

On the fundamental theorem of compact and noncompact surfaces

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Abstract

In this article an overview of the history of surface topology is given. From the Euler-formula, to the Kerékjártó Theorem we follow the development process of the fundamental theorem of compact and non-compact surfaces. We refer to the works of Riemann, Möbius, Jordan, Klein and others, but our main focus point is to show the work of the Hungarian mathematician, Béla Kerékjártó.

Key Words: history of surface topology, classification of surfaces, homeomorphism, topological invariants, Kerékjártó, compact and noncompact surfaces

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1. Introduction

We can observe many similarities between the developmental process of mathematics and the teaching process of a particular topic. Among others, Imre Lakatos was dealt with these similarities. In his work, *Proofs and Refutations* [16] he demonstrated the creative and informal nature of a real mathematical discovery, and suggested a way of teaching in which the concepts and theorems are thought by reproducing the historical steps.

*“Mathematics develops, according to Lakatos, ... by a process of conjecture, followed by attempts to ‘prove’ the conjecture (i.e. to reduce it to other conjectures) followed by criticism via attempts to produce counter-examples both to the conjectured theorem and to the various steps in the proof.”*¹

The development of topology, especially of surface topology is similar to the process of teaching mathematics in two aspects: First, the development of surface

¹J. Worrall: Imre Lakatos (1922-1974) : philosopher of mathematics and philosopher of science, *Z. Allgemeine Wissenschaftstheorie* 5 (2) (1974), 211-217.

topology gives an instructive example of the difficulties of formulating an intuitive and practical problem into a precise mathematical model. Such problems in topology include: The definition of the topological transformation: From the intuitive idea of “a change in form without tearing and sticking together” researchers reached the mathematical concept of the one to one and continuous mapping. The definition of the surface: from the intuitive idea through the concept of triangulation mathematicians developed the concept of two dimensional topological manifolds. Second, studying the developmental process of surface topology we can observe definitions and theorems that precede the birth of a concrete result. In our paper we illustrate this process describing the fundamental theorem of compact and non-compact surfaces. Since the development of the theorem includes the same phases that students go through when formulating and solving a problem, the history of fundamental theorem of surface topology is a useful parallel to the process of teaching and learning mathematics.

The starting point of the theory of 2-dimensional manifolds as well as for many topological theories was the Euler-theorem (1750). Euler’s famous formula is for a polyhedron: $v - e + f = 2$, where v is the number of the vertices, e is the number of edges and f is the number of faces.

The generalisation by L’Huilier (1811) led to the first known result on a topological invariant.

Another type of generalization was made by Schläfli and Poincare. The Euler-formula was extended to n -dimensional spaces by Schläfli, and proved by Poincaré (1893): $N_0 - N_1 + N_2 - \dots + (-1)^{n-1}N_{n-1} = 1 - (-1)^n$, where $N_0 = v$, $N_1 = e$, $N_2 = f$, and N_k is analogically the number of k -dimensional figures. [6]

Riemann examined the connectivity of surfaces in 1851 and 1857. The Euler-formula for an n -connected polyhedron is the following: $v - e + f = 3 - n$, where n is the connectivity number of the polyhedron. [11]

In the 1860-s Möbius(1863, 1865) and Jordan (1866) worked independently from each other on the problem of topological equivalent surfaces. They elaborated the classification of compact orientable surfaces. Although Listing (1862) mentioned first the so called Möbius-strip as an example of a one-sided surface, Möbius (1865) described its properties in terms of non-orientability.

In the 1870-s, Schläfli and Klein discussed on the orientability of the real 2-dimensional subspaces of the projective space. Klein introduced the concept of relative and absolute properties of a manifold, and identified orientability as an absolute property. In 1882 he described the so called Klein-bottle, one of the most famous non-orientable closed surfaces.

The classification of non-orientable compact surfaces was published in the paper of van Dyck in 1888. Seeing that neither the concept of homeomorfism nor the concept of an abstract surface was completely precise, the first essentially rigorous proof of the classification theorem for compact surfaces was given only in 1907 by Dehn and Heegaard. After Brahana’s exact algebraic proof in 1921 some additional proofs were made in the 20. century too.

The problem of non-compact surfaces didn’t occur in works in the 19. century.

The starting point for the compactification was perhaps the concept of the projective plane. Kerékjártó in 1923 gave the classification theorem of non-compact surfaces after introducing boundary components to compactify the open surface. Richards (1962) and Goldmann (1971) proved more precisely the theorem and gave some consequences to it.

Of course there exists the generalisation of the theory of surfaces for higher dimensions, but this is not the topic of the present paper.

2. Compact surfaces

One of the earliest topological results is the Euler-formula² from 1750 ($v - e + f = 2$). Euler was looking for a relation similar to that, which exists between the numbers of vertices and sides of a polygon ($v = s$). At that time the concept of polyhedron was an intuitive extending of the five Platonic solids (pyramids, prisms etc.). Before the appearance of the Euler-theorem it was no reason to give a precise definition.

After trying to prove the theorem, and in connection with this, after the appearance of different counterexamples, it was necessary to define a polyhedron. Cauchy's proof in 1813³, and L'Huilier's well-known counterexamples⁴ led the examinations to the direction of topology. The novelty of Cauchy's proof was that he considered a polyhedron not a rigid body, but a surface. He omitted one of the faces of polyhedron, and embedded it in the plane admitting some deformations of edges and faces, and examined a connected graph on the plane. At that time the concepts of function and geometrical transformation was not developed to the level, that Cauchy would reach the idea of topological transformation.

Observing transparent crystals L'Huilier tried to prove the Euler-theorem for polyhedron with holes. He became the result, that the number $v - e + f$ is not always 2, there exists surfaces with other numbers too.

Works of Poinot, Vandermonde, Cauchy, Poincare, and others led to further counterexamples, and to the generalization of the Euler-theorem, as well as to the concept of Euler-characteristics.

In the 1850-s and 1860-s the surface topology was developed through the works of Möbius, Jordan, Riemann and Listing. Möbius and Jordan gave a definition of topological transformation, and with Riemann used the concept of surfaces more generally as others before.

²L. Euler: Opera Omnia I, Bd. 26, 71-93.

³A. L. Cauchy: Recherches sur les polyédres. In: Journal de l'École Polytechnique, 1813, 68-86.

⁴S. A. J. L'Huilier: Mémoire sur la polyédrométrie, In: Annales de Mathématiques, Pures et Appliquées, 1812-13, 179, 194.

2.1. Topological transformation

Let A, B two sets of points, and $f : A \rightarrow B$ a function. We can see that f is a mapping of the set A into the set B . The distance between two points in the sets A and B is defined.

A function $f : A \rightarrow B$ is continuous at a point $x_0 \in A$, if for $\forall \epsilon > 0$, $\exists \delta > 0$ such that whenever x differs from x_0 by less than δ , $f(x)$ differs from $f(x_0)$ by less than ϵ . If a mapping $f : A \rightarrow B$ is continuous at every point $x_0 \in A$, then we say that f is continuous.

A mapping $f : A \rightarrow B$ is said to be bijective, if the preimage of every point of B is exactly one point of A . For a bijective mapping $f : A \rightarrow B$ we can define the inverse mapping $f^{-1} : B \rightarrow A$.

A mapping $f : A \rightarrow B$ is said to be a homeomorphism or topological transformation, if it is both bijective, and f as well as its inverse f^{-1} are continuous. Intuitively, homeomorphism is a mapping of a set on another set that involves no tearing (the continuity condition) and no gluing together (the bijective condition).

Two figures A and B are homeomorphic or topologically equivalent if there exists a mapping $f : A \rightarrow B$, which is homeomorphism. For example the cube is homeomorphic to a sphere. (See Figure 1.)

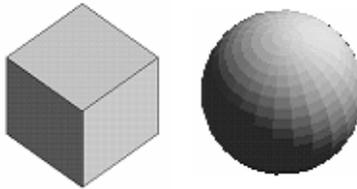


Figure 1.

Properties of figures unchanged by homeomorphisms are called topological properties, or topological invariants. One of the first known topological invariant of a surface S was the Euler-characteristic, the number $\chi(S) = v - e + f$.

2.2. Concept of surfaces

There are many definitions of surface, in geometry, in differential geometry, in topology. The modern topological definition is the following: A surface is a connected two-dimensional manifold.

Our terminology must be start from the generalization of the Euler-theorem: The relation $v - e + f = 2$ is true for any polyhedron whose surface is homeomorphic to a sphere and each of whose faces is homeomorphic to a disk. A figure is called surface without boundary, if each of its points x has a neighbourhood homeomorphic to a disk. (A neighbourhood of a point x is a set, whose points differ from x less than a given positive number.)

To our further review of history of surface topology we need the definition of triangulation. Triangulation is a method dividing a surface in set of triangles, which are in bijective relation to planar triangles. We dealt with triangulable surfaces only.

A surface is a set of triangles satisfying the following properties:

1. The inner points of a triangle belong to this triangle only.
2. Every edge of a triangle belongs to exactly two triangles, which do not have any other common points besides this edge.
3. Every vertex of a triangle belongs to a finite, cyclically ordered set of triangles, in which set every two consecutive triangles have a common edge containing this vertex.
4. For every pair of triangles there exists a not necessarily unique finite sequence of triangles, in which the first and last elements are the given triangles, and every two consecutive triangles have exactly one common edge.

If the number of triangles is finite then the surface is called closed, otherwise, it is called open.

A bordered surface is a finite set of triangles satisfying the following properties:

- 1'. The inner points of a triangle belong to this triangle only.
- 2'. Every edge of a triangle belongs to exactly one triangle, or belongs to exactly two triangles, which do not have any other common points besides this edge.
- 3'. Every vertex of a triangle belongs to a finite, cyclically or linearly ordered set of triangles, in which set every two consecutive triangles have a common edge containing this vertex.
- 4'. For every pair of triangles there exists a not necessarily unique finite sequence of triangles, in which the first and last elements are the given triangles, and every two consecutive triangles have exactly one common edge. [14]

A boundary of a surface consists of edges of triangles belonging to exactly one triangle. The boundary contains finite number of simple closed curves without common points, they are called boundary components.

2.3. Johann Benedict Listing (1808-1882)

He wrote two important works related to topology. In 1847 in his paper *Vorstudien zur Topologie* he used the term topology instead of Analysis Situs. This means that his main focus was on connection, relative position and continuity.

In 1862 in his other work *Der Census Räumlicher Complexe oder Verallgemeinerung des Eulerschen Satzes von der Polyedern* the Möbius-strip as an example appeared, and he denoted, that it has “quite different properties”, and it's bound by one closed curve but did not describe exactly this one-sided surface. He generalized the Euler-formula to a sphere and to surfaces homeomorphic to a sphere. [12]

2.4. Bernhard Riemann (1826-1866)

The interpretation of complex numbers and functions of a variable complex quantity in Riemann's works lead to the concept of connectivity of surfaces. In his

dissertation *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*, in 1851 he defined surfaces which cover a domain in the complex plane, the “*Riemann-surfaces*”.

He recognised the importance of topological ideas and applied topological methods to his problems in complex analysis. In his study *Theorie der Abelschen Functionen* in 1857 he introduced the connectivity number as a topological invariant. He said that surface S is simply connected, if it falls in two parts by any cross cut (a line which runs through the interior of the surface without self-intersections, and joins one boundary point to another). For not simply connected surfaces he described a cutting method which gives the minimal number of cutting resulting only simply connecting surfaces. If the surface is closed, also without boundary, Riemann made a surface with boundary through a perforation.

According to this method we give a closed curve on the surface, and give the second curve so, that it joins two not necessary different points of the first curve. Drawing of curves should be continued until, till it is possible to draw a new curve without intersecting of the sequence of previous curves. For example the sphere is simply connected surface, and the connectivity number of the torus is 3.

He recognised the relation between the Euler-characteristic and the number of connectivity: $n = 3 - \chi(S)$. [12]

2.5. August Ferdinand Möbius (1790-1868)

In our point of view Möbius had two important papers. The first, *Theorie der elementaren Verwandtschaft* in 1863 described the concept of topological transformation as an “*elementary relationship*” in the following intuitive way: Two points of a figure which are infinitely near each other are corresponding to two points of the other figure which are infinitely near each other too.

“Zwei geometrische Figuren sollen einander elementar verwandt heissen, wenn jedem nach allen Dimensionen unendlich kleinen Elemente der einen Figur ein dergleichen Element in der anderen dergestalt entspricht, dass von je zwei an eiander gremzenden Elementen der einen Figur die zwei ihnen entsprechende Elemente der anderen ebenfalls zusammenstossen; oder, was dasselbe ausdrückt: wenn je einem Punkte der einen Figur ein Punct der anderen also entspricht, dass von je zwei einander unendlich nahen Puncten der einen auch die ihnen entsprechenden der anderen einander unendlich nahe sind.” [17]

Möbius examined elementary relationships of closed and bordered surfaces which are without self-intersection in the Eucliden plane or space. He showed that each such surface can be constructed from two elementary equivalent surfaces each with n boundary components which are pasted together at the boundary components. He called n the class of the surface. On a surface n th class we can draw $n - 1$ closed curves not decomposing it.

He said the classification theorem the following way: Two closed surfaces are elementary equivalents if and only if they belong to the same class.

“Je zwei geschlossene Fläche φ und φ' , welche zu derselben Klasse gehören, sind elementar verwandt. Dagegen sind zwei zu verschiedenen Klassen gehörige

geschlossene Flächen nicht in elementarer Verwandtschaft. [17]

Möbius generalized the Euler-theorem from the direction of surface topology and not from the direction of counterexamples. He gave the normal forms of closed or bordered, orientable surfaces. The relation between the number n and the Euler-characteristic is: $\chi = 2(2 - n)$.

Since all surfaces were considered as embedded into R^3 , non-orientable surfaces were found quite late. In his paper *Über die Bestimmung des Inhaltes eines Polyeders* in 1858 (printed in 1865) Möbius defined the orientation of surfaces, and described one-sided and two sided surfaces.

He defined an orientation of a surface (polyhedron) nearly the following way: Let S be, a triangulable closed or bordered surface. Give orientation to the edges of triangles so that if H_1 and H_2 are neighbouring triangles (with one common edge), then their common edge has different orientations. (See Figure 2.)

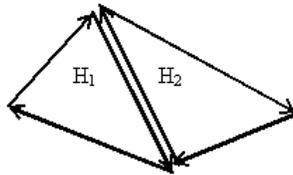


Figure 2.

Den Perimetern der ein Polyeder umgrenzenden Flächen können solche Sinne beigelegt werden, dass für jede Kante des Polyeders die zwei Richtungen, welche derselben, als der gemeinschaftlichen Kante zweier Polyederflächen, in Folge der Sinne dieser zwei Flächen zukommen, einander entgegengesetzt sind. [18]

Möbius called this construction “rule of edges” (*Der Gesetz der Kanten*). If there exists a $H_1, H_2, \dots, H_n, H_1$ closed chain containing neighbouring triangles of S , and the orientation of H_1 triangle is changed, then the surface S is called non-orientable. If such a chain doesn’t exist, then S is called an orientable surface.

Möbius found a surface, by which the “rule of edges” could not apply. This was the Möbius-strip, the first example of a non-orientable surface. It is a bordered surface which is obtained by taking a rectangular strip, twisting it once and gluing its ends together. (See Figure 3.)

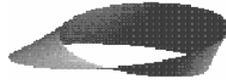


Figure 3.

This is a one-sided surface, if we go over a Möbius-strip with a “paintbrush”, then we return to the starting point on the “opposite side”.

“Auch hat diese Fläche nur eine Seite; denn wenn man sie von einer beliebigen Stelle aus mit einer Farbe zu überstreichen anfängt und damit fortfährt, ohne mit

dem Pinsel über die Grenzlinie hinaus auf die andere Seite überzugehen, so werden nichtsdestoweniger zuletzt an jeder Stelle die zwei daselbst einander gegenüberliegenden Seiten der Fläche gefärbt sein.” [18]

2.6. Camille Jordan (1838-1921)

Independently from Möbius Jordan defined the topological transformation as mapping, and classified the orientable surfaces too. In his work *Sur la deformation des surfaces* in 1866 he wrote the following theorem: The maximal number of recurrent cuts which do not dissect the surface into disconnected pieces, and the number of boundary components are invariant properties and classifies uniquely the compact orientable surfaces. [12]

2.7. Felix Klein (1849-1925)

He distinguished absolute and relative properties of surfaces. For example non-orientability is an absolute, but one-sidedness is a relative property. The definition of one-sidedness involves not only the surface, but also its disposition in space. Orientability depends only on the surface. (*Über den Zusammenhang der Flächen*, 1875) In his paper *Über Riemanns Theorie der algebraischen Functionen und ihre Integrale*, in 1882 Klein gave the normal forms of closed surfaces and described a non-orientable closed surface, the Klein-bottle. It is impossible to embed this surface in three-dimensional Euclidean space without self-intersection. [22] (See Figure 4.)

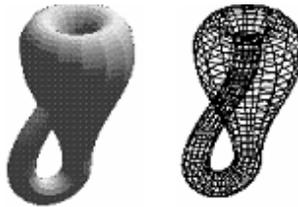


Figure 4.

2.8. Walter von Dyck (1856-1934)

In his study *Beiträge zur Analysis Situs*, in 1888 he dealt with absolute properties of compact orientable and non-orientable 2-dimensional manifolds. He defined surfaces with recurrent and non-recurrent indicatrix. We draw a small circle around a point of the surface, which is not a boundary point, and oriented it. The circle with its orientation (clockwise or anticlockwise) is called an indicatrix. If there is a closed path on the surface, whose traversal reverses the orientation of the indicatrix, the surface with recurrent indicatrix is non-orientable, and the surface with non-recurrent indicatrix is orientable.

He described the cross-cap, a surface homeomorphic to a Möbius-strip. (See Figure 5.) (We cut a square from a half-sphere, and glue together the diagonally opposite vertices of the square.) This representation of the Möbius-strip has a self intersection, but its boundary curve is homeomorphic to a circle, hence cross-caps can be glued into holes in a sphere.

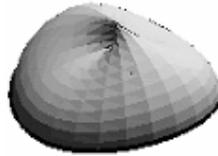


Figure 5.

He proved the following **fundamental theorem of compact surfaces**:

Two closed or bordered triangulable surfaces are topological equivalents if and only if, they have the same number of boundary curves, the same Euler-characteristic and are either both orientable or non-orientable.

Classification of closed or bordered surfaces:

Let $H(p, r)$ a surface which is derived from a sphere with r holes by adding p handles. (See Figure 6.)

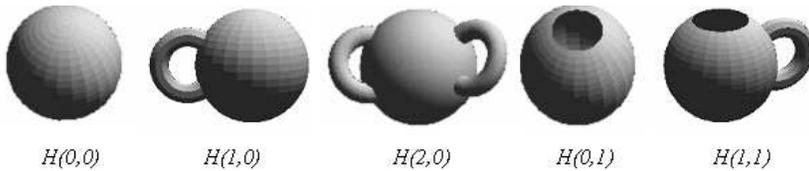


Figure 6.

Let $C(q, r)$ be a surface which is derived from a sphere with r holes by adding q cross-caps. (See Figure 7.)

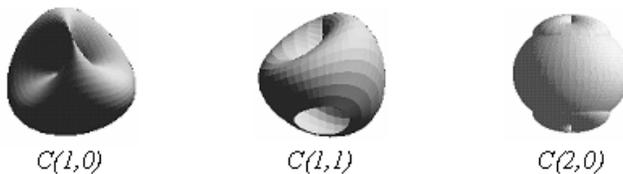


Figure 7.

It is proved, that every closed or bordered surface belongs to one and only one of these classes. [7]

Roughly speaking, the number of handles or cross-caps is called the genus of the surface. This concept appeared in Riemann's work already as the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it. The genus is related to the Euler-characteristic. More precisely, if the compact surface is orientable, the genus: $g = 1/2(2 - \chi - r)$, where χ is the Euler-characteristic, r the number of boundary components. The genus of a non-orientable surface is $g = 2 - r - \chi$. The genus of a surface is one of the oldest known topological invariants and much of topology has been created in order to generalize this concept of surface topology.

Examples of the connection between invariant properties: (See Table 1.) [15]

The surface	χ	g	r	Orientable	Class
sphere	2	0	0	yes	H(0,0)
cube	2	0	0	yes	H(0,0)
torus	0	1	0	yes	H(1,0)
Möbius-strip	0	1	1	no	C(1,1)
sphere with 1 cross-cap	1	1	0	no	C(1,1)
Klein-bottle	0	2	0	no	C(2,0)
disk	1	0	1	yes	H(0,1)
half-sphere	1	0	1	yes	H(0,1)
sphere with 2 handles	-2	2	0	yes	H(2,0)
ring	0	0	2	yes	H(0,2)
cylinder	0	0	2	yes	H(0,2)
sphere with 1 handle and 1 hole	-1	1	1	yes	H(1,1)
cross-cup	0	1	1	no	C(1,0)

Table 1.

2.9. Max Dehn (1878-1952) and Poul Heegaard (1871-1948)

In an encyclopaedia article, in 1907 they elaborated the axiomatic structure of combinatorial topology, this approach allowed them to establish a normal form of surfaces and give the first rigorous proof of the fundamental theorem of compact surfaces. (*Analysis Situs, Enzyklopädie der Mathematischen Wissenschaften*).

2.10. Henry R. Brahana

In his doctoral thesis, *Systems of circuits on two-dimensional manifolds* (1921) he gave an algebraic proof of the classification of closed two-dimensional surfaces, and gave a method of reducing any two-dimensional manifold to one of the known polygonal normal forms through a series of transformations by cutting and joining them. [4]

3. Noncompact surfaces

3.1. Béla Kerékjártó (1898-1946)

The work of the Hungarian mathematician, Béla Kerékjártó according to the non-compact surfaces was the last step in the process of development of surface topology. The process started with the Euler-theorem for polyhedron and went through the classification of compact orientable, then compact non-orientable surfaces till the classification of non-compact surfaces.

Kerékjártó was born in 1898 in Budapest, and died in 1946 in Gyöngyös. He received his Phd in 1920, and 2 years later he became a Full Professor at the University of Szeged. In 1922-23 he was a visiting professor at the University of Göttingen, where he wrote his book *Vorlesungen über Topologie*. This was the first research monograph, and the first textbook on this topic. A chapter of this book contains the theorem of open surfaces, which is known as Kerékjártó's Theorem. With this theorem the problem of topological equivalence of compact and non-compact surfaces is completely solved.

Kerékjártó's main idea was that he defined the ideal boundary of an open surface. This compactification process is a generalisation of the projective closure of the Euclidean space. With the help of these ideal points he compactified the open surface to a closed surface.

An ideal point of a surface S is a nested sequence $G_1 \supset G_2 \supset G_3 \supset \dots$ of connected, unbounded regions (open connected sets) in S satisfying the following properties:

- The boundary curves of G_k regions are simple closed curves of S , for $\forall k \in \mathbb{N}$
- The sequence of regions doesn't have any common points.

This sequence of regions defines an ideal point, a boundary point. (See Figure 8.)

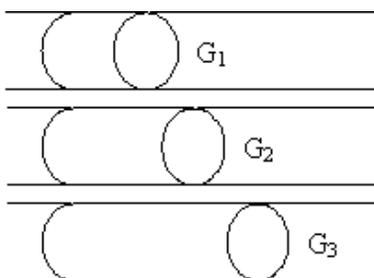


Figure 8.

For example the ideal boundary of a disk can be realized as a circle or the Euclidian plane can be compactified with one point.

Two G_1, G_2, \dots and G'_1, G'_2, \dots sequences of regions define the same ideal point if for $\forall k \in \mathbb{N}$ there is a corresponding integer n such that $G'_n \supset G_k$ and $G_n \supset G'_k$.

The invariants of compact surfaces can be generalized to open surfaces.

We distinguish planar and orientable ideal points. A closed surface is planar, if every Jordan curve separates it. The ideal points of an open surface are planar or orientable, if G_k , the element of the sequence of regions are planar or orientable for k sufficiently large. [14] An open surface S is of finite genus, if there is a bounded subsurface A such that $S - A$ is homeomorphic to a subset of the Euclidean plane. Otherwise S is of infinite genus. We have four orientability classes of surfaces: If the surface S is non-orientable, then it is either finitely or infinitely non-orientable. S is infinitely non-orientable, if there is no subset A so that $S - A$ is orientable. If S is finitely non-orientable, we distinguish odd or even non-orientability according to whether every sufficiently large bounded subsurface has an odd or even genus. [20]

Kerékjártó's Theorem

Let S' and S'' be two open triangulable surfaces of the same genus and orientability class. Then S' and S'' are homeomorphic if and only if their ideal boundaries are homeomorphic, and the sets of planar and orientable ideal points are homeomorphic too.

Examples of open surfaces that are homeomorphic include:

- I. The Euclidean plane and a sphere perforated in one point
- II. Grate-surface and a sphere with infinite handles (See Figure 9.)

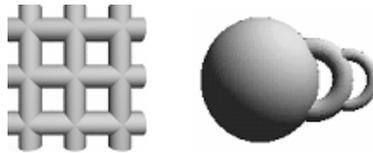


Figure 9.

- III. The Euclidean plane with one cross-cap and a cross-cap-surface perforated in one point

We can construct every surface from the following five bordered surface-elements: (See Figure 10.)

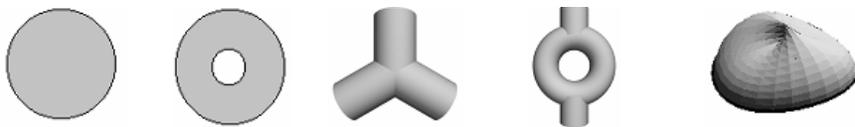


Figure 10.

The way of the construction: We take one from these surface-elements, then stick an other one to all its every boundary curves, so we get a new bordered surface. We repeat this construction so that we have different surfaces stuck to different curves. [14]

3.2. Further results

Of course the formulation of Kerékjártó's Theorem in 1922 did not correspond to the level of exactness in modern abstract mathematics. New versions and proofs of the theorem have been published. The ideal points for open surfaces are generalized as boundary components of a surface imbedded in a topological space.

- Kerékjártó Béla: *A nyílt felületek topológiájáról* (1931)

He gave a new definition of ideal points. [13]

- Ian Richards: *On the classification of noncompact 2-manifolds* (1960)

He described a complete topological classification of non-compact triangulable surfaces, and gave a concrete model for arbitrary surface, similar to the classical normal form. He reconceptualised the Kerékjártó's Theorem and made more precise the proof. [20]

- M. E. Goldman: *An algebraic classification of noncompact 2-manifolds* (1971)

He proved a theorem, which is for an open surface an algebraic version of the Kerékjártó Classification Theorem. [9]

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