Annales Mathematicae et Informaticae
32 *(2005) pp. 189–201.*

Homogeneity properties of subadditive functions

Pál Burai and Árpád Száz

Institute of Mathematics, University of Debrecen
e-mail: buraip@math.klte.hu
e-mail: szaz@math.klte.hu

Abstract

We collect, supplement and extend some well-known basic facts on various homogeneity properties of subadditive functions.

Key Words: Homogeneous and subadditive functions, seminorms and preseminorms.

AMS Classification Number: 39B72

1. Introduction

Subadditive functions, with various homogeneity properties, play important roles in many branches of mathematics. First of all, they occur in the Hahn-Banach theorems and the derivation of vector topologies. (See, for instance, [2] and [14] .)

A positively homogeneous subadditive function is usually called sublinear. While, an absolutely homogeneous subadditive function may be called a seminorm. However, some important subadditive functions are only preseminorms.

Moreover, it is also worth noticing that subbadditive functions are straightforward generalizations of the real-valued additive ones. Therefore, the study of additive functions should, in principle, be preceded by that of the subadditive ones.

Subadditive functions have been intensively studied by several authors. Their most basic algebraic and analytical properties have been established by R. Cooper [5] , E. Hille [7, pp. 130–145] , R. A. Rosenbaum [17] , E. Berz [1] , M. Kuczma [9, pp. 400–423] and J. Matkowski [11] .

In this paper, we are only interested in the most simple homogeneity properties of subadditive functions. Besides collecting some well-known basic facts, for instance, we prove the following theorem.

Theorem 1.1. *If p is a quasi-subadditive function of a vector space X over $\mathbb{Q}$, and moreover $x \in X$ and $0 \neq k \in \mathbb{Z}$, then*

(1) $\frac{1}{k}\, p\,(l\,x\,) \leq p\left(\frac{l}{k}\,x\right)$ *for all* $l \in \mathbb{Z}$;

(2) $p\left(\frac{l}{k}\,x\right) \leq l\,p\left(\frac{1}{k}\,x\right) \geq \frac{l}{k}\,p\,(x)$ *for all* $0 < l \in \mathbb{Z}$;

(3) $p\left(\frac{l}{k}\,x\right) \geq l\,p\left(\frac{1}{k}\,x\right) \leq \frac{l}{k}\,p\,(x)$ *for all* $0 > l \in \mathbb{Z}$.

Remark 1.2. If p is a subodd subadditive function of X, then p is additive. Therefore, the corresponding equalities are also true.

While, if p is an even subadditive function of X, then we can only prove that

$$\frac{1}{|\,k\,|}\, p\,(\,l\,x\,) \leq p\left(\frac{l}{k}\,x\right) \leq |\,l\,|\,p\left(\frac{1}{k}\,x\right)$$

for all $x \in X$, $0 \neq k \in \mathbb{Z}$ and $l \in \mathbb{Z}$.

2. Superodd and subhomogeneous functions

Definition 2.1. A real-valued function p of a group X will be called

(1) subodd if $p\,(-x) \leq -p\,(x)$ for all $x \in X$;

(2) superodd if $-p\,(x) \leq p\,(-x)$ for all $x \in X$.

Remark 2.2. Note that thus p may be called odd if it is both subodd and superodd.

Moreover, p is superodd if and only if $-p$ is subodd. Therefore, superodd functions need not be studied separately.

However, because of the forthcoming applications, it is more convenient to study superodd functions. By the above definition, we evidently have the following

Proposition 2.3. If p is a superodd function of a group X, then

(1) $0 \leq p\,(0)$; (2) $-p\,(-x\,) \leq p\,(x)$ for all $x \in X$.

Hint. Clearly, $-p\,(0) \leq p\,(-0\,) = p\,(0)$. Therefore, $0 \leq 2\,p\,(0)$, and thus (1) also holds.

Remark 2.4. Note that if p is subodd, then just the opposite inequalities hold. Therefore, if in particular p is odd, then the corresponding equalities are also true.

Analogously to Definition 2.1, we may also naturally introduce the following

Definition 2.5. A real-valued function p of a group X will be called

(1) $\mathbb{N}$–subhomogeneous if $p(nx) \leq n\,p(x)$ for all $n \in \mathbb{N}$ and $x \in X$;

(2) $\mathbb{N}$–superhomogeneous if $n\,p(x) \leq p(nx)$ for all $n \in \mathbb{N}$ and $x \in X$.

Remark 2.6. Note that thus p may be called $\mathbb{N}$–homogeneous if it is both $\mathbb{N}$–subhomogeneous and $\mathbb{N}$–superhomogeneous.

Moreover, p is $\mathbb{N}$–superhomogeneous if and only if $-p$ is $\mathbb{N}$–subhomogeneous. Therefore, $\mathbb{N}$–superhomogeneous functions need not be studied separately.

Concerning $\mathbb{N}$–subhomogeneous functions, we can easily establish the following

Proposition 2.7. If p is an $\mathbb{N}$–subhomogeneous function of a group X, then

$$p(kx) \leq -k\,p(-x).$$

for all $x \in X$ and $0 > k \in \mathbb{Z}$.

Proof. Under the above assumptions, we evidently have $p(kx) = p\big((-k)(-x)\big) \leq (-k)\,p(-x) = -k\,p(-x)$. □

Now, as an immediate consequence of Definition 2.5 and Proposition 2.7, we can also state

Proposition 2.8. If p is an $\mathbb{N}$–subhomogeneous function of a group X and $x \in X$, then

(1) $\frac{1}{k}\,p(kx) \leq p(x)$ for all $0 < k \in \mathbb{Z}$;

(2) $-p(-x) \leq \frac{1}{k}\,p(kx)$ for all $0 > k \in \mathbb{Z}$.

Moreover, by using this proposition, we can easily prove the following

Theorem 2.9. *If p is an $\mathbb{N}$–subhomogeneous function of a vector space X over $\mathbb{Q}$, and moreover $x \in X$ and $l \in \mathbb{Z}$, then*

(1) $\frac{1}{k}\,p\big(l x\big) \leq p\left(\frac{l}{k}\,x\right)$ *for all* $0 < k \in \mathbb{Z}$;

(2) $-\frac{1}{k}\,p\big(-l x\big) \leq p\left(\frac{l}{k}\,x\right)$ *for all* $0 > k \in \mathbb{Z}$.

Proof. If $0 < k \in \mathbb{Z}$, then by Proposition 2.8 it is clear that

$$\frac{1}{k}\,p\big(l x\big) = \frac{1}{k}\,p\left(k\left(\frac{l}{k}\,x\right)\right) \leq p\left(\frac{l}{k}\,x\right).$$

While, if $0 > k \in \mathbb{Z}$, then by the above inequality it is clear that

$$-\frac{1}{k}\,p\big(-l x\big) = \frac{1}{-k}\,p\big(l(-x)\big) \leq p\left(\frac{l}{-k}\,(-x)\right) = p\left(\frac{l}{k}\,x\right).$$

□

Remark 2.10. Note that if p is $\mathbb{N}$–superhomogeneous, then just the opposite inequalities hold. Therefore, if in particular p is $\mathbb{N}$–homogeneous, then the corresponding equalities are also true.

3. Subadditive and quasi-subadditive functions

Following the terminology of Hille [7, p. 131] and Rosenbaum [17, p. 227], we may also naturally have the following

Definition 3.1. A real-valued function p of a group X will be called

(1) subadditive if $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$;

(2) superadditive if $p(x) + p(y) \leq p(x+y)$ for all $x, y \in X$.

Definition 3.2. Note that thus p is additive if and only if it is both subadditive and superadditive.

Moreover, p is superadditive if and only if $-p$ is subadditive. Therefore, superadditive functions need not be studied separately.

The appropriateness of Definitions 2.1 and 2.5 is apparent from the following theorem whose proof can also be found in Kuczma [9, p. 401].

Theorem 3.3. *If p is a subadditive function of a group X, then p is superodd and $\mathbb{N}$–subhomogeneous.*

Proof. Clearly, $p(0) = p(0+0) \leq p(0) + p(0)$, and thus $0 \leq p(0)$. Moreover, if $x \in X$, then we have

$$0 \leq p(0) = p(x + (-x)) \leq p(x) + p(-x).$$

Therefore, $-p(x) \leq p(-x)$, and thus p is superodd.

Moreover, if $p(nx) \leq np(x)$ for some $n \in \mathbb{N}$, then we also have

$$p\big((n+1)x\big) = p(nx + x) \leq p(nx) + p(x) \leq np(x) + p(x) = (n+1)p(x).$$

Hence, by the induction principle, it is clear that $p(nx) \leq np(x)$ for all $n \in \mathbb{N}$. Therefore, p is $\mathbb{N}$–subhomogeneous. □

Remark 3.4. Note that if p is superadditive, then p is subodd and $\mathbb{N}$–superhomogeneous. Therefore, if in particular p is additive then p is odd and $\mathbb{N}$–homogeneous.

Because of Theorem 3.3, we may also naturally introduce the following

Definition 3.5. A real-valued function p of a group X will be called

(1) quasi-subadditive if it is superodd and $\mathbb{N}$–subhomogeneous;

(2) quasi-superadditive if it is subodd and $\mathbb{N}$–superhomogeneous.

Remark 3.6. Note that thus p may be called quasi-additive if it is both quasi-subadditive and quasi-superadditive.

Moreover, p is quasi-superadditive if and only if $-p$ is quasi-subadditive. Therefore, quasi-superadditive functions need not be studied separately.

Now, in addition to Propositions 2.7 and 2.8, we can also prove the following

Proposition 3.7. If p is a quasi-subadditive function of a group X and $x \in X$, then

(1) $k\,p(x) \leq p(k\,x)$ for all $0 > k \in \mathbb{Z}$;

(2) $\frac{1}{k}\,p(k\,x) \leq p(x)$ for all $0 \neq k \in \mathbb{Z}$.

Proof. If $0 > k \in \mathbb{Z}$, then by the corresponding definitions we have

$$-p(k\,x) \leq p(-(k\,x)) = p((-k)\,x) \leq (-k)\,p(x) = -k\,p(x).$$

Therefore,

$$k\,p(x) \leq p(k\,x), \qquad \text{and hence} \qquad \frac{1}{k}\,p(k\,x) \leq p(x).$$

Moreover, from Proposition 2.8 we know that the latter inequality is also true for $0 < k \in \mathbb{Z}$. □

Now, by using the above proposition, we can easily prove the following counterpart of Theorem 2.9.

Theorem 3.8. *If p is a quasi–subadditive function of a vector space X over $\mathbb{Q}$, and moreover $x \in X$ and $0 \neq k \in \mathbb{Z}$, then*

(1) $\frac{1}{k}\,p(l\,x) \leq p\left(\frac{l}{k}\,x\right)$ *for all* $l \in \mathbb{Z}$;

(2) $p\left(\frac{l}{k}\,x\right) \leq l\,p\left(\frac{1}{k}\,x\right) \geq \frac{l}{k}\,p(x)$ *for all* $0 < l \in \mathbb{Z}$;

(3) $p\left(\frac{l}{k}\,x\right) \geq l\,p\left(\frac{1}{k}\,x\right) \leq \frac{l}{k}\,p(x)$ *for all* $0 > l \in \mathbb{Z}$.

Proof. If $l \in \mathbb{Z}$, then by Proposition 3.7 (2), it is clear that

$$\frac{1}{k}\,p(l\,x) = \frac{1}{k}\,p\left(k\left(\frac{l}{k}\,x\right)\right) \leq p\left(\frac{l}{k}\,x\right).$$

Moreover, if $0 < l \in \mathbb{Z}$, then by using the $\mathbb{N}$–subhomogeneity of p and the $l = 1$ particular case of Theorem 3.8 (1) we can see that

$$p\left(\frac{l}{k}\,x\right) = p\left(l\,\frac{1}{k}\,x\right) \le l\,p\left(\frac{1}{k}\,x\right) \quad \text{and} \quad \frac{l}{k}\,p(x) = l\,\frac{1}{k}\,p(x) \le l\,p\left(\frac{1}{k}\,x\right).$$

While, if $0 > l \in \mathbb{Z}$, then by using Proposition 3.7 (1) and the $l = 1$ particular case of Theorem 3.8 (1) we can see that

$$l\,p\left(\frac{1}{k}\,x\right) \le p\left(l\,\frac{1}{k}\,x\right) = p\left(\frac{l}{k}\,x\right) \quad \text{and} \quad l\,p\left(\frac{1}{k}\,x\right) \le l\,\frac{1}{k}\,p(x) = \frac{l}{k}\,p(x).$$

□

Remark 3.9. Note that if p is quasi-superadditive, then just the opposite inequalities hold. Therefore, if p is in particular quasi-additive, then the corresponding equalities are also true.

4. Some further results on subadditive functions

Whenever p is subadditive, then in addition to Theorem 3.8 we can also prove the following

Theorem 4.1. *If p is a subadditive function of a group X, then for any $x, y \in X$ we have*

(1) $-p(-(x-y)) \le p(x) - p(y) \le p(x-y)$;

(2) $-p(-(-y+x)) \le p(x) - p(y) \le p(-y+x)$.

Proof. We evidently have

$$p(x) = p(x-y+y) \le p(x-y) + p(y),$$

and hence also

$$p(y) \le p(y-x) + p(x) = p(-(x-y)) + p(x).$$

Therefore, (1) is true.

Moreover, quite similarly we also have

$$p(x) = p(y-y+x) \le p(y) + p(-y+x),$$

and hence also

$$p(y) \le p(x) + p(-x+y) = p(x) + p(-(-y+x)).$$

Therefore, (2) is also true. □

Now, as a useful consequence of the above theorem, we can also state

Corollary 4.2. *If p is a subadditive function of a group X, then for any $x, y \in X$ we have*

(1) $\big| p(x) - p(y) \big| \leq \max \big\{ p(x-y),\ p(-(x-y)) \big\}$;

(2) $\big| p(x) - p(y) \big| \leq \max \big\{ p(-y+x),\ p(-(-y+x)) \big\}$.

Proof. If
$$M = \max \{ p(x-y),\ p(-(x-y)) \},$$
then by Theorem 4.1 we have
$$p(x) - p(y) \leq p(x-y) \leq M \qquad \text{and} \qquad -(p(x)-p(y)) \leq p\big(-(x-y)\big) \leq M.$$
Therefore, $|p(x) - p(y)| \leq M$, and thus (1) is true. The proof of (2) is quite similar. □

By using Theorem 4.1, we can easily prove the following improvement of Kuczma's [9, Lemma 9, p. 402]. (See also Cooper [5, Theorem IX, p. 430].)

Theorem 4.3. *If p is a real-valued function of a group X, then the following assertions are equivalent:*

(1) *p is additive;*

(2) *p is odd and subadditive.*

(3) *p is subodd and subadditive.*

Proof. If (1) holds, then by Remark 3.4 it is clear that p is odd, and thus (2) also holds. Therefore, since (2) trivially implies (3), we need actually show that (3) implies (1).

For this, note that if (3) holds, then by Remark 2.4 and Theorem 4.1 we have
$$p(x) + p(y) \leq p(x) + (-p(-y)) = p(x) - p(-y) \leq p(x - (-y)) = p(x+y)$$
for all $x, y \in X$. Hence, by the subadditivity of p, it is clear that (1) also holds. □

From the above theorem, by using Remark 3.9, we can immediately get

Corollary 4.4. *If p is a subodd subadditive function of a vector space X over $\mathbb{Q}$, then $p(rx) = r\,p(x)$ for all $r \in \mathbb{Q}$ and $x \in X$.*

Hence, it is clear that in particular we also have

Corollary 4.5. *If p is a subodd subadditive function of $\mathbb{Q}$, then $p(r) = p(1)\,r$ for all $r \in \mathbb{Q}$.*

5. Even superodd and subhomogeneous functions

Because of quasi-subadditive functions, it is also worth studying even superodd and $\mathbb{N}$–subhomogeneous functions.

Definition 5.1. A real-valued function p of a group X will be called even if $p(-x) = p(x)$ for all $x \in X$.

Remark 5.2. Now, in contrast to Definition 2.1, the subeven and supereven functions need not be introduced. Namely, we have evidently the following

Proposition 5.3. If p is a real-valued function of a group X, then the following assertions are equivalent:

(1) p is even;

(2) $p(-x) \leq p(x)$ for all $x \in X$;

(3) $p(x) \leq p(-x)$ for all $x \in X$.

Hint. If (3) holds, then for each $x \in X$ we also have $p(-x) \leq p(-(-x)) = p(x)$. Therefore, $p(-x) = p(x)$, and thus (1) also holds.

Remark 5.4. Note that a counterpart of the above proposition fails to hold for odd functions. Namely, if for instance $p(x) = |x|$ for all $x \in \mathbb{R}$, then p is superodd, but not odd.

By using Definition 5.1, in addition to Proposition 2.3, we can also easily establish the following extension of Cooper's [5, Theorem X, p. 430]. (See also Kuczma [9, Lemma 8, p. 402].)

Proposition 5.5. If p is an even superodd function of a group X, then $0 \leq p(x)$ for all $x \in X$.

Proof. Namely, if $x \in X$, then $-p(x) \leq p(-x) = p(x)$. Therefore, $0 \leq 2p(x)$, and thus $0 \leq p(x)$ also holds. $\square$

Remark 5.6. Hence, it is clear that if p is an even subodd function of X, then $p(x) \leq 0$ for all $x \in X$.

Therefore, if in particular p is an even and odd function of X, then we necessarily have $p(x) = 0$ for all $x \in X$.

Moreover, by using Proposition 2.7 and Theorem 2.9, we can also easily prove the following counterparts of Proposition 3.7 and Theorem 3.8.

Proposition 5.7. If p is an even $\mathbb{N}$–subhomogeneous function of a group X, then

$$p(kx) \leq |k|\, p(x)$$

for all $x \in X$ and $0 \neq k \in \mathbb{Z}$.

Proof. If $0 < k \in \mathbb{Z}$, then the corresponding definitions we evidently have $p(kx) \leq kp(x) = |k|\,p(x)$.

While, if $0 > k \in \mathbb{Z}$, then by Proposition 2.7 and the corresponding definitions we also have $p(kx) \leq -kp(-x) = |k|\,p(x)$. □

Theorem 5.8. *If p is an even $\mathbb{N}$–subhomogeneous function of a vector space X over $\mathbb{Q}$, then*

$$\frac{1}{|k|}\,p(lx) \leq p\Big(\frac{l}{k}\,x\Big) \leq |l|\,p\Big(\frac{1}{k}\,x\Big)$$

for all $x \in X$ and $k, l \in \mathbb{Z}$ with $k, l \neq 0$.

Proof. If $k > 0$, then Theorem 2.9 (1) and Proposition 5.7 it is clear that

$$\frac{1}{|k|}\,p(lx) = \frac{1}{k}\,p(lx) \leq p\Big(\frac{l}{k}\,x\Big) = p\Big(l\,\frac{1}{k}\,x\Big) \leq |l|\,p\Big(\frac{1}{k}\,x\Big).$$

While, if $k < 0$, then by Theorem 2.9 (2) and Proposition 5.7, it is clear that

$$\frac{1}{|k|}\,p(lx) = -\frac{1}{k}\,p(-lx) \leq p\Big(\frac{l}{k}\,x\Big) = p\Big(l\,\frac{1}{k}\,x\Big) \leq |l|\,p\Big(\frac{1}{k}\,x\Big).$$

□

Remark 5.9. To compare the above theorem with Theorem 3.8, note that by the $l = 1$ particular case of Theorem 5.8 now we also have

$$\frac{|l|}{|k|}\,p(x) \leq |l|\,p\Big(\frac{1}{k}\,x\Big).$$

Finally, we note that by Corollary 4.2 we can also state the following

Proposition 5.10. If p is an even subadditive function of a group X, then

$$|p(x) - p(y)| \leq \min\{p(x-y),\ p(-y+x)\}.$$

for all $x, y \in X$.

6. Homogeneous subadditive functions

Definition 6.1. A real-valued function p of a vector space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ will be called

(1) homogeneous if $p(\lambda x) = \lambda p(x)$ for all $\lambda \in \mathbb{K}$ and $x \in X$;

(2) positively homogeneous if $p(\lambda x) = \lambda p(x)$ for all $\lambda > 0$ and $x \in X$;

(3) absolutely homogeneous if $p(\lambda x) = |\lambda|\,p(x)$ for all $\lambda \in \mathbb{K}$ and $x \in X$.

Remark 6.2. Note that if p is homogeneous (absolutely homogeneous), then p is, in particular, odd (even) and positively homogeneous.

Moreover, if p is positively homogeneous, then in particular we have $p(0) = p(2 \cdot 0) = 2p(0)$, and hence $p(0) = 0$. Therefore, $p(0x) = p(0) = 0 = 0p(x)$ is also true.

Now, as some useful characterizations of positively and absolutely homogeneous functions, we can also easily prove the following two propositions.

Proposition 6.3. If p is a real-valued function of a vector space X over $\mathbb{R}$, then the following assertions are equivalent:

(1) p is positively homogeneous;

(2) $p(\lambda x) \le \lambda p(x)$ for all $\lambda > 0$ and $x \in X$;

(3) $\lambda p(x) \le p(\lambda x)$ for all $\lambda > 0$ and $x \in X$.

Proposition 6.4. If p is a real-valued function of a vector space X over $\mathbb{K}$, then the following assertions are equivalent:

(1) p is absolutely homogeneous;

(2) $p(\lambda x) \le |\lambda| p(x)$ for all $0 \ne \lambda \in \mathbb{K}$ and $x \in X$;

(3) $|\lambda| p(x) \le p(\lambda x)$ for all $0 \ne \lambda \in \mathbb{R}$ and $x \in X$.

Hint. If (3) holds, then for any $0 \ne \lambda \in \mathbb{R}$ and $x \in X$ we also have

$$p(\lambda x) = |\lambda| \left| \frac{1}{\lambda} \right| p(\lambda x) \le |\lambda| p\Big(\frac{1}{\lambda} \lambda x \Big) = |\lambda| p(x).$$

Therefore, the corresponding equality is also true. Moreover, from Remark 6.2, we can see that $p(0x) = p(0) = 0 = |0| p(x)$. Therefore, (1) also holds.

In addition to the above propositions, it is also worth establishing the following

Theorem 6.5. *If p is a real-valued function of a vector space X over $\mathbb{R}$, then*

(1) p is homogeneous if and only if p is odd and positively homogeneous;

(2) p is absolutely homogeneous if and only if p is even and positively homogeneous.

Hint. If p is even and positively homogeneous, then for any $\lambda < 0$ and $x \in X$ we also have $p(\lambda x) = p\big(-\lambda(-x)\big) = -\lambda p(-x) = |\lambda| p(x)$. Hence, by the second part of Remark 6.2, it is clear that p is absolutely homogeneous.

Remark 6.6. >From Remark 6.2 and Theorem 4.3, we can see that a homogeneous subadditive function is necessarily linear.

Therefore, only some non-homogeneous subbadditive functions have to be studied separately. The most important ones are the norms.

Definition 6.7. A real-valued, absolutely homogeneous, subadditive function p of a vector space X is called a seminorm on X.

In particular, the seminorm p is called a norm if $p(x) \neq 0$ for all $x \in X \setminus \{0\}$.

Remark 6.8. Note that if p is a seminorm on X, then by Remark 6.2, Theorem 3.3 and Proposition 5.5, we necessarily have $0 \leq p(x)$ for all $x \in X$.

Definition 6.9. A real-valued subadditive function p of a vector space X over $\mathbb{K}$ is called a preseminorm on X if

(1) $\lim\limits_{\lambda \to 0} p(\lambda x) = 0$ for all $x \in X$;

(2) $p(\lambda x) \leq p(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$.

In particular, the preseminorm p is called a prenorm if $p(x) \neq 0$ for all $x \in X \setminus \{0\}$.

Remark 6.10. By Remark 6.8, it is clear that every seminorm p on X is, in particular, a preseminorm.

Moreover, if p is a preseminorm on X, then defining $p^*(x) = \min\{1, p(x)\}$ $\big($or $p^*(x) = p(x)/\big(1 + p(x)\big)\big)$ for all $x \in X$, it can be shown that p^* is a preseminorm on X such that p^* is not a seminorm.

Most of the following basic properties of preseminoms have also been established in [18]. The simple proofs are included here for the reader's convenience.

Theorem 6.11. *If p is a preseminorm on a vector space X over $\mathbb{K}$ and $x \in X$, then*

(1) $p(0) = 0$ *(2)* $0 \leq p(x)$;

(3) $p(\lambda x) = p(|\lambda| x)$ *for all* $\lambda \in \mathbb{K}$;

(4) $|p(x) - p(y)| \leq p(x - y)$ *for all* $y \in X$;

(5) $p(\lambda x) \leq p(\mu x)$ *for all* $\lambda, \mu \in \mathbb{K}$ *with* $|\lambda| \leq |\mu|$;

(6) $p(\lambda x) \leq n\, p(x)$ *for all* $\lambda \in \mathbb{K}$ *and* $n \in \mathbb{N}$ *with* $|\lambda| \leq n$;

(7) $\frac{1}{|k|} p(l x) \leq p\left(\frac{l}{k} x\right) \leq |l|\, p\left(\frac{1}{k} x\right)$ *for all* $k, l \in \mathbb{Z}$ *with* $k \neq 0$.

Proof. By Definition 6.9 (1), we evidently have

$$p(0) = \lim_{\lambda \to 0} p(0) = \lim_{\lambda \to 0} p(\lambda 0) = 0.$$

Moreover, if $\lambda, \mu \in \mathbb{K}$ such that $|\lambda| \leq |\mu|$ and $\mu \neq 0$, then by using Definition 6.9 (2) we can see that $p\big(\lambda x\big) = p\big((\lambda/\mu)\mu x\big) \leq p\big(\mu x\big)$. Hence, since $|\lambda| \leq |\mu|$ and $\mu = 0$ imply $\lambda = 0$, it is clear (5) is also true.

Now, by (5) and the inequalities $|\lambda| \leq ||\lambda|| \leq |\lambda|$, it is clear that in particular we also have $p(\lambda x) \leq p\big(|\lambda| x\big) \leq p(\lambda x)$. Therefore, (3) is also true.

Moreover, if $n \in \mathbb{N}$ such that $|\lambda| \leq n$, then by (5) and Theorem 3.3, it is clear that $p(\lambda x) \leq p(n x) \leq n p(x)$ also holds.

Finally, to complete the proof, we note that by (3) p is, in particular, even. Therefore, by Propositions 5.3 and 5.10 and Theorem 5.8, assertions (2), (4) and (7) are also true. □

Remark 6.12. From the above proof, it is clear that if p is a subadditive function of a vector space X over $\mathbb{K}$ such that in addition to Definition 6.9 (2) we only have $\inf_{\lambda \neq 0} p(\lambda x) \leq 0$ for all $x \in X$, then p is already a preseminorm.

Finally, we note that by using Theorem 6.11 (2) and (6) we can also prove

Corollary 6.13. *If p is a nonzero preseminorm on a one-dimensional vector space X over $\mathbb{K}$, then p is necessarily a prenorm on X.*

Proof. Namely, if this not the case, then there exists $x \in X$ such that $x \neq 0$ and $p(x) = 0$. Hence, by using $\dim(X) = 1$, we can see that $X = \mathbb{K} x$. Moreover, if $\lambda \in \mathbb{K}$, then by choosing $n \in \mathbb{N}$ such that $|\lambda| \leq n$ we can see that $0 \leq p(\lambda x) \leq n p(x) = 0$, and thus $p(\lambda x) = 0$. Therefore, p is identically zero, which is a contradiction. □

Remark 6.14. The importance of preseminorms lies mainly in the fact that in contrast to seminorms, a nonzero preseminorm can be bounded by Remark 6.10.

Thus, by an idea of Fréchet, any sequence $(p_n)_{n=1}^{\infty}$ preseminorms on X can be replaced by a single preseminorm $q = \sum_{n=1}^{\infty} (1/2^n) p_n^*$ which induces the same topology on X.

In this respect, it is also worth mentioning that, in contrast to seminorms, each vector topology on X can be derived from a family preseminorms on X. (See, for instance, [14].)

References

[1] Berz, E., Sublinear functions on $\mathbb{R}$, *Aequationes Math.*, Vol. 12 (1975), 200–206.

[2] Buskes, G., The Hahn–Banach theorem surveyed, *Dissertationes Math.*, Vol. 327, (1993) 1–49.

[3] Bruckner, A. M., Minimal superadditive extensions of superadditive functions, *Pacific J. Math.*, Vol. 10 (1960), 1155–1162.

[4] Bruckner, A. M., Some relationships between locally superadditive functions and convex functions, *Proc. Amer. Math. Soc.*, Vol. 15 (1964), 61–65.

[5] Cooper, R., The converses of the Cauchy–Hölder inequality and the solutions of the inequality $g(x+y) \leq g(x) + g(x)$, *Proc. London Math. Soc.*, Vol. 26 (1927), 415–432.

[6] Gajda, Z., Kominek, Z., On separation theorems for subadditive and superadditive functions, *Studia Math.*, Vol. 100 (1991), 25–38.

[7] Hille, E., Functional analysis and Semi-Groups, *Amer. Math. Soc. Coll. Publ.*, Vol. 31, New York 1948.

[8] Jolly, R. F., Concerning periodic subadditive functions, *Pacific J. Math.*, Vol. 15 (1965), 159–169.

[9] Kuczma, M., An Introduction to the Theory of Functional Equations and Inequalities, *Państwowe Wydawnictwo Naukowe*, Warszawa 1985.

[10] Laatsch, R. G., Extensions of subadditive functions, *Pacific J. Math.*, Vol. 14 (1964), 209–215.

[11] Matkowski, J., Subadditive functions and a relaxation of the homogeneity condition of seminorms, *Proc. Amer. Math. Soc.*, Vol. 117 (1993), 991–1001.

[12] Matkowski, J., On subadditive functions and ψ–additive mappings, *Central European J. Math.*, Vol. 4 (2003), 435–440.

[13] Matkowski, J., Światkowski, T., On subadditive functions, *Proc. Amer. Math. Soc.*, Vol. 119 (1993), 187–197.

[14] Metzler, R., Nakano, H., Quasi-norm spaces, *Trans. Amer. Math. Soc.*, Vol. 123 (1966), 1–31.

[15] Nikodem, K., Sadowska, E., Wasowicz, S., A note on separation by subadditive and sublinear functions, *Ann. Math. Sil.*, Vol. 14 (2000), 33–39.

[16] Rathore, S. P. S., On subadditive and superadditve functions, *Amer. Math. Monthly*, Vol. 72 (1965), 653–654.

[17] Rosenbaum, R. A., Sub-additive functions, *Duke Math. J.*, Vol. 17 (1950), 227–247.

[18] Száz, Á., Preseminormed spaces, *Publ. Math.*, Debrecen Vol. 30 (1983), 217–224.

[19] Száz, Á., The intersection convolution of relations and the Hahn–Banach type theorems, *Ann. Polon. Math.*, Vol. 69 (1998), 235–249.

[20] Száz, Á., Preseminorm generating relations and their Minkowski functionals, *Publ. Elektrotehn. Fak., Ser. Math., Univ. Beograd*, Vol. 12 (2001), 16–34.

[21] Wong, J. S. W., A note on subadditive functions, *Proc. Amer. Math. Soc.*, Vol. 44 (1974), 106.

Burai Pál
H-4010 Debrecen, Pf. 12, Hungary

Száz Árpád
H-4010 Debrecen, Pf. 12, Hungary