Some special cases of a general convergence rate theorem in the law of large numbers

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Abstract

Tómács in [6] proved a general convergence rate theorem in the law of large numbers for arrays of Banach space valued random elements. We shall study this theorem in case Banach space of type $\varphi$ and for two special arrays.

Key Words: Convergence rates; Arrays of Banach space valued random variables; Banach space of type $\varphi$

1. Introduction and notation

Let $\mathbb{N}$ be the set of the positive integers and $\mathbb{R}$ the set of real numbers. Let $\Phi_0$ denote the set of functions $f: [0, \infty) \to [0, \infty)$, that are nondecreasing. A function $f \in \Phi_0$ is said to satisfy the $\Delta_2$-condition ($f \sim \Delta_2$) if there exists a constant $c > 0$ such that $f(2t) \leq cf(t)$ for all $t > 0$.

Let $B$ be a real separable Banach space with norm $\|\cdot\|$ and zero element $0$. If $X$ is a $B$-valued random variable (r.v.) and $E\|X\| < \infty$ then $EX$ stands for the Bochner integral of $X$.

Throughout the paper let $\{k_n, n \in \mathbb{N}\}$ be a strictly increasing sequence of positive integers. Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of $B$-valued r.v.’s. It is rowwise independent, if $X_{n1}, \ldots, X_{nk_n}$ are independent r.v.’s for any fixed $n \in \mathbb{N}$. Let $S_{k_n} = \sum_{k=1}^{k_n} X_{nk}$. If $k_n = n$ for all $n$, then we denote $S_{k_n}$ by $S_n$. This corresponds to the case of ordinary sequences.

The array $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ is said to be bounded in probability if for all $\varepsilon > 0$ there exists $A > 0$ such that $P(\|X_{nk}\| \geq A) < \varepsilon$ for all $n \in \mathbb{N}$ and $k = 1, \ldots, k_n$.

The following remark give a sufficient condition for the boundedness in probability.
Remark 1.1. If there exists a constant $M > 0$ such that $E\|X_{nk}\| \leq M$ for every $n \in \mathbb{N}, k = 1, \ldots, k_n$, then the array $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ is bounded in probability. (The reader can readily verify this statement.)

Definition 1.2 (Gut [2]). We say that the array $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ is weakly mean dominated (w.m.d.) by the r.v. $X$, if for some $\gamma > 0$,

$$
\frac{1}{k_n} \sum_{k=1}^{k_n} P(\|X_{nk}\| > t) \leq \gamma P(|X| > t) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad n \in \mathbb{N}.
$$

The following theorem a general convergence rate theorem, which is proved in [6].

Theorem 1.3 (Tómács [6], Theorem 3.1). Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, n\}$ be an array of rowwise independent $B$-valued r.v.'s which is w.m.d. by the r.v. $X$. Assume that there exists a sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers such that $\{\|S_n\| / \gamma_n, n \in \mathbb{N}\}$ is bounded in probability. Let $\alpha, \vartheta, \varphi \in \Phi_0$, and assume that $\alpha$ is not bounded, $\vartheta, \varphi \sim \Delta_2$, $\vartheta \neq 0$. Let $\beta(n) = \varphi(\alpha(n + 1)) - \varphi(\alpha(n)), n = 0, 1, 2, \ldots$. Assume that $\mathbb{E}\varphi(|X|) < \infty$, $\mathbb{E}\vartheta(|X|) < \infty$ and $\lim_{n \to \infty} \alpha(n)/\gamma_n = \infty$.

Let either $\mu(n) = \beta(n - 1)$ for all $n \in \mathbb{N}$ or $\mu(n) = \beta(n)$ for all $n \in \mathbb{N}$. In second case assume that there exists a constant $c > 0$ such that for $n \in \mathbb{N}$ large enough $c\beta(n) \leq \beta(n - 1)$.

Let $n_0 \in \mathbb{N}$ be such that $\vartheta(\alpha(n)) > 0$ for all $n \geq n_0$. If there exist $j \in \mathbb{N}$ and $r > 0$ such that

$$
\sum_{n=n_0}^{\infty} \frac{\mu(n)}{n} \left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))}\right)^{2^j} < \infty,
$$

then

$$
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} P(\|S_n\| > \varepsilon \alpha(n)) < \infty \quad \text{for all} \quad \varepsilon > 0.
$$

In the following two corollaries of Theorem 1.3 we use some special notations: Following Gut [1], introduce the functions $\psi$ and $M_r$ with

$$
\psi(t) = \text{Card}\{n \in \mathbb{N} : k_n \leq t\} \quad \text{for} \quad t \geq 0,
$$

and

$$
M_r(t) = \begin{cases} 
\sum_{i=1}^{[t]} k_i^{r-1} & \text{if} \quad t \geq 1 \\
1 & \text{if} \quad 0 \leq t < 1,
\end{cases}
$$

where $r \in \mathbb{R}$, Card$A$ is the cardinality of the set $A$ and $[.]$ denotes the integer function. Let $M = M_2$. Let $f \circ g$ be the composite function of functions $f$ and $g$.

Remark 1.4. $M_r \circ \psi \in \Phi_0$ and

$$
(M_r \circ \psi)(t) = M_r(\psi(t)) = \begin{cases} 
\sum_{i=1}^{k_n} k_i^{r-1} = M_r(n), & \text{if} \quad k_n \leq t < k_{n+1}, \\
\sum_{i=1}^{k_1} k_i^{r-1} = M_r(1), & \text{if} \quad 0 \leq t < k_1.
\end{cases}
$$
The following corollary is a generalization of Theorem 6.2 of Fazekas [5].

**Corollary 1.5** (Tómács [6], Corollary 3.2). Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\} \) be an array of rowwise independent \( B \)-valued r.v.'s which is w.m.d. by the r.v. \( X \). Let \( M \circ \psi \sim \Delta_2 \), \( r, s, t > 0 \), \( rs > t \). Assume that \( \{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbb{N}\} \) is bounded in probability. Furthermore, if \( r > 2 \) we assume that \( \{M(n)/M(n-1), n \in \mathbb{N}\} \) is bounded. If \( E M^{r/2}(\psi(|X|^{t/r})) < \infty \) and \( E |X|^s < \infty \), then

\[
\sum_{n=1}^{\infty} \frac{M(n)}{(M(n))^{r/2-1}} P \left( \frac{\|S_{k_n}\|}{\varepsilon k_n^{1/t}} > \varepsilon k_n^{1/r} \right) < \infty \quad \text{for all } \varepsilon > 0.
\]

The following corollary is a version of Corollary 4.1 of Hu et al. [3].

**Corollary 1.6** (Tómács [6], Corollary 3.3). Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\} \) be an array of rowwise independent \( B \)-valued r.v.'s which is w.m.d. by the r.v. \( X \). Let \( r \in \mathbb{R}, 0 < t < s \) and \( M_r \circ \psi \sim \Delta_2 \). Assume that \( \{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbb{N}\} \) is bounded in probability. If \( E M_r(\psi(|X|^t)) < \infty \) and \( E |X|^s < \infty \), then

\[
\sum_{n=1}^{\infty} k_n^{s-2} P \left( \frac{\|S_{k_n}\|}{\varepsilon k_n^{1/2}} > \varepsilon k_n^{1/2} \right) < \infty \quad \text{for all } \varepsilon > 0.
\]

In Section 2 we give a sufficient condition for the boundedness in probability and in Section 3 we study two concrete sequences \( k_n \) in Corollary 1.5 and 1.6.

## 2. The boundedness in probability in case Banach space of type \( \varphi \)

If \( B \) has an appropriate geometric property, then a moment condition can imply the boundedness of \( \{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\} \).

**Definition 2.1.** A function \( \varphi : [0, \infty) \to [0, \infty) \) is said to be an *Orlicz function* if it is continuous, convex, \( \varphi(0) = 0 \), \( \varphi(t) > 0 \) for \( t > 0 \) and \( \lim_{t \to \infty} \varphi(t) = \infty \). For an Orlicz function \( \varphi \) the *Orlicz space* \( l_\varphi(B) \) consists of those \( B \)-valued sequences \( \{u_n, n \in \mathbb{N}\} \) for which

\[
\sum_{n=1}^{\infty} \varphi \left( \frac{\|u_n\|}{a} \right) < \infty \quad \text{for some } a > 0.
\]

Let \( \varepsilon_1, \varepsilon_2, \ldots \) be independent r.v.’s with \( P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2 \) for all \( n \in \mathbb{N} \). \( B \) is said to be of type \( \varphi \), if \( \sum_{n=1}^{\infty} \varepsilon_n u_n \) converges in probability for all \( \{u_n, n \in \mathbb{N}\} \in l_\varphi(B) \).
Definition 2.2. An Orlicz function $\varphi$ is said to satisfy the $\Delta^0_2$-condition ($\varphi \sim \Delta^0_2$) if there exist constants $c > 0$ and $t_0 > 0$ such that $\varphi(2t) \leq c\varphi(t)$ is satisfied for all $0 \leq t \leq t_0$.

Lemma 2.3. Let $\varphi$ be an Orlicz function and $\varphi \sim \Delta^0_2$. $B$ is of type $\varphi$ iff there exists a constant $c > 0$ such that

$$E \left\| \sum_{k=1}^{n} X_k \right\| \leq cE \inf_{y > 0} \left\{ \frac{1}{y} \left( 1 + \sum_{k=1}^{n} \varphi(y \|X_k\|) \right) \right\}$$

for all $n \in \mathbb{N}$ and every independent $B$-valued r.v.’s $X_1, \ldots, X_n$ with $EX_k = 0$, $k = 1, \ldots, n$.

For the proof see Fazekas [4].

The following lemma is a generalization of Lemma 2.1 of Gut [2] and Lemma 2.7 (b) of Fazekas [5].

Lemma 2.4 (Tómács [6], Lemma 4.4). Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of $B$-valued r.v.’s which is w.m.d. by the r.v. $X$ and constant $\gamma$. If $\varphi \in \Phi_0$ then

$$\frac{1}{k_n} \sum_{k=1}^{k_n} E \varphi(\|X_{nk}\|) \leq \max\{1, \gamma\} E \varphi(|X|).$$

The following theorem show that in Theorem 1.3 we can write moment conditions instead of the boundedness of $\{\|S_{k_n}\| / \gamma_{k_n}, n \in \mathbb{N}\}$ if $B$ is of type $\varphi$.

Theorem 2.5. Let $\varphi \in \Phi_0$ be a submultiplicative Orlicz function, $\varphi \sim \Delta^0_2$ and let $B$ be a space of type $\varphi$. Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of rowwise independent $B$-valued r.v.’s which is w.m.d. by the r.v. $X$. Assume that the sequence $\{k_n \varphi(1 / \gamma_{k_n}), n \in \mathbb{N}\}$ is bounded for some sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers. If $EX_{nk} = 0$ for every $n \in \mathbb{N}$, $k = 1, \ldots, k_n$ and $E \varphi(|X|) < \infty$, then $\{\|S_{k_n}\| / \gamma_{k_n}, n \in \mathbb{N}\}$ is bounded in probability.

Proof. By Lemma 2.3 and 2.4 there exists a constant $c > 0$ such that

$$E \frac{\|S_{k_n}\|}{\gamma_{k_n}} \leq \frac{c}{\gamma_{k_n}} E \inf_{y > 0} \left\{ \frac{1}{y} \left( 1 + \sum_{k=1}^{k_n} \varphi(y \|X_{nk}\|) \right) \right\}$$

$$\leq cE \left( 1 + \sum_{k=1}^{k_n} \varphi(\|X_{nk}\| / \gamma_{k_n}) \right)$$

$$\leq c(1 + \varphi(1 / \gamma_{k_n}) \max\{1, \gamma\} k_n E \varphi(|X|)).$$

Thus Remark 1.1 implies the statement. □
3. Convergence rate theorems for two concrete sequences $k_n$

Lemma 3.1. \( f \sim \Delta_2 \) iff there exist constants \( k > 1 \) and \( c > 0 \) such that

\[
f(kt) \leq cf(t) \quad \text{for all} \quad t > 0.
\]

(3.1)

Proof. If \( f \sim \Delta_2 \) then in case \( k = 2 \) we get (3.1). Now suppose that there exist constants \( k > 1 \) and \( c > 0 \) such that the inequality (3.1) is true for all \( t > 0 \). Then we can obtain with induction that

\[
f(k^n t) \leq c^n f(t) \quad \text{for all} \quad t > 0 \quad \text{and for all} \quad n \in \mathbb{N}.
\]

It follows that there exists \( n_0 \in \mathbb{N} \) such that

\[
f(2^n t) \leq f(k^{n_0} t) \leq c^{n_0} f(t) \quad \text{for all} \quad t > 0.
\]

Thus we get \( f \sim \Delta_2 \). \( \Box \)

The reader can readily verify the following lemma.

Lemma 3.2. Let \( g: [k_1, \infty) \to \mathbb{R} \) be a nondecreasing function which has the property that \( g(k_n) \geq M_r(n) \) for all \( n \in \mathbb{N} \). Then \( M_r(\psi(x)) \leq g(x) \) for all \( x \geq k_1 \).

Lemma 3.3. Let \( r \in \mathbb{R} \). Assume that there exists strictly increasing sequence \( \{a_n, n \in \mathbb{N}\} \) of positive integers and there exist constants \( k > 1, c > 0 \) such that

\[
\frac{k_n}{k_{n+1}} \leq \frac{1}{k} \quad \text{and} \quad \frac{M_r(a_n)}{M_r(n-1)} \leq c \quad \text{for all} \quad n \in \mathbb{N}.
\]

Then \( M_r \circ \psi \sim \Delta_2 \).

Proof. Assume that \( k_n \leq t < k_{n+1} \). Then Remark 1.4 implies

\[
M_r(\psi(kt)) \leq M_r(\psi(k_{n+1})) \leq M_r(\psi(k_{n+1})) = M_r(a_{n+1}) \leq cM_r(n) = cM_r(\psi(t)).
\]

Similarly if \( 0 < t < k_1 \) then

\[
M_r(\psi(kt)) \leq M_r(\psi(kk_1)) \leq M_r(\psi(k_1)) = M_r(a_1) \leq cM_r(0) = cM_r(\psi(t)).
\]

It follows that \( M_r(\psi(kt)) \leq cM_r(\psi(t)) \) for all \( t > 0 \). Thus, by Lemma 3.1 we get the statement. \( \Box \)

Lemma 3.4. Let \( l \in \mathbb{N} \). Then

\[
\lim_{n \to \infty} \frac{1^k + 2^k + \cdots + (ln)^k}{1^k + 2^k + \cdots + (n-1)^k} = \begin{cases} 
  l^{k+1}, & \text{if} \quad k > -1, \\
  1, & \text{if} \quad k \leq -1.
\end{cases}
\]
Proof. It is easy to see that
\[ x^{k+1} - (x - 1)^{k+1} \leq (k + 1) x^k \leq (x + 1)^{k+1} - x^{k+1} \quad \text{for all } \ x \geq 1, \ k \geq 0 \]
and
\[ (x + 1)^{k+1} - x^{k+1} \leq (k + 1) x^k \leq x^{k+1} - (x - 1)^{k+1} \quad \text{for all } \ x \geq 1, \ -1 < k < 0. \]
Apply these inequalities for \( x = 1, 2, \ldots, n \). Then we have
\[ \lim_{n \to \infty} \frac{1 + 2^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1} \quad \text{for all } \ k > -1, \]
which implies the statement for \( k > -1 \).

It is well known that \( \frac{1}{1} + \frac{1}{c^2} + \cdots + \frac{1}{c^n} \) is convergent if \( c > 1 \). It follows that the statement is true in case \( k < -1 \) as well.

Finally in case \( k = -1 \) the inequalities
\[ 1 + \frac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}} < \frac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}} < 1 + \frac{l}{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}} \]
imply the statement. \( \square \)

Lemma 3.5. Let \( k_1, d \in \mathbb{N} \), \( q \in \mathbb{N} \setminus \{1\} \). If \( k_n = k_1 q^{n-1} \) or \( k_n = k_1 n^d \) then
\( M_r \circ \psi \sim \Delta_2 \) for all \( r \in \mathbb{R} \).

Proof. In the first case, when \( k_n = k_1 q^{n-1} \), let \( a_n = n + 1 \) and \( k = q \). Then
\[ \frac{k_n}{k_{a_n}} = \frac{k_1 q^{n-1}}{k_1 q^n} = \frac{1}{q} \leq \frac{1}{k} \]
Let \( Q = q^r-1 \) and assume that \( r > 1 \). In this case \( |1/Q| < 1 \), thus we get
\[ \frac{M_r(a_n)}{M_r(n-1)} = \frac{M_r(n + 1)}{M_r(n-1)} = \frac{1 + Q + \cdots + Q^n}{1 + Q + \cdots + Q^{n-2}} = \frac{Q^2 - \frac{1}{Q^{n-2}}}{1 - \frac{1}{Q^{n-2}}} \to Q^2. \]
If \( r < 1 \) then \( 1/Q > 1 \), thus
\[ \frac{M_r(a_n)}{M_r(n-1)} = \frac{Q^2 - \frac{1}{Q^{n-2}}}{1 - \frac{1}{Q^{n-2}}} \to 1. \]
If \( r = 1 \) then \( Q = 1 \), so
\[ \frac{M_r(a_n)}{M_r(n-1)} = \frac{n + 1}{n - 1} \to 1. \]
Thus we get that \( \frac{M_r(a_n)}{M_r(n-1)} \) is bounded for all \( r \in \mathbb{R} \). Hence conditions of Lemma 3.3 are satisfied, which implies the statement.
In the second case, when \( k_n = k_1 n^d \), let \( a_n = 2n \) and \( k = 2^d \). Then
\[
\frac{k_n}{k_n} = \frac{k_1 n^d}{k_1 (2n)^d} = \frac{1}{2^d} \leq \frac{1}{k}.
\]

On the other hand it follows from Lemma 3.4 that \( \frac{M(n)}{M(n-1)} \) is bounded for all \( r \in \mathbb{R} \). So Lemma 3.3 implies the statement. \( \square \)

**Theorem 3.6.** Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_1 n^d\} \) \( (k_1, d \in \mathbb{N} \) are fixed) be an array of rowwise independent \( B \)-valued r.v.’s which is w.m.d. by the r.v. \( X \). Let \( t > 0, r \geq 2d/(d+1), s > t/r \) and \( v = \max\{s, t(d+1)/(2d)\} \). If \( \{\|S_{k_1 n^d}\|/n^d, n \in \mathbb{N}\} \) is bounded in probability and \( E |X|^v < \infty \), then
\[
\sum_{n=1}^{\infty} n^{(d+1)(r/2-1)} P \left( \|S_{k_1 n^d}\| > \varepsilon n^{dr/t} \right) < \infty \quad \text{for all} \quad \varepsilon > 0.
\]

**Proof.** We shall prove that conditions of Corollary 1.5 are satisfied. Let \( k_n = k_1 n^d \). Then by Lemma 3.4 \( \{M(n)/M(n-1), n \in \mathbb{N}\} \) is bounded. Let \( Y = M^{r/2}(\psi([X[^t/r]])) \). Now we turn to the proof of \( EY < \infty \). It is well known that
\[
1^d + \cdots + n^d = a_1 n^{d+1} + a_2 n^d + \cdots + a_{d+2}
\]

for some \( a_1, a_2, \ldots, a_{d+2} \in \mathbb{R} \). Let
\[
g: [k_1, \infty) \to \mathbb{R}, \quad g(x) = \sum_{i=1}^{d+2} |a_i| (k_1^{-2} x^{d+2-i})^{1/d}.
\]

Then \( g \) is nondecreasing, \( g(k_n) \geq M(n) \) and \( g(x) \leq \text{const.} x^{(d+1)/d} \) for all \( x \geq k_1 \). Therefore by Lemma 3.2 we have
\[
M^{r/2}(\psi(x^{t/r})) \leq \text{const.} x^{(d+1)/(2d)} \quad \text{for all} \quad x^{t/r} \geq k_1.
\]

It follows that
\[
Y = YI([X[^t/r] < k_1]) + YI([X[^t/r] \geq k_1]) \leq k_1^{r/2} + \text{const.} |X[^{(d+1)/(2d)}],
\]
where \( I(A) \) denotes the indicator function of the set \( A \). So \( EY < \infty \). By Lemma 3.5 \( M \circ \psi \sim \Delta_2 \). It is easy to see that the other conditions of Corollary 1.5 hold true as well, on the other hand \( M(n) \geq \text{const.} n^{d+1} \). So this theorem is consequence of Corollary 1.5. \( \square \)

**Theorem 3.7.** Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_1 q^{n-1}\} \) \( (k_1 \in \mathbb{N}, q \in \mathbb{N} \setminus \{1\} \) are fixed) be an array of rowwise independent \( B \)-valued r.v.’s which is w.m.d. by the r.v. \( X \). Let \( w \geq 0, t > 0, s > t \) and \( v = \max\{s, t(w+1)\} \). If \( \{\|S_{k_1 q^{n-1}}\|/q^{n/t}, n \in \mathbb{N}\} \) is bounded in probability and \( E |X|^v < \infty \), then
\[
\sum_{n=1}^{\infty} q^{nu} P \left( \|S_{k_1 q^{n-1}}\| > \varepsilon q^{n/t} \right) < \infty \quad \text{for all} \quad \varepsilon > 0.
\]
Proof. We shall prove that conditions of Corollary 1.6 are satisfied. Let $k_n = k_1 q^{n-1}$, $r = w + 2$ and $Y = M_r(\psi(|X|^t))$. Then $M_r(n) = k_1^{r-1} Q^{1-1/Q-1}$, where $Q = q^{r-1}$. Let
\[ g: [k_1, \infty) \to \mathbb{R}, \quad g(x) = k_1^{r-1} \frac{Q^{1+\log(x/k_1)/\log q - 1}}{Q - 1}. \]
Then $g$ is nondecreasing, $g(k_n) = M_r(n)$ and $g(x) \leq \text{const.} x^{r-1}$ for all $x \geq k_1$. Therefore by Lemma 3.2 we have
\[ M_r(\psi(x^t)) \leq \text{const.} x^{t(w+1)} \quad \text{for all} \quad x^t \geq k_1. \]
It follows that
\[ Y = Y I(|X|^t < k_1) + Y I(|X|^t \geq k_1) \leq k_1^{r-1} + \text{const.} |X|^t(w+1). \]
So $EY < \infty$. By Lemma 3.5 $M_r \circ \psi \sim \Delta_2$. The other conditions of Corollary 1.6 hold true as well. Thus Corollary 1.6 implies the statement.

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