

A remark on Rainwater's theorem

Olav Nygaard

Agder University College
olav.nygaard@hia.no

Abstract

We define Rainwater sets as subsets of the dual of a Banach space for which Rainwater's theorem holds and show that (I)-generating subsets have this property. We apply this observation to give a proof of James' theorem when the dual unit ball is sequentially compact in its weak-star topology.

Key Words: Rainwater set, James boundary, (I)-generation

AMS Classification Number: 46B20

1. (I)-generating sets and Rainwater's theorem

For a subset B in the dual unit ball B_{X^*} of a Banach space X , in [2] a property was localized between the properties $\overline{\text{conv}}(B) = B_{X^*}$ and $\overline{\text{conv}}^{w^*}(B) = B_{X^*}$:

Definition 1.1. B is said to (I)-generate B_{X^*} if whenever B is written as a countable union, $B = \bigcup B_i$, then $B_{X^*} = \overline{\text{conv}}(\bigcup_i \overline{\text{conv}}^{w^*}(B_i))$.

Note the following equivalent definition: Whenever B is written as an increasing countable union $B = \bigcup B_i \uparrow$, then $\bigcup_i \overline{\text{conv}}^{w^*}(B_i)$ is norm-dense in B_{X^*} . (I)-generation of course makes sense in any w^* -compact convex subset of X^* .

Recall that a set $B \subset B_{X^*}$ is called a *James boundary* if, for every $x \in X$, the maximum over B_{X^*} is attained on B . As a standard example, for any Banach space X the extreme points of B_{X^*} is a James boundary. The fundamental result from [2] is the following:

Theorem 1.2 ([2, Thm. 2.3]). *If B is a James boundary, then B (I)-generates B_{X^*} . The same is true for a James boundary in any w^* -compact convex subset of X^* .*

Note how this theorem both generalizes and sharpens the Krein-Milman theorem in this situation. It generalizes because it works for any James boundary and

sharpens because (I)-generation is a stronger property than $\overline{\text{conv}}^{w^*}(B) = B_{X^*}$ as a simple example in [2] shows. If B is separable and (I)-generates, then we already have $\overline{\text{conv}}(B) = B_{X^*}$.

In 1963 (see [3] or [1, p. 155]) the following theorem was published under the pseudonym J. Rainwater: *For a bounded sequence in a Banach space X to converge weakly it is enough that it converges pointwise on the extreme points of the unit ball in the dual, B_{X^*} .* The proof is an application of Choquet's theorem. Later on S. Simons (see [4] or [5]) gave a completely different argument to show that Rainwater's theorem is true with any James boundary.

Definition 1.3. Let X be a Banach space. A subset B of B_{X^*} is called a Rainwater set if every bounded sequence that converges pointwise on B converges weakly.

Rainwater's original theorem then reads: The extreme points of B_{X^*} is a Rainwater set. Simons' more general version reads: Any James boundary is a Rainwater set. We want in this little note just to remark the simple but general fact that (I)-generating sets are Rainwater sets and give an application of this observation to a proof of James' theorem in a rather wide class of Banach spaces.

Theorem 1.4. *Let X be a Banach space. Suppose B (I)-generates B_{X^*} . Then B is a Rainwater set.*

Proof. Let (x_i) be a bounded sequence in X . Let M be such that $\|x_i\|, \|x\| \leq M$ for all i . Pick an arbitrary $x^* \in B_{X^*}$ and let $\varepsilon > 0$. Define

$$B_i = \{y^* \in B : \forall j \geq i, |y^*(x_j - x)| < \varepsilon\}.$$

Then, since $y^*(x_i) \rightarrow y^*(x)$ for every $y^* \in B$, (B_i) is an increasing covering of B .

Since B (I)-generates, there is a y^* in some $\overline{\text{conv}}^{w^*}(B_N)$ such that $\|x^* - y^*\| < \varepsilon$. Note that for every $y^* \in \overline{\text{conv}}^{w^*}(B_N)$, $j \geq N$ implies that $|y^*(x_j - x)| \leq \varepsilon$. Now, the triangle inequality show that for $j \geq N$

$$\begin{aligned} |x^*(x_j - x)| &\leq |x^*(x_j) - y^*(x_j)| + |y^*(x_j) - y^*(x)| + |y^*(x) - x^*(x)| \\ &\leq (1 + 2M)\varepsilon, \end{aligned}$$

and hence (x_i) converges weakly to x . □

Note that Simons' version of Rainwater's theorem follows from Theorem 1.4 and Theorem 1.2. Remark also that completeness is not needed in Definition 1.3 and also not in Theorem 1.4.

2. A proof of James' theorem when the dual unit ball is weak-star sequentially compact

Recall James famous characterization of reflexive spaces: *If every $x^* \in X^*$ attains its supremum over B_X , then X is reflexive.* In other words, if S_X is a

James boundary for $B_{X^{**}}$, then $B_X = B_{X^{**}}$. We now prove this result when B_{X^*} is sequentially compact in its weak-star topology. Such spaces are discussed in [1, Chapter XIII], the basic result being the Amir-Lindenstrauss theorem telling us that any subspace of a weakly compactly generated space is of this type.

Here is the argument: Suppose every $x^* \in X^*$ attains its supremum over S_X . Then S_X is a James boundary of $B_{X^{**}}$. Thus, from Theorem 1.2 and 1.4, X is a Grothendieck space, that is, weak and w^* -convergence of (bounded) sequences coincide in X^* . Since B_{X^*} is w^* -sequentially compact it is weakly sequentially compact and hence, by Eberlein's theorem, weakly compact. Hence X^* , and thus X , is reflexive.

Whether it is true in general that X is reflexive whenever S_X (I)-generates $B_{X^{**}}$ is to my best knowledge an open question. Let us end this little note by analyzing this problem a little more:

Definition 2.1. A Banach space X where S_X (I)-generates $B_{X^{**}}$ is called an (I)-space.

By Theorem 1.4 it is clear that (I)-spaces are Grothendieck spaces. The point in the proof of James' theorem in the sequentially weak-star compact dual unit ball case is that Grothendieck together with weak-star compact dual unit ball imply reflexivity, by Eberlein's theorem. The standard example of a non-reflexive Grothendieck space is ℓ_∞ . A starting point in characterizing (I)-spaces should be to decide whether ℓ_∞ is an (I)-space or not. But even this is a hard task since we have no description of ℓ_∞^{**} .

References

- [1] J. DIESTEL, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics, vol. 92, Springer, 1984.
- [2] V. P. FONF AND J. LINDENSTRAUSS, *Boundaries and generation of convex sets*, Israel. J. Math. **136** (2003) 157-172.
- [3] J. RAINWATER, *Weak convergence of bounded sequences*, Proc. Amer. Math. Soc. **14** (1963) 999.
- [4] S. SIMONS *A convergence theory with boundary*. Pacific J. Math. **40** (1972) 703-708
- [5] S. SIMONS *An Eigenvector Proof of Fatou's Lemma for Continuous Functions*. The Math. Int. **17** (No 3) (1995) 67-70

Olav Nygaard

Faculty of Mathematics and Science
 Agder University College
 Gimlemoen 25J
 Servicebox 422
 4604 Kristiansand
 Norway