denoted by $Z(f|J)$.

Dyn and Micchelli [4], p. 324-327 have established a zero bound for polynomial splines via connection matrices as in remark 2.3 under the basic assumptions (2.6) and (2.7). A carefull examination of their proof shows that it can be adapted to the space $\mathcal{S}_n(\mathcal{U},\mathcal{A}^+,M,X)$. The reason is that also for ECT-spaces there holds a Budan-Fourier-Theorem [30], p. 371. From this as in [4] a Boudan-Fourier-Theorem for $\mathcal{S}_n(\mathcal{U},\mathcal{A}^+,M,X)$ can be derived (see theorem 3.3 of [25]), and this in turn yields the following

Theorem 3.3. *Let $f \in \mathcal{S}_n(\mathcal{U},\mathcal{A}^+,M,X)$ with $X = (x_0,\ldots,x_{k+1})$ being a partition of a compact interval $[a,b]$. Under the basic assumptions* (2.5),(2.6) *and* (2.7) *if f is not identically zero then*

$$Z(f|[x_0,x_{k+1}]) \le n - 1 + \mu, \qquad \mu = \sum_{i=1}^{k} \mu_i.$$

For the particular case that all multiplicities are zero also Barry [1] has given this bound.

Corollary 3.4. *If $f \in \mathcal{S}_n(\mathcal{U},\mathcal{A}^+,M,X)$ is not the zero function and vanishes identically on $([a,x_1)$ and on $(x_k,b])$ then*

$$Z(f|(x_1,x_k)) \le \max\{\mu - n - 1, 0\}.$$

It is the situation of corollary 3.4 that is needed for constructing a B-spline basis for the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ or rECT-splines. For the space of piecewise ordinary polynomials of order n via totally positive connection matrices, Dyn and Micchelli [4] have constructed such a basis. Again, a careful inspection of their proof shows that it carries over to rECT-splines yielding the following theorem.

Theorem 3.5. *Suppose that $n \geq 2$ and $[a,b] \subset \mathbb{R}$ is compact. Under the basic assumptions* (2.5),(2.6) *and* (2.7) *with $1 \leq \mu_i \leq n-1$ for $i = 1, \dots, k$, then there is a basis $(N_j^n)_{j=-n+1}^{\mu}$ of the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{ext})$ having the properties*

$$
\begin{aligned}
&N_j(x) := N_j^n(x) := N_j(x|\xi_j, \dots, \xi_{j+n}) && \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{ext})\\
&N_j(x) > 0 && x \in (\xi_j, \xi_{j+n})\\
&N_j(x) = 0 && x \notin [\xi_j, \xi_{j+n}]\\
&N_j^{(l)}(\xi_j+) = 0 \quad \text{for } l = 0, \dots, n-1-\mu_j^+, && D_+^{n-\mu_j^+} N_j(\xi_j) > 0,\\
&N_j^{(l)}(\xi_{j+n}-) = 0 \quad \text{for } l = 0, \dots, n-1-\mu_{j+n}^-, && D_-^{n-\mu_{j+n}^-} N_j(\xi_{j+n}) < 0,\\
&\sum_{j=-n+1}^{\mu} N_j(x) = 1 && x \in [a,b].
\end{aligned}
$$

Here $\mu_j^{\pm} := \#\{l \geq 0 : \xi_j = \xi_{j \pm l}\}$ denote the right and left multiplicities of a knot ξ_j in the sequence $(\xi_l)_{l=-n+1}^{\mu+n}$.

Another proof of theorem 3.5 based upon right sided generalized divided differences can be found in [24]. It should be remarked that for arbitrary knot sequences total positivity of the connection matrices is a sufficient condition to ensure existence of a local support basis forming a nonnegative partition of unity. As shown by Mazure [14] it is not necessary. It is an open problem to give conditions which are necessary and sufficient for existence of such a basis. Given arbitrary nonsingular connection matrices, for Chebycheff splines local support bases forming a partition of unity exist for knot sequences which are dense in the set of all possible knot sequences, as is shown recently by Prautzsch [28].

4. Interpolation properties of the spline spaces $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$

Consider the spline space $\mathcal{S}_n = \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ with $X = (x_i)_{i=0}^{k+1}$ a partition of a compact interval $[a,b]$. Assume that there are given d *nodes* or *interpolation points* y_j,

$$
Y = (y_1, \dots, y_d) \quad \text{where} \quad x_0 \leq y_1 \leq y_2 \leq \dots \leq y_d \leq x_{k+1}. \tag{4.1}
$$

Here d denotes the dimension (2.8) of the space $\mathcal{S}_n$. Since its elements are continuous functions that are piecewise generalized polynomials of order n we assume that

$$\nu_j := \max\,\{l \geq 0 : y_j = y_{j-1} = \ldots = y_{j-l}\} \leq n-1, \quad j = 1, \ldots, d, \tag{4.2}$$

i.e. each node has multiplicity not greater than n.

For every node y_j there is a unique integer h such that

$$y_j = x_h, \quad h \in \{0, \ldots, k+1\} \quad \text{or} \quad y_j \in \mathrm{int} J_h, \quad h \in \{0, \ldots, k\}.$$

When $y_j = x_h$ with $h \in \{1, \ldots, k\}$ we suppose that

$$\nu_j + \mu_h \leq n-1, \qquad j = 1, \ldots, d. \tag{4.3}$$

This condition is called the *accumulation condition.* It allows that nodes are knots. Only finite endpoints or knots of multiplicity 0 may be nodes of multiplicity n. If a node y_j equals an inner knot x_h whose multiplicity μ_h is not zero then the accumulation condition guarantees that for every $f \in \mathcal{S}_n$ the rECT-derivative of highest order $L^{[h]}_{\nu_j} f(y_j+)$ does exist. Then also $D_+^{\nu_j} f(y_j+)$ exists.

We consider the problem $H(\mathcal{S}_n, Y_+, f)$ of *right sided Hermite interpolation* (cf. (1.13))

$$H(\mathcal{S}_n, Y_+, f): \quad \begin{cases} \text{given } y_1 \leq \ldots \leq y_d, \quad y_j \in [a,b], \\ \text{given } f \in C_+^N([a,b];\mathbb{R}) \text{ with } N = \max_{j=1,\ldots,d} \nu_j, \\ \text{find } s \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \text{ such that} \\ D_+^{\nu_j} s(y_j) = D_+^{\nu_j} f(y_j), \ j = 1, \ldots, d. \end{cases}$$

The following theorem gives conditions which are necessary and sufficient for this problem to have a unique solution.

Theorem 4.1. *We make the assumptions* (2.5),(2.6),(2.7),(4.1),(4.2) *and* (4.3). *Then the following assertions are equivalent*

(i) $H(\mathcal{S}_n, Y+, f)$ *has a unique solution for every admissible function* f.

(ii)

$$y_i < \xi_i < y_{i+n} \qquad i = 1, \ldots, \mu, \quad \mu := \sum_{l=1}^{k} \mu_l.$$

(iii)

$$y_i \in M_i \qquad i = 1, \ldots, d,$$

where

$$M_i := \begin{cases} [x_0, \xi_i) & i = 1, \ldots, n \\ (\xi_{i-n}, \xi_i) & i = n+1, \ldots, d-n \\ (\xi_{i-n}, x_{k+1}] & i = d-n+1, \ldots, d. \end{cases}$$

The conditions (ii) and (iii) are called the *mixing conditions of the first* resp. *second kind.*

Theorem 4.1 generalizes in part the interpolation theorems of Schoenberg and Whitney [29] for ordinary polynomial splines with simple knots, of Karlin and Ziegler [9] for Chebycheffian splines with multiple knots and an interpolation theorem of Dyn and Micchelli [4] for polynomial splines via totally positive connection matrices. It is consistent with theorems 4.67 and 9.33 of Schumaker [30] on right sided Hermite interpolation by ordinary polynomial splines or by Tchebycheffian splines, respectively, since all our interpolation functions are continuous and the nodes satisfy the conditions (4.2) and (4.3). For the same reasons it is also consistent with the particular case $q = d$ of the more general result of Lyche and Schumaker [11] on modified Hermite interpolation by LB-splines.

In case $M = (0)$ when all inner knots have multiplicity zero the mixing conditions of both kinds are void. We then have

Corollary 4.2. *Under the assumptions of theorem 4.1 the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, (0), X)$ is an rET-space of order n consisting of continuous functions. It has a basis $p_1, \ldots, p_n$ such that*

$$V \begin{vmatrix} p_1, \ldots, p_n \\ y_1, \ldots, y_n \end{vmatrix}_+ := det \ \left(D_+^{\nu_j} p_i(y_j)\right) > 0$$

for all $y_1 \leq \ldots \leq y_n$ in $[a, b]$.

Corollary 4.3. *Under the assumptions of theorem 4.1 for all systems of nodes $y_1 \leq \ldots \leq y_d$ in $[a, b]$*

$$V := V \begin{vmatrix} b_1, \ldots, b_d \\ y_1, \ldots, y_d \end{vmatrix}_+ \geq 0$$

with strict inequality iff the mixing conditions hold. Here $b_1, \ldots, b_d$ is the basis of generalized truncated powers constructed in section 2.

Corollary 4.4. *Under the assumptions of theorem 4.1 and of theorem 3.2 we have*

$$V \begin{vmatrix} N_{-n+1}, \ldots, N_{d-n} \\ y_1, \ldots, y_d \end{vmatrix}_+ \geq 0 \tag{4.4}$$

with strict inequality iff the mixing conditions hold. Here $(N_j)_{j=-n+1}^{d-n}$ is the basis of theorem 3.5.

It is an open problem if the generalized Vandermonde matrix of (4.4) for simple nodes is totally positive as in the particular case of ordinary polynomial B-splines.

Corollary 4.5. *Under the assumptions of theorem 4.1 and assuming that each connection matrix can be partitioned according to $A^{[i]} = diag(1, A_1^{[i]}, A_2^{[i]})$ where $A_1^{[i]}$*

and $A_2^{[i]}$ *are square matrices of dimensions* $n-m-1$ *and* m*, respectively, that both satisfy* (2.7)*, then the space* $\mathcal{S}_n(\mathcal{U},\mathcal{A}^+,(0),X)$ *is an rET-space of order* n *that has an rET-subspace of order* $n-m$*. There is a basis* $p_1,\ldots,p_n$ *of* $\mathcal{S}_n(\mathcal{U},\mathcal{A}^+,(0),X)$ *such that*

$$V\left|\begin{matrix} p_1,\ldots,p_{n-m} \\ y_1,\ldots,y_{n-m}\end{matrix}\right|_r := det\ (D_r^{\nu_j}p_l(y_j)) > 0$$

for all $y_1 \le \ldots \le y_{n-m}$ *in* $[a,b]$*.*

Corollary 4.6. *Under the assumptions of corollary 4.5 every nontrivial* $f \in$ *span* $\{p_1,....,p_{n-m}\} \subset \mathcal{S}_n(\mathcal{U},\mathcal{A}^+,(0),X)$ *has at most* $n-m-1$ *zeros in* $[a,b]$*.*

Corollary 4.7. *Assuming* (2.5) *and that each connection matrix* $A^{[i]}$ *is a nonsingular positive diagonal matrix with* $a_{11}^{[i]}=1$*, then the space* $\mathcal{S}_n(\mathcal{U},\mathcal{A}^+,(0),X)$ *is an rECT-space of order* n*, i.e. this space has a basis* $p_1,\ldots,p_n$ *such that for* $m=1,\ldots,n$

$$V\left|\begin{matrix} p_1,\ldots,p_m \\ y_1,\ldots,y_m\end{matrix}\right|_r := det\ (D_r^{\nu_j}p_l(y_j)) > 0$$

for all $y_1 \le \ldots \le y_m$ *in* $[a,b]$*.*

In the situation of corollary 4.7 every interpolation problem $H(\mathcal{S}_n, Y+, f)$ can be solved recursively either using Newton's method via generalized divided differences [17], [18], [21], or using the generalized Neville-Aitken algorithm [19], [20]. This proves particular usefull in computing the spline weights recursively that occur in the recurrence relation for rECT-B-splines (see (7.5) below).

5. Pólya-polynomials and Marsden's identity generalized to ECT-splines

As in the preceding sections we adopt the general assumptions (2.5), (2.6) and (2.7). Let $\mathcal{S}_n(\mathcal{U},\mathcal{A}^+,\boldsymbol{\xi}_{\text{ext}}) = \mathcal{S}_n(\mathcal{U},\mathcal{A}^+,M,X)$ be an rECT-spline space as in section 2. Assuming for the weights of every local ECT-system (1.9) we set

$$\mathcal{C}_{\mathcal{A}^+} := (C^{[i]})_{i\in K_X} \qquad \text{with} \qquad C^{[i]} = \text{diag}(A^{[i]}, I_{\mu_i})$$
$$\mathcal{E}_{\mathcal{A}^+} := (E^{[i]})_{i\in K_X} \qquad \text{with} \qquad E^{[i]T} := R^{-1}\left(C^{[i]}\right)^{-1} R$$

where $R = R_n$ is the $n-$dimensional orthogonal matrix defined by

$$R^T := \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & (-1) & 0 \\ 0 & 0 & \ldots & (-1)^2 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ (-1)^{n-1} & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}.$$

We use the spaces

$$\mathcal{P}_n := \mathcal{P}_n(\mathcal{U}, \mathcal{C}_{\mathcal{A}^+}, X) := \mathcal{S}_n(\mathcal{U}, \mathcal{C}_{\mathcal{A}^+}, (0), X) := \{f : f \in C_r^{n-1}([a,b];\mathbb{R}), \tag{5.1}$$
$$f|_{J_i} \in \text{ span } U^{[i]},\ f \text{ is } (U^{[i-1]}, U^{[i]}, C^{[i]}) - \text{smooth at } x_i,\ i \in K_X\} \quad \text{and}$$
$$\mathcal{P}_n^* := \mathcal{P}_n(\mathcal{U}^*, \mathcal{E}_{\mathcal{A}^+}, X) := \check{\mathcal{S}}_n(\mathcal{U}^*, \mathcal{E}_{\mathcal{A}^+}, (0), X) := \{f : f \in C_l^{n-1}([a,b];\mathbb{R}), \tag{5.2}$$
$$f|_{\check{J}_i} \in \text{ span } U^{[i]^*},\ f \text{ is } (U^{[i-1]^*}, U^{[i]^*}, E^{[i]}) - \text{smooth at } x_i,\ i \in K_X\}.$$

Clearly, $\mathcal{P}_n \subset \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$. From the assumption (2.6) it follows that

$$E^{[i]} = \text{ diag } (I_{\mu_i}, \bar{E}^{[i]}_{n-1-\mu_i}, 1), \quad \bar{E}^{[i]}_{n-1-\mu_i} = R^{-1}_{n-1-\mu_i} \left(\bar{A}^{[i]}\right)^{-T} R_{n-1-\mu_i}. \tag{5.3}$$

is a n-dimensional square matrix which satisfies (2.7).

Later we will also use spaces

$$\mathcal{P}_{n+1} := \mathcal{S}_{n+1}(\hat{\mathcal{U}}, \hat{\mathcal{C}}_{\mathcal{A}^+}, (0), X)$$

having $\mathcal{P}_n$ as a subspace and

$$\mathcal{P}^*_{n+1} := \check{\mathcal{S}}_{n+1}(\hat{\mathcal{U}}^*, \hat{\mathcal{E}}_{\mathcal{A}^+}, (0), X)$$

having $\mathcal{P}^*_n$ as a subspace. They are defined by the extensions

$$\hat{\mathcal{U}} = \left(\hat{U}^{[i]}\right)_{i \in K_X},\ \hat{\mathcal{C}}_{\mathcal{A}^+} = \left(\hat{C}^{[i]}\right)_{i \in K_X},\ \hat{\mathcal{U}}^* = \left(\hat{U}^{[i]*}\right)_{i \in K_X},\ \hat{\mathcal{E}}_{\mathcal{A}^+} = \left(\hat{E}^{[i]}\right)_{i \in K_X}$$

where $\hat{U}^{[i]} = (u_1^{[i]}, \ldots, u_n^{[i]}, u_{n+1}^{[i]})$ is an ECT-system generated by the weights $(w_1^{[i]}, \ldots, w_{n+1}^{[i]}) = (\mathbf{1}, w_2^{[i]}, \ldots, w_n^{[i]}, \hat{w}_{n+1}^{[i]})$ and $\hat{U}^{[i]*} = (u_1^{[i]*}, \ldots, u_n^{[i]*}, u_{n+1}^{[i]*})$ is an ECT-system generated by the weights $(w_1^{[i]*}, \ldots\ldots, w_{n+1}^{[i]*}) = (\mathbf{1}, w_n^{[i]}, \ldots, w_2^{[i]}$ $, w_{n+1}^{[i]*})$. Here $0 < \hat{w}_{n+1}^{[i]} \in C^0(\bar{J}_i;\mathbb{R})$ and $0 < w_{n+1}^{[i]*} \in C^0(\bar{J}_i;\mathbb{R})$ may be chosen arbitrarily, where now we assume that

$$w_j^{[i]} \in C^{\max\{n+1-j, j-1\}}(J_i;\mathbb{R}), \qquad j = 2, \ldots, n.$$

The connection matrices for $\mathcal{P}_n$ resp. for $\mathcal{P}^*_{n+1}$ for every $i \in K_X$ are defined by $\hat{C}^{[i]} :=$ diag $(C^{[i]}, 1)$ resp. $\hat{E}^{[i]} := \text{diag}(E^{[i]}, 1)$. Here $\hat{E}^{[i]} = \text{diag}(I_{\mu_i}, \overline{E}^{[i]}_{n-1-\mu_i}, 1, 1)$ if $A^{[i]}$ may be partitioned as in (2.6).

According to corollary 4.2 $\mathcal{P}_n$ resp. $\mathcal{P}^*_n$ is an rET- resp. lET-space of order n each and $\mathcal{P}^*_{n+1}$ is an lET-space of order $n+1$. By corollary 4.5 the space $\mathcal{P}^*_{n+1}$ has a basis

$$q_1, \ldots, q_{n+1} \tag{5.4}$$

such that for $\nu = 0, 1, 2$ the system $q_1, \ldots, q_{n-1+\nu}$ is an lET-sytem of order $n-1+\nu$. Such a basis is obtained by fixing in any knot interval $\bar{J}_i$ the local ECT-system (1.8) in canonical form with respect to any $c \in [x_i, x_{i+1}]$ $(s^{[i]*}_{1,n+1}(x,c), \ldots, s^{[i]*}_{n+1,n+1}(x,c))$,

$x \in (x_i, x_{i+1}]$, and extending these functions by the connection equations of $\mathcal{P}^*_{n+1}$ to the left and right of J_i. Since $s^{[i]*}_{j,n} = s^{[i]*}_{j,n+1}, \quad j = 1, \dots, n$, the basis $q_1, \dots, q_{n+1}$ of $\mathcal{P}^*_{n+1}$ constructed this way under the hypothesis (2.6) indeed yields in the sections $q_1, \dots, q_{n-1}$ and $q_1, \dots, q_n$ lET-systems of orders $n-1$ and n, respectively.

Generalized Pólya polynomials are defined for $j \in J_\varphi$ by

$$M_j(y) = M^n_j(x) = \\ = M_j(y|\xi_{j+1}, \dots, \xi_{j+n-1}) = (-1)^{n-1} rq_n \begin{bmatrix} q_1, \dots, q_{n-1} \\ \xi_{j+1}, \dots, \xi_{j+n-1} \end{bmatrix}_l (y),$$

denoting by $rf \begin{bmatrix} u_1,\dots,u_n \\ t_1,\dots,t_n \end{bmatrix}_l (y) := f(y) - pf \begin{bmatrix} u_1,\dots,u_n \\ t_1,\dots,t_n \end{bmatrix}_l (y)$ the interpolation remainder where $pf \begin{bmatrix} u_1,\dots,u_n \\ t_1,\dots,t_n \end{bmatrix}_l (y)$ is the solution of the Hermite interpolation problem $H(U, T-, f)$.

$M_j \in \mathcal{P}^*_n$ has exactly $n-1$ zeros $\xi_{j+1}, \dots, \xi_{j+n-1}$, counting left multiplicities, and no other zeros, and M_j has leading coefficient $(-1)^{n-1}$ in every interval $\check{J}_i$. Therefore M_j is positive for $x < \xi_{j+1}$. It is not hard to show that every n consecutive generalized Pólya polynomials $(M_j)^{l+n-1}_{j=l}$, $l \in J_\varphi$, form a basis of $\operatorname{span}\{q_1, \dots, q_n\}$.

Barry [1] has constructed *de Boor-Fix functionals*

$$\Lambda_j(x)[f] := \sum_{p=0}^{n-1} (-1)^{n-1-p} L^{[r]}_p f(x) \cdot L^{[r]*}_{n-1-p} M_j(x),\ x \in J_r,\ \xi_j < x < \xi_{j+n}.$$

Actually, it is easily derived from (1.10) that under our general assumptions (2.5), (2.6) and (2.7) for every $j \in J_\varphi$ and $f \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{\text{ext}})$ the function

$$x \mapsto \Lambda_j(x)[f]$$

is a constant function of $x \in (\xi_j, \xi_{j+n})$. As a consequence,

$$\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \ni f \mapsto \Lambda_j(f) := \Lambda_j(x)[f], \quad x \in (\xi_j, \xi_{j+n})$$

is a well defined linear functional for every $j \in J_\varphi$. It follows that they are dual functionals for the rECT-B-spline basis (N_j) of order n. This is due to Barry [1], cf. also [24].

Theorem 5.1.

$$\Lambda_j(N_i) = \delta_{i,j} \qquad i, j \in J_\varphi.$$

The following theorem will be basic both for the definition of the rECT-B-splines in terms of generalized divided differences and for their general recursion relation. It's proof is a straightforward extension of the proof due to Dyn and Micchelli [4] of the similar result for polynomial splines via connection matrices.

Theorem 5.2. *Under the assumptions* (2.5)*,* (2.6)*,* (2.7) *there exists a unique function* $[a,b] \times [a,b] \ni (x,y) \mapsto h(x,y)$ *such that*

(i) for each $y \in [a,b]$ $\quad h(\,\cdot\,,y) \in \mathcal{P}_n$*,*

(ii) for each $x \in [a,b]$ $\quad h(x,\,\cdot\,) \in \mathcal{P}_n^*$*.*

(iii) Whenever for some $l \in K_X$ $\;x \in J_l$ *and* $y \in \check{J}_l$ *then*

$$\begin{aligned} h(x,y) &= w_1^{[l]}(x) h_{n-1}(x,y;w_2^{[l]},\dots,w_n^{[l]}) = s_n^{[l]}(x,y) \\ &= \sum_{k=1}^{n} (-1)^{n-k} s_k^{[l]}(x,c) \cdot s_{n+1-k,n}^{[l]*}(y,c) \\ &= w_1^{[l]}(x) \sum_{k=1}^{n} h_{k-1}(x,c;w_2^{[l]},\dots,w_k^{[l]}) h_{n-k}(y,c;w_{k+1}^{[l]},\dots,w_n^{[l]}) \end{aligned}$$

with $c \in \bar{J}_l$ *arbitrary.*

(iv) For $i \in K_X$ *fixed,* $c \in \bar{J}_i$ *and* $j = 1,\dots,n$ *let* $p_j(\,\cdot\,,c) \in \mathcal{P}_n$ *be defined by*

$$p_j(x,c) = w_1^{[i]}(x) h_{j-1}(x,c;w_2^{[i]},\dots,w_j^{[i]}) = s_j^{[i]}(x,c), \; x \in J_i,$$

and for $i \in K_X$ *fixed,* $c \in \bar{J}_i$ *and* $j = 1,\dots,n$ *let* $q_j(\,\cdot\,,c) \in \mathcal{P}_n^*$ *be defined by*

$$q_j(y,c) = h_{j-1}(y,c;w_n^{[i]},\dots,w_{n+2-j}^{[i]}) = s_{j,n}^{[i]*}(y,c), \; y \in \check{J}_i. \tag{5.5}$$

Then the function h *has the representation*

$$h(x,y) = \sum_{k=1}^{n} (-1)^{n-k} p_k(x,c) q_{n+1-k}(y,c), \quad (x,y) \in [a,b] \times [a,b], \; c \in \bar{J}_i \tag{5.6}$$

where the right hand side is independent of i *and of* $c \in \bar{J}_i$*.*

The function h will be called *generating function* of rECT-B-splines for reasons that will become clear soon.

Remark 5.3. If $U^{[i]} = (\mathbf{1}, x, \dots, x^{n-1})|_{J_i}$ for all i then $h(x,y) = \frac{(x-y)^{n-1}}{(n-1)!}$ whenever $x \in J_l$ and $y \in \check{J}_l$ for some l and (5.6) reduces to formula (3.67) of [4]. If moreover $A^{[i]} = I_{n-\mu_i}$ for all i then (5.6) reduces to the binomial theorem

$$\frac{(x-y)^{n-1}}{(n-1)!} = \sum_{k=1}^{n} (-1)^{n-k} \frac{(x-c)^{k-1}}{(k-1)!} \frac{(y-c)^{n-k}}{(n-k)!}.$$

When $U^{[i]} = U|_{J_i}$ where $U = (u_1,\dots,u_n)$ is an ECT-system on $[a,b]$ and for all $i \in \mathbb{Z}$ $A^{[i]} = I_{n-\mu_i}$ is an identity matrix then (5.6) reduces to Marsden's identity for Tchebycheff splines [30] p. 382.

The next theorem is a generalization of Marsden's identity to rECT-B-splines.

Theorem 5.4. *Under the assumptions* (2.5), (2.6) *and* (2.7) *for the function h of theorem 5.2 there holds*

$$h(x,y) = \sum_{i \in J_\varphi} N_i(x) M_i(y) \qquad \textit{for all} \quad (x,y) \in [a,b] \times [a,b].$$

Remark 5.5. Theorems 5.1 and 5.4 are equivalent in the sense that each is a consequence of the other [24].

6. rECT-B-splines defined by generalized divided differences

Let

$$g(x,y) := \begin{cases} h(x,y) & x \geq y \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

where h is the function of theorem 3.5 By the properties of h, with y fixed, as a function of x, $g(x,y)$ belongs piecewise, for $x \geq y$ and for $x < y$, to $\mathcal{P}_n \subset \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$. If x is fixed, as a function of y, $g(x,y)$ belongs piecewise, for $x \geq y$ and for $x < y$, to $\mathcal{P}_n^*$. The function g is separately, with respect to x, in $C_r^{n-1}([a,b];\mathbb{R})$, and with respect to y, in $C_l^{n-1}([a,b];\mathbb{R})$, since h has this property, and the $(n-1)$st ECT-derivative of g with respect to x at $x = y$ has the characteristic jump discontinuity of a Green's function: for every i

$$\lim_{x \to y-} L_\nu^{[i]} g(x,y)| = 0 \qquad \nu = 0, \ldots, n-1, \quad \text{if } x_i < y \leq x_{i+1}$$

$$\lim_{x \to y+} L_\nu^{[i]} g(x,y)| = \begin{cases} 0 & \nu = 0, \ldots, n-2 \\ 1 & \nu = n-1 \quad \text{if } x_i \leq y < x_{i+1}. \end{cases}$$

We call (6.1) the *Green's function with respect to the spaces* $\mathcal{P}_n(\mathcal{U}, \mathcal{C}_{\mathcal{A}^+}, X)$ and $\mathcal{P}_n(\mathcal{U}^*, \mathcal{E}_{\mathcal{A}^+}, X)$.

Definition 6.1. For $j \in J_\varphi$ and $x \in [a,b]$

$$\begin{aligned} \tilde{N}_j^n(x) &:= \tilde{N}(x|\xi_j, \ldots, \xi_{j+n}) \\ &:= (-1)^n \left(\begin{bmatrix} q_1, \ldots, q_n \\ \xi_{j+1}, \ldots, \xi_{j+n} \end{bmatrix}_l g(x, \cdot\,) - \begin{bmatrix} q_1, \ldots, q_n \\ \xi_j, \ldots, \xi_{j+n-1} \end{bmatrix}_l g(x, \cdot\,) \right). \end{aligned} \tag{6.2}$$

Here we have made use of the notation (1.14) for the left sided generalized divided differences of a function, and the functions $q_1, \ldots, q_n$ are those defined in Theorem 5.2.

Theorem 6.2. *For $j \in J_\varphi$ and $x \in [a,b]$*

$$\tilde{N}_j^n(x) = (-1)^n f_{n,j} \cdot \begin{bmatrix} q_1,\dots,q_{n+1} \\ \xi_j,\dots,\xi_{j+n} \end{bmatrix}_l g(x,\,\cdot\,), \tag{6.3}$$

$$f_{n,j} = \begin{bmatrix} q_1,\dots,q_n \\ \xi_{j+1},\dots,\xi_{j+n} \end{bmatrix}_l q_{n+1} - \begin{bmatrix} q_1,\dots,q_n \\ \xi_j,\dots,\xi_{j+n-1} \end{bmatrix}_l q_{n+1}.$$

Here $q_1,\dots,q_{n+1} \in \mathcal{P}_{n+1}^*$ are the functions defined by (5.4) and $q_1,\dots,q_n$ are those of Theorem 5.2.

Remark 6.3. In case of ordinary polynomial splines of order n where all connection matrices are identity matrices (6.3) simplifies to

$$\tilde{N}_j^n(x) = (-1)^n(\xi_{j+n} - \xi_j)[\xi_j,\dots,\xi_{j+n}]_l(x - \,\cdot\,)_+^{n-1}$$

where $[\xi_j,\dots,\xi_{j+n}]_l f$ denotes the ordinary left sided divided difference of the function $f \in C_l^{n-1}(J;\mathbb{R})$ with respect to the polynomials of degree n at most. In case of Tchebycheffian splines of order n where all connection matrices are identity matrices (6.3) extends Lyche's definition (6.2) of Tchebycheffian B-splines [10].

In [24] it is proved

Theorem 6.4. *For $j \in J_\varphi$ $\qquad \tilde{N}_j^n = N_j^n$.*
Here N_j^n are the rECT-B-splines of theorem 3.5.

Remark 6.5. It is definition 6.1 which, under suitable assumptions, leads to a recursive method for computing ECT-B-splines and ECT-spline curves developed in section 7 [25]. In [31] cardinal ECT-B-splines with simple knots defined by connection matrices are computed directly according to theorem 6.2 Actually, there the left sided generalized divided differences are computed directly via a certain characteristic polynomial wherein also the Taylor's expansion (1.7) with respect to an ECT-system and that with respect to its dual are involved.

7. Computing rECT-B-splines recursively

Recursive methods for computing B-splines (normalized to form a nonnegative partition of unity) for splines of particular classes are well known. For ordinary polynomial splines (with connection matrices that all are identity matrices) best known is the deBoor-Mansion-Cox recurrence relation (7.3),(7.4) which is a two-term recursion. Using a contour integral approach Walz [32] has proved more general more-term recursions. For Tchebycheff splines there is a two-term recursion due to Lyche [10] where the spline weights are expressed as quotients of determinants. For a wide class of Tchebycheff splines, LB-splines and complex splines Dyn and Ron [5] have given four-term recursions.

It should be noticed that our constructive approach does not cover trigonometric B-splines. Stable two-term recursions for ordinary trigonometric B-splines (with all connection matrices equal to identity matrices) are due to Lyche and Winther [12] and more general ones due to Walz [33].

It is Lyche's approach to Tchebycheff B-splines [10] that can be extended to rECT-B-splines (and similarly to lECT-B-splines). Two ideas are basic in establishing the recurrence relation. One is due to Lyche defining auxiliary B-splines of lower orders which are used as intermediate results in computing the B-splines N_l^n of order n. As before we adopt the assumptions (2.5),(2.6) and (2.7).

Definition 7.1. For $n \in \mathbb{N}$, $k = 1, \ldots, n$ and $x \in \mathbb{R}$ and $j \in \mathbb{Z}$ let

$$N_j^{k,n}(x) := \tag{7.1}$$
$$\begin{cases} (-1)^n \left(\left[\begin{matrix} q_1(\cdot,c),\ldots,q_n(\cdot,c) \\ \xi_{j+1},\ldots,\underbrace{x,\ldots,x}_{n-k},\ldots,\xi_{j+k} \end{matrix} \right]_l g(x,\cdot) \right. \\ \left. - \left[\begin{matrix} q_1(\cdot,c),\ldots,q_n(\cdot,c) \\ \xi_j,\ldots,\underbrace{x,\ldots,x}_{n-k},\ldots,\xi_{j+k-1} \end{matrix} \right]_l g(x,\cdot) \right) & \text{if } \xi_j < \xi_{j+k} \text{ and } \xi_j \le x < \xi_{j+k} \\ 0 & \text{otherwise} \end{cases}$$

Here g is defined by (6.1) and the $q_l(y,c)$ are defined by (5.5). It can be shown [25] that $N_j^{k,n}(x)$ is independent of $c \in \bar{J}_i$ and of $i \in \mathbb{Z}$. We call these functions *auxiliary B-splines of suborder* k of the B-spline N_l^n of order n such that $\xi_l \le \xi_j < \xi_{j+k} \le \xi_{l+n}$.

The B-splines of lowest orders are

$$N_j^{1,1}(x) = N_j^1(x) = N(x|\xi_j, \xi_{j+1}) = \begin{cases} 1 & \text{if } \xi_j \le x < \xi_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to show that the auxiliary B-splines have similar properties as the B-splines themselves (see theorem 3.2) though, at least in general, they do not belong to $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{\text{ext}})$. The second basic idea dates back to Popoviciu [26]. He has modified generalized divided differences by introducing further nodes as is done in definition 7.1. This idea was elaborated for ECT-systems by Lyche [10] and for lET-systems by Mühlbach and Tang in lemmata 4.2 and 4.3 of [25].

Remark 7.2. In case of ordinary polynomial splines of order n the $N_j^{k,n}$ are the polynomial B-splines of order k:

$$N_j^{k,n}(x) = (-1)^k (\xi_{j+k} - \xi_j)[\xi_j, \ldots, \xi_{j+k}]_l (x - \cdot)_+^{k-1} = N_j^{k,k}(x), \qquad k = 1, \ldots, n. \tag{7.2}$$

It is well known that the polynomial B-splines (7.2) can be computed by the de Boor-Mansion-Cox recursion

$$N_j^{k+1,n}(x) = \lambda_j^{k,n}(x) N_j^{k,n}(x) + \mu_{j+1}^{k,n}(x) N_{j+1}^{k,n}(x), \quad k = 1, \ldots, n-1 \tag{7.3}$$

starting with

$$N_j^{1,n}(x) = N_j^{1,1}(x) = \begin{cases} 1 & \text{if } \xi_j \leq x < \xi_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

where the coefficients are the Neville-Aitken weights of polynomial interpolation that are independent of n

$$\lambda_j^{k,n}(x) = \frac{x - \xi_j}{\xi_{j+k} - \xi_j}, \quad \mu_{j+1}^{k,n}(x) = 1 - \lambda_{j+1}^{k,n}(x) = \frac{\xi_{j+k+1} - x}{\xi_{j+k+1} - \xi_{j+1}}. \tag{7.4}$$

They occur in the Neville-Aitken interpolation formula

$$\begin{aligned} p_{k+1} f[\xi_j, \ldots, \xi_{j+k}](x) = & \frac{x - \xi_j}{\xi_{j+k} - \xi_j} p_k f[\xi_{j+1}, \ldots, \xi_{j+k}](x) \\ & + \frac{\xi_{j+k} - x}{\xi_{j+k} - \xi_j} p_k f[\xi_j, \ldots, \xi_{j+k-1}](x) \end{aligned}$$

where $p_l f[z_1, \ldots, z_l]$ is that polynomial of order l interpolating the function f at the nodes $z_1, \ldots, z_l$ in the sense of Hermite. The equality (7.2) relies on the factorization of algebraic polynomials. For arbitrary ECT-systems $\boldsymbol{s}^{[i]}$ there is no similar interpretation of the auxiliary ECT-B-splines of suborder k as in (7.2). Only for particular ECT-systems, for instance of rational functions with prescribed poles there is a similar interpretation (see section 8). However, under suitable assumptions also in the general case the auxiliary ECT-B-splines of suborder k can be computed recursively by a de Boor-like recursion. It turns out that the spline weight factors also in the general case can be interpreted in terms of interpolation theory with respect to an lET-system $q_1, \ldots, q_n$. They are again certain generalized Neville-Aitken weights.

Theorem 7.3. *Suppose that $N_j^{k,n}(x)$ for $k = 1, \ldots, n$ and all $j \in \mathbb{Z}$ is defined by* (7.1)*. Assume that the basis* (5.5) *$q_1(\,\cdot\,, c), \ldots, q_n(\,\cdot\,, c)$ of $\mathcal{P}_n(\mathcal{U}^*, \mathcal{E}_{\mathcal{A}^+}, X)$ is an lET-system on $[a, b]$ of order n that has the property that also $q_1(\,\cdot\,, c), \ldots, q_{n-1}(\,\cdot\,, c)$ is an lET-systems on $[a, b]$. Here $c \in \bar{J}_i$ and $i \in \mathbb{Z}$ are arbitrary. In view of* (5.3) *according to corollary 4.5 this holds true due to the basic assumption* (2.6)*. Then for $k = 1, \ldots, n-1$, $x \in \mathbb{R}$ and $j \in \mathbb{Z}$*

$$N_j^{k+1,n}(x) = \lambda_j^{k,n}(x) \cdot N_j^{k,n}(x) + \mu_{j+1}^{k,n}(x) \cdot N_{j+1}^{k,n}(x) \tag{7.5}$$

with the initialization

$$N_j^{1,n}(x) = \begin{cases} 1 & \textit{if } \xi_j \leq x < \xi_{j+1} \\ 0 & \textit{for all other } x, \end{cases}$$

and $N_j^{1,n}(x) \equiv 0$ iff $\xi_j = \xi_{j+1}$, where the spline weights can be computed by the following formulas:

$$\lambda_j^{k,n}(x) \equiv 0 \qquad \textit{if } \xi_j = \ldots = \xi_{j+k}$$

$$\lambda_j^{k,n}(x) := \begin{cases} 0 & \text{if } x = \xi_j \\ \dfrac{D_l^{\mu^+(\xi_j)-1} rq_n[\xi_{j+1},\ldots,\underbrace{x,\ldots,x}_{n-k},\ldots,\xi_{j+k-1}]_l(\xi_j)}{D_l^{\mu^+(\xi_j)-1} rq_n[\xi_{j+1},\ldots,\underbrace{x,\ldots,x}_{n-k-1},\ldots,\xi_{j+k}]_l(\xi_j)} & \text{if } \xi_j < x \le \xi_{j+k} \end{cases}$$

where $\mu^+(\xi_j)$ *is the multiplicity of* ξ_j *in* $(\xi_j, \xi_{j+1}, \ldots, \xi_{j+k})$ *and the interpolation remainders are with respect to the lET-system* $q_1(\,\cdot\,, c), \ldots, q_{n-1}(\,\cdot\,, c)$ *with* $c \in \mathbb{R}$ *arbitrary. Moreover*

$$\mu_{j+1}^{k,n}(x) \equiv 0 \qquad \text{if } \xi_{j+1} = \ldots = \xi_{j+k+1}$$

$$\mu_{j+1}^{k,n}(x) = \begin{cases} 1 & \text{if } x = \xi_{j+1} \\ \dfrac{D_l^{\mu^-(\xi_{j+k+1})-1} rq_n[\xi_{j+2},\ldots,\underbrace{x,\ldots,x}_{n-k},\ldots,\xi_{j+k}]_l(\xi_{j+k+1})}{D_l^{\mu^-(\xi_{j+k+1})-1} rq_n[\xi_{j+1},\ldots,\underbrace{x,\ldots,x}_{n-k-1},\ldots,\xi_{j+k}]_l(\xi_{j+k+1})} & \\ & \text{if } \xi_{j+1} < x \le \xi_{j+k+1} \end{cases}$$

where $\mu^-(\xi_{j+k+1})$ *is the multiplicity of* ξ_{j+k+1} *in* $(\xi_{j+1}, \ldots, \xi_{j+k+1})$*.*

Moreover, we have

$$0 < \lambda_j^{k,n}(x) < 1 \qquad \text{if } \xi_j < x < \xi_{j+k},$$

and $x \mapsto \lambda_j^{k,n}(x)$ *is left continuous everywhere and strictly increasing from 0 to 1 for* $\xi_j < x < \xi_{j+k}$*. Similarly, we have*

$$0 < \mu_{j+1}^{k,n}(x) < 1 \qquad \text{if } \xi_{j+1} < x < \xi_{j+k+1},$$

and $x \mapsto \mu_{j+1}^{k,n}(x)$ *is left continuous everywhere and strictly decreasing from 1 to 0 for* $\xi_{j+1} < x < \xi_{j+k+1}$*.*

Remark 7.4. The spline weight factors can be interpreted in terms of interpolation theory, cf. [25] remark 4.2.

Remark 7.5. If for every i the connection matrix $A^{[i]}$ is a diagonal matrix with only positive diagonal elements then the basis $q_1(\,\cdot\,, c), \ldots, q_n(\,\cdot\,, c)$ is an lECT-system, and all weights $\lambda_j^{k,n}(x), \mu_{j+1}^{k,n}(x)$ can be computed recursively since the remainders $rq_n[y_1, \ldots, y_n]_l(y)$ can be computed recursively (cf. [19]).

Remark 7.6. It should be observed that theorem 7.3 also covers the case of *Bézier-ECT-splines.* This type of splines arises if we consider a compact interval $[a, b]$, a finite partition $X = (x_i)_{i=0}^{k+1}$ of $[a, b]$, $a = x_0 < x_1 < \ldots < x_k < x_{k+1} = b$, into knot intervals $J_i = [x_i, x_{i+1})$, $i = 0, \ldots, k-1$, with the last knot interval

$J_k = [x_k, x_{k+1}]$ resp. $\check{J}_i = (x_i, x_{i+1}]$, $i = 1, \dots, k$, with the first knot interval $\check{J}_0 = [x_0, x_1]$, with multiplicities $\mu_0 = \mu_{k+1} = n$, $\mu_i = 0$ for $i = 1, \dots, k$, with local ECT-systems (2.2) on $\bar{J}_i$ generated by weights (2.3) satisfying (2.5), with full connection matrices $A^{[i]} \in \mathbb{R}^{n \times n}$ that satisfy (2.6) and (2.7). Notice that the ECT-system $U^{[0]}$ on $\bar{J}_0 = [x_0, x_1]$ may be extended as an ECT-system to a larger interval $\hat{J}_0 = [x_0 - \delta, x_1]$ for $\delta > 0$ simply by extending the weights $w_j^{[0]}$ to $\hat{J}_0$ maintaining the smoothness properties (2.3).

According to theorem 7.3 for $\xi_j \le x < \xi_{j+1}$ the B-spline curve of order n

$$s(x) = \sum_{l=j-n+1}^{j} c_l \cdot N_l^n(x)$$

where the *control points* $c_{j-n+1}, \dots, c_j \in \mathbb{R}^s$ $(s \in \mathbb{N})$ are given can be computed recursively by the following de Boor-like algorithm.

Algorithm 7.7
initialisation:

$$c_l^{1,n}(x) := c_l \qquad l = j-n+1, \dots, j$$

algorithm:

$$c_i^{k+1,n}(x) = \lambda_i^{n-k,n}(x) \cdot c_i^{k,n}(x) + (1 - \lambda_i^{n-k,n}(x)) \cdot c_{i-1}^{k,n}(x),$$
$$\xi_j \le x < \xi_{j+1},\ i = j-n+k+1, \dots, j,\ k = 1, \dots, n-1.$$

output:

$$c_j^{n,n}(x) = s(x), \qquad \xi_j \le x < \xi_{j+1}.$$

Moreover, at level k $(k = 1, \dots, n)$ there holds

$$s(x) = \sum_{l=j-n+k}^{j} c_l^{k,n}(x) \cdot N_l^{n+1-k,n}(x), \qquad \xi_j \le x < \xi_{j+1}.$$

8. Examples

Example 8.1. In case of ordinary polynomial splines of order n with all connection matrices equal to identity matrices we have

$$q_j(y, c) = \frac{(y-c)^{j-1}}{(j-1)!} \qquad j = 1, \dots, n$$

and from theorem 7.3 for $\xi_j \le x < \xi_{j+1}$ and $l = j-k, \dots, j$ the spline weights

$$\lambda_l^{k,n}(x) = \frac{D_l^{\mu^+(\xi_l)-1} rq_n[\xi_{l+1}, \dots, \underbrace{x, \dots, x}_{n-k}, \dots, \xi_{l+k-1}]_l(\xi_l)}{D_l^{\mu^+(\xi_l)-1} rq_n[\xi_{l+1}, \dots, \underbrace{x, \dots, x}_{n-k-1}, \dots, \xi_{l+k}]_l(\xi_l)} = \frac{x - \xi_l}{\xi_{l+k} - \xi_l},$$

$$\mu_l^{k,n}(x) = 1 - \lambda_l^{k,n}(x) = \frac{\xi_{l+k} - x}{\xi_{l+k} - \xi_l}.$$

In this example algorithm 7.7 reduces to the de Boor algorithm computing points of B-spline curves in case of ordinary polynomial splines of order n when all connection matrices are identity matrices.

Example 8.2. In case of Tchebycheff splines of order n with all connection matrices equal to identity matrices we have

$$q_j(y,c) = s^*_{j,n}(y,c) \qquad j = 1, \dots, n$$

and from theorem 7.1 for $\xi_j \le x < \xi_{j+1}$ and $l = j-k, \dots, j$ the spline weights

$$\lambda_j^{k,n}(x) = \frac{D_l^{\mu^+(\xi_j)-1} rq_n[\xi_{j+1}, \dots, \underbrace{x, \dots, x}_{n-k}, \dots, \xi_{j+k-1}]_l(\xi_j)}{D_l^{\mu^+(\xi_j)-1} rq_n[\xi_{j+1}, \dots, \underbrace{x, \dots, x}_{n-k-1}, \dots, \xi_{j+k}]_l(\xi_j)}$$

$$\mu_j^{k,n}(x) = 1 - \lambda_j^{k,n}(x)$$

giving new interpretations to the weights due to Lyche [10].

Example 8.3. For the global ECT-system $s^*_{1,n}(y,c), \dots, s^*_{n,n}(y,c)$ (1.11), (1.12) on $[a,b]$ of example 1.2 where all connection matrices are identity matrices the N_j^n are are Chebycheff-B-splines with respect to this ECT-system. In this case the functions $q_1(\,\cdot\,,x), \dots, q_n(\,\cdot\,,x)$ of theorem 5.2 are known as the rational functions (1.4), (1.5), (1.6). This ECT-system being also a Cauchy-Vandermonde system with respect to the poles $b_1 = \infty, \dots, b_{n-2} = \infty$, $b_{n-1} = a - \varepsilon\,, b_n = b + \varepsilon$ allows to compute the spline weights of theorem 5.2 and of algorithm 7.7 explicitly using the explicit expression (42) of [22] of the interpolation remainder in terms of the nodes and the poles. If $\xi_j \le x < \xi_{j+1},\ j \in \{-n+1, \dots, \mu\}$ according to theorem 5.2 for $n \in \mathbb{N}$ at level $k = 1, \dots, n$ for $m = j-k+1, \dots, j$

$$\lambda_m^{k,n}(x) := \left\{ \lim_{\tilde{\xi}_m \to \xi_m - 0} \frac{rq_n[\xi_{m+1}, \dots, \underbrace{x, \dots, x}_{n-k}, \dots, \xi_{m+k-1}]_l(\tilde{\xi}_m)}{rq_n[\xi_{m+1}, \dots, \underbrace{x, \dots, x}_{n-k-1}, \dots, \xi_{m+k}]_l(\tilde{\xi}_m)} \quad \text{if } \xi_j < x < \xi_{j+1} \right. \tag{8.1}$$

$$= \frac{\xi_m - x}{\xi_m - \xi_{m+k}} \frac{b + \varepsilon - \xi_{m+k}}{b + \varepsilon - x} \qquad \text{if } \xi_j \le x < \xi_{j+1}$$

and

$$\mu_m^{k,n}(x) = 1 - \lambda_m^{k,n}(x) = \frac{\xi_{m+k} - x}{\xi_{m+k} - \xi_m} \frac{b + \varepsilon - \xi_{m+k}}{b + \varepsilon - x}. \tag{8.2}$$

The spline weights (8.1) and (8.2) agree with the weights given by Gresbrand [7] where splines with the ordinary smoothness conditions constructed from Cauchy-Vandermonde systems with respect to arbitrary given poles $b_1, b_2, \ldots, b_n$ outside $[a, b]$ are considered. This shows that also in the simple case of Chebycheff ECT-B-splines with one pole of order $n-1$ at $b+\varepsilon$ the auxiliary ECT-B-splines are the ECT-B-splines of lower orders.

More analytical and some numerical examples can be found in [25]. Of course, it will depend on the applications what kind of splines a designer will choose. The family of ECT-B-splines provides a real alternative to the classical polynomial B-splines. In fact, they allow more freedoms without increasing computational costs too much.

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