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Pointwise very strong approximation as a generalization of Fejér's summation theorem

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Abstract

We will present an estimation of the $H^q_{k_r}f$ mean as a approximation versions of the Totik type generalization(see [6]) of the result of G. H. Hardy, J. E. Littlewood. Some results on the norm approximation will also given.

Key Words: very strong approximation, rate of pointwise strong summability

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1. Introduction

Let L^p $(1 < p < \infty)$ $[resp. C]$ be the class of all 2π–periodic real–valued functions integrable in the Lebesgue sense with p–th power [continuous] over $Q = [-\pi, \pi]$ and let $X = X^p$ where $X^p = L^p$ when $1 < p < \infty$ or $X^p = C$ when $p = \infty$. Let us define the norm of $f \in X^p$ as

$$\|f\|_{X^p} = \|f(x)\|_{X^p} = \begin{cases} \left(\int_Q \mid f(x) \mid^p dx\right)^{1/p} & when\ 1 < p < \infty\ , \\ \sup_{x\in Q} \mid f(x) \mid & when\ \ p = \infty. \end{cases}$$

Consider the trigonometric Fourier series

$$Sf(x) = \frac{a_o(f)}{2} + \sum_{k=0}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx) = \sum_{k=0}^{\infty} C_k f(x)$$

and denote by $S_k f$, the partial sums of Sf. Let

$$H_{k_r}^q f(x) := \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} |S_{k_\nu} f(x) - f(x)|^q \right\}^{\frac{1}{q}}, \qquad (q>0)$$

where $0 \leqslant k_0 < k_1 < k_2 < \ldots < k_r \quad (\geqslant r)$.

The pointwise characteristic

$$\begin{aligned} \overline{w}_x f(\delta)_p &:= \sup_{0<h\leqslant\delta} \left\{ \frac{1}{h} \int_0^h |\varphi_x(t)|^p \, dt \right\}^{1/p}, \\ \text{where} \quad \varphi_x(t) &:= f(x+t) + f(x-t) - 2f(x) \end{aligned}$$

constructed on the base of definition of Lebesgue points ($L^1 - points$) was firstly used as a measure of approximation, by S.Aljančič, R.Bojanic and M.Tomić [1]. This characteristic was very often used, but it appears that such approximation cannot be comparable with the norm approximation beside when $X = C$. In [5] there was introduced the slight modified function:

$$w_x f(\delta)_p := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p \, dt \right\}^{1/p}.$$

We can observe that for $p \in [1, \infty)$ and $f \in C$

$$w_x f(\delta)_p \leqslant \overline{w}_x f(\delta)_p \leqslant \omega_C f(\delta)$$

and also, with $\widetilde{p} > p$ for $f \in X^{\widetilde{p}}$, by the Minkowski inequality

$$\|w_{\cdot} f(\delta)_p\|_{X^{\widetilde{p}}} \leqslant \omega_{X^{\widetilde{p}}} f(\delta),$$

where $\omega_X f$ is the modulus of continuity of f in the space $X = X^{\widetilde{p}}$ defined by the formula

$$\omega_X f(\delta) := \sup_{0<|h|\leqslant\delta} \|f(\cdot + h) - f(\cdot)\|_X.$$

It is well-known that $H_n^q f(x) -$ means tend to 0 at the $L^p - points$ of $f \in L^p$ $(1 < p \leqslant \infty)$. In [3] this fact was by G. H. Hardy, J. E. Littlewood proved as a generalization of the Fejér classical result on the convergence of the $(C,1)$ - means of Fourier series. Here we present an estimation of the $H_{k_r}^q f(x)$ means as a approximation version of the Totik type (see [6]) generalization of the result of G. H. Hardy, J. E. Littlewood. We also give some corollaries on norm approximation.

By K we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same of each occurrence.

2. Statement of the results

Theorem 2.1. *If* $f \in L^p$ $(1 < p \leqslant 2)$ *, then, for indices* $0 \leqslant k_0 < k_1 < k_2 < ... < k_r$ $(\geqslant r)$,

$$H_{k_r}^q f(x) \leqslant 2\left\{\sum_{k=r}^{k_r} \frac{w_x f(\frac{\pi}{k+1})_1}{k+1}\right\} + 6\left\{\frac{1}{(r+1)^{p-1}} \sum_{k=0}^{r} \frac{\left(w_x f(\frac{\pi}{k+1})_p\right)^p}{(k+1)^{2-p}}\right\}^{1/p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Applying the inequality for the norm of the modulus of continuity of f we can immediately derive from the above theorem the next one.

Theorem 2.2. *If* $f \in L^p$ $(1 < p \leqslant 2)$ *, then for indices* $0 \leqslant k_0 < k_1 < k_2 < ... < k_r$ $(\geqslant r)$,

$$\left\|H_{k_r}^q f(\cdot)\right\|_{L^p} \leqslant 2\left\{\sum_{k=r}^{k_r} \frac{\omega_{L^p} f(\frac{\pi}{k+1})}{k+1}\right\} + 6\left\{\frac{1}{(r+1)^{p-1}} \sum_{k=0}^{r} \frac{\left(\omega_{L^p} f(\frac{\pi}{k+1})\right)^p}{(k+1)^{2-p}}\right\}^{1/p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.3. In the special case $k_\nu = \nu$ for $\nu = 0, 1, 2, ..., r$, the first term in the above estimates is superfluous.

Next, we consider a function w_x of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0) = 0$, $w_x(\delta_1 + \delta_2) \leqslant w_x(\delta_1) + w_x(\delta_2)$ for any $0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \delta_1 + \delta_2$ and let

$$L^p(w_x) = \left\{g \in L^p : w_x g(\delta)_p \leqslant w_x(\delta)\right\}.$$

In this class we can derive the following

Theorem 2.4. *Let* $f \in L^p(w_x)$ $(1 < p \leqslant 2)$ *and* $0 \leqslant k_0 < k_1 < k_2 < ... < k_r$ $(\geqslant r)$. *If* w_x *satisfy , for some* $A > 1$ *the condition* $\limsup_{\delta \to 0+} \left(\frac{w_x(A\delta)}{w_x(\delta)}\right)^p < A^{p-1}$, *then*

$$H_{k_r}^q f(x) \leqslant K w_x\left(\frac{\pi}{r+1}\right) \log \frac{k_r + 1}{r+1}.$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In the same way for subclass

$$L^p(\omega) = \{g \in L^p : \omega_{L^p} f(\delta) \leqslant \omega(\delta), \quad \text{with modulus of continuity} \quad \omega\}$$

we can obtain

Theorem 2.5. *Let* $f \in L^p(\omega)$ $(1 < p \leqslant 2)$ *and* $0 \leqslant k_0 < k_1 < k_2 < \ldots < k_r$ $(\geqslant r)$. *If* ω *satisfy , for some* $A > 1$ *and an integer* $s \geqslant 1$, *the condition* $\limsup_{\delta \to 0+} \frac{\omega(A\delta)}{\omega(\delta)} < A^s$, *then*

$$\left\| H^q_{k_r} f(\cdot) \right\|_{L^p} \leqslant K\omega\left(\frac{\pi}{r+1}\right) \log \frac{k_r+1}{r+1},$$

where $\frac{1}{p} + \frac{1}{q} = 1$

For the proof of Theorem 2.2 we will need the following lemma of N. K. Bari and S. B. Stechkin [2].

Lemma 2.6. *If a continuous and non-decreasing on* $[0,\infty)$ *function* w *satisfies conditions:* $w(0) = 0$ *and* $\limsup_{\delta \to 0+} \frac{w(A\delta)}{w(\delta)} < A^s$ *for some* $A > 1$ *and an integer* $s \geqslant 1$, *then*

$$u^s \int_u^\pi \frac{w(t)}{t^{s+1}} dt \leqslant Kw(u) \quad \text{for} \quad u \in (0,\pi],$$

where the constant K *depend only on* w *and in other way the fulfillment of the above inequality for all* $u \in (0,\pi]$ *imply the existence of a constant* $A > 1$ *for which* $\limsup_{\delta \to 0+} \frac{w(A\delta)}{w(\delta)} < A^s$ *with some integer* $s \geqslant 1$.

3. Proofs of the results

We only prove Theorems 2.1 and 2.4.

Proof of Theorem 2.1. Let as usually

$$\begin{aligned} H^q_{k_r} f(x) &= \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^\pi \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q} \\ &\leqslant A_{k_r} + B_{k_r} + C_{k_r}, \end{aligned}$$

where $D_{k_\nu}(t) = \frac{\sin \frac{(2k_\nu+1)t}{2}}{2 \sin \frac{t}{2}}$,

$$A_{k_r}(\delta) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^\delta \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q},$$

$$B_{k_r}(\gamma,\delta) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_\delta^\gamma \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q}$$

and

$$C_{k_r}(\gamma) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_\gamma^\pi \varphi_x(t) D_{k_\nu}(t)\, dt \right|^q \right\}^{1/q},$$

with $\delta = \frac{\pi}{k_r+1}$ and $\gamma = \frac{\pi}{r+1}$.

Since $k_\nu \leqslant k_r$, for $\nu = 0, 1, 2, ..., r$,we conclude that $|D_{k_\nu}(t)| \leqslant k_r + 1$ and $|D_{k_\nu}(t)| \leqslant \frac{\pi}{2|t|}$. Hence

$$A_{k_r}(\delta) \leqslant \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[\frac{k_r+1}{\pi} \int_0^\delta |\varphi_x(t)|\, dt \right]^q \right\}^{1/q} = w_x f(\delta)_1$$

and

$$B_{k_r}(\gamma,\delta) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[\frac{1}{2} \int_\delta^\gamma \frac{|\varphi_x(t)|}{t} dt \right]^q \right\}^{1/q} = \frac{1}{2} \int_\delta^\gamma \frac{|\varphi_x(t)|}{t} dt.$$

Integrating by parts, we obtain

$$\begin{aligned} B_{k_r}(\gamma,\delta) &= \frac{1}{2} \left\{ w_x f(t)_1 \left|_{t=\delta}^{\gamma} \right. + \int_\delta^\gamma \frac{w_x f(t)_1}{t} dt \right\} \\ &= \frac{1}{2} w_x f(\gamma)_1 - \frac{1}{2} w_x f(\delta)_1 + \frac{1}{2} \int_{r+1}^{k_r+1} \frac{w_x f(\pi/u)_1}{u} du \end{aligned}$$

and by simple calculation we have

$$\begin{aligned} B_{k_r}(\gamma,\delta) &\leqslant \frac{1}{2} w_x f(\gamma)_1 - \frac{1}{2} w_x f(\delta)_1 + \frac{1}{2} \sum_{k=r+1}^{k_r} \int_k^{k+1} \frac{w_x f(\pi/u)_1}{u} du \\ &\leqslant \frac{1}{2} w_x f(\gamma)_1 - \frac{1}{2} w_x f(\delta)_1 + \frac{1}{2} \sum_{k=r+1}^{k_r} \frac{k+1}{k} \frac{w_x f(\pi/k)_1}{k} \end{aligned}$$

$$\begin{aligned} &\leqslant \frac{1}{2} w_x f(\gamma)_1 - \frac{1}{2} w_x f(\delta)_1 + \frac{1}{2} \left(1 + \frac{1}{r+1}\right) \sum_{k=r}^{k_r-1} \frac{w_x f(\pi/k)_1}{k} \\ &\leqslant w_x f(\gamma)_1 + 2 \sum_{k=r}^{k_r-1} \frac{w_x f(\pi/k)_1}{k}. \end{aligned}$$

Putting $D_{k_\nu}(t) = \frac{1}{2} \sin(k_\nu t) \cot \frac{t}{2} + \frac{1}{2} \cos(k_\nu t)$, by the Hausdorff–Young inequality,

$$\begin{aligned} & C_{k_r}(\gamma) \\ \leqslant\ & \frac{1}{2(r+1)^{1/q}} \left\{ \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_\gamma^\pi \varphi_x(t) \cot \frac{t}{2} \sin(k_\nu t)\, dt \right|^q \right\}^{1/q} \\ & + \frac{1}{2(r+1)^{1/q}} \left\{ \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_\gamma^\pi \varphi_x(t) \cos(k_\nu t)\, dt \right|^q \right\}^{1/q} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\frac{1}{\pi}\int_{\gamma}^{\pi}\left|\varphi_x\left(t\right)\cot\frac{t}{2}\right|^p dt\right\}^{1/p} \\
&\quad+\frac{1}{2\left(r+1\right)^{1/q}}\left\{\frac{1}{\pi}\int_{\gamma}^{\pi}\left|\varphi_x\left(t\right)\right|^p dt\right\}^{1/p} \\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\int_{\gamma}^{\pi}\left|\frac{\varphi_x\left(t\right)}{t/\pi}\right|^p dt\right]^{1/p}+\pi^{1/p}w_x f(\pi)_p\right\}
\end{aligned}
$$

and by partial integration,

$$
\begin{aligned}
&C_{k_r}\left(\gamma\right) \\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\frac{\left[w_x f(t)_p\right]^p}{t^{p-1}}\Big|_{t=\gamma}^{\pi}+p\int_{\gamma}^{\pi}\left|\frac{w_x f(t)_p}{t}\right|^p dt\right]^{1/p}\right. \\
&\left.+\pi^{1/p}w_x f(\pi)_p\right\} \\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\pi^{1-p}\left[w_x f(\pi)_p\right]^p+p\int_{1}^{r+1}\left|\frac{w_x f(\pi/u)_p}{\pi/u}\right|^p\frac{\pi}{u}du\right]^{1/p}\right. \\
&\left.+\pi^{1/p}w_x f(\pi)_p\right\}.
\end{aligned}
$$

Therefore, analogously as before,

$$
\begin{aligned}
&C_{k_r}\left(\gamma\right) \\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\pi^{1-p}\left[w_x f(\pi)_p\right]^p+p\pi^{1-p}\sum_{k=1}^{r}\int_{k}^{k+1}\frac{\left[w_x f(\pi/u)_p\right]^p}{u^{2-p}}du\right]^{1/p}\right. \\
&\left.+\pi^{1/p}w_x f(\pi)_p\right\} \\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\pi^{1-p}\left[w_x f(\pi)_p\right]^p+p\pi^{1-p}\sum_{k=1}^{r}\frac{k+1}{k}\frac{\left[w_x f(\pi/k)_p\right]^p}{k^{2-p}}\right]^{1/p}\right. \\
&\left.+\pi^{1/p}w_x f(\pi)_p\right\} \\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\left(1+p\right)\pi^{1-p}\sum_{k=1}^{r}\frac{\left[w_x f(\pi/k)_p\right]^p}{k^{2-p}}\right]^{1/p}+\pi^{1/p}w_x f(\pi)_p\right\} \\
&\leqslant K\left\{\frac{1}{\left(r+1\right)^{p-1}}\sum_{k=1}^{r}\frac{\left[w_x f(\pi/\left(k+1\right))_p\right]^p}{\left(k+1\right)^{2-p}}\right\}^{1/p}.
\end{aligned}
$$

Finally, since

$$
w_x f(\gamma)_1 \leqslant w_x f(\gamma)_p\left\{\frac{p}{\left(r+1\right)^p}\sum_{k=0}^{r}\frac{1}{\left(k+1\right)^{1-p}}\right\}^{1/p}
$$

$$\leqslant \quad \left\{ \frac{p}{(r+1)^{p-1}} \sum_{k=1}^{r} \frac{[w_x f(\pi/(k+1))_p]^p}{(k+1)^{2-p}} \right\}^{1/p},$$

our result follows. □

Proof of Theorem 2.4. It is clear that if $f \in L^p(w_x)$ $(1 < p \leqslant 2)$ then $w_x f(\delta)_1 \leqslant w_x f(\delta)_p \leqslant w_x(\delta)$. Thus, by Theorem 2.1,

$$H_{k_r}^q f(x) \leqslant 2 \left\{ \sum_{k=r}^{k_r} \frac{w_x(\frac{\pi}{k+1})}{k+1} \right\} + 6 \left\{ \frac{1}{(r+1)^{p-1}} \sum_{k=0}^{r} \frac{\left(w_x(\frac{\pi}{k+1})\right)^p}{(k+1)^{2-p}} \right\}^{1/p}$$

and, by the monotonicity of w_x and simple inequality $w_x(\pi) \leqslant 2w_x(\frac{\pi}{2})$, we obtain

$$\begin{aligned} H_{k_r}^q f(x) \quad &\leqslant \quad 2 \left\{ \sum_{k=r}^{k_r} \frac{w_x(\frac{\pi}{k+1})}{k+1} \right\} \\ &\quad + 6 \left\{ \frac{1}{(r+1)^{p-1}} \left((w_x(\pi))^p + \sum_{k=1}^{r} \frac{\left(w_x(\frac{\pi}{k+1})\right)^p}{(k+1)^{2-p}} \right) \right\}^{1/p} \end{aligned}$$

$$\begin{aligned} &\leqslant \quad 2 \left\{ \sum_{k=r}^{k_r} \frac{w_x(\frac{\pi}{k+1})}{k+1} \right\} + 6 \left\{ \frac{5}{(r+1)^{p-1}} \sum_{k=1}^{r} \frac{\left(w_x(\frac{\pi}{k+1})\right)^p}{(k+1)^{2-p}} \right\}^{1/p} \\ &\leqslant \quad 2 \left\{ w_x(\frac{\pi}{r+1}) \sum_{k=r}^{k_r} \frac{1}{k+1} \right\} \\ &\quad + 6 \left\{ \frac{5}{(r+1)^{p-1}} \sum_{k=1}^{r} \int_k^{k+1} \frac{(w_x(\frac{\pi}{t}))^p}{t^{2-p}} dt \right\}^{1/p} \end{aligned}$$

$$\begin{aligned} &\leqslant \quad 2 \left\{ w_x(\frac{\pi}{r+1}) \int_r^{k_r+1} \frac{1}{t} dt \right\} \\ &\quad + 6 \left\{ \frac{5}{(r+1)^{p-1}} \pi^{p-1} \int_{\frac{\pi}{r+1}}^{\pi} \frac{(w_x(u))^p}{u^{p-2}} \frac{du}{u^2} \right\}^{1/p} \end{aligned}$$

$$\begin{aligned} &\leqslant \quad 2w_x(\frac{\pi}{r+1}) \log \frac{k_r+1}{r} \\ &\quad + 6 \left\{ 5 \left(\frac{\pi}{r+1} \right)^{p-2} \frac{\pi}{r+1} \int_{\frac{\pi}{r+1}}^{\pi} \frac{(w_x(u))^p u^{2-p}}{u^2} du \right\}^{1/p} \end{aligned}$$

Now, we observe that, by our assumption , the function $(w_x(u))^p\, u^{2-p}$ satisfy the condition

$$\limsup_{\delta\to 0+} \frac{(w_x(A\delta))^p\,(A\delta)^{2-p}}{(w_x(\delta))^p\,(\delta)^{2-p}} = A^{2-p} \limsup_{\delta\to 0+} \frac{(w_x(A\delta))^p}{(w_x(\delta))^p} < A^{2-p}A^{p-1} = A$$

i.e. the condition of Lemma 2.6 with $s = 1$. Therefore

$$\frac{\pi}{r+1}\int_{\frac{\pi}{r+1}}^{\pi} \frac{(w_x(u))^p\,u^{2-p}}{u^2}du \leqslant \left(w_x(\frac{\pi}{r+1})\right)^p \left(\frac{\pi}{r+1}\right)^{2-p}.$$

Hence

$$\begin{aligned} H_{k_r}^q f(x) &\leqslant 2w_x(\frac{\pi}{r+1})\log\frac{k_r+1}{r} \\ &\quad +6\left\{5\left(\frac{\pi}{r+1}\right)^{p-2}\left(w_x(\frac{\pi}{r+1})\right)^p\left(\frac{\pi}{r+1}\right)^{2-p}\right\}^{1/p} \\ &\leqslant \left(2+6\;5^{1/p}\right) w_x(\frac{\pi}{r+1})\log\frac{k_r+1}{r}, \end{aligned}$$

and our result is proved. □

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