# SHAPE MODIFICATION OF CUBIC B-SPLINE CURVES BY MEANS OF KNOT PAIRS 

Róbert Tornai (Debrecen, Hungary)


#### Abstract

The effect of the modification of not consecutive knot values on the shape of B -spline curves is examined in this paper. It is known that an envelope of the one-parameter family of B-spline curves of order $k$, obtained by the modification of a knot, is also a B-spline curve of the same control polygon and of order $k-m$, where $m$ is the multiplicity of the modified knot. An extension of shape modification methods are provided for cubic B-spline curves, that utilize this envelope. This paper extends the possibilities for choosing the new position of a point of the curve by allowing to modify knots that are not consecutive.


AMS Classification Number: 68U05

## 1. Introduction

Computer aided design widely use B-spline curves and their rational generalizations (NURBS curves) that play central role today. Besides, they are used in computer graphics and animations. These curves are excellent tools in design systems to create new objects, but the modification and shape control of the existing objects are also essential.

The data structure of a B-spline curve of order $k$ is fairly simple. It only consists of control points and knot values. Hence shape control methods can modify such curves only by altering these data. One of the most comprehensive books of this field is [9] where shape modifications, based on control point repositioning are also described. Some publications discuss shape modifications, e.g., [10] which present constraint-based curve manipulations of curves of arbitrary degree and basis functions. [11] proposes direct modification of free-form curves by displacement functions, which method comprises knot refinement and removal, control point repositioning and degree elevation.

Some aspects of knot modification is also been studied, like in [12] where the effect of knot variation is examined from numerical point of view. Several papers and articles investigate the choice of knot values in curve approximation and interpolation, cf. the recently published [13] and the references therein.

It is an obvious fact, that the modification of the knot vector affects the shape of the curve. Some results concerning the geometric aspects of knot modifications have already been presented by Juhász and Hoffmann for B-spline curves in [2], [3]
and [6], where the main result was the following: the one-parameter family of Bspline curves of order $k$, resulted by the modification of a knot, possesses an envelope which is also a B-spline curve of the same control polygon and of order $k-m$, where $m$ is the multiplicity of the modified knot. This envelope can be used for geometric constraint-based shape modification of cubic B-spline curves. This property forms the basis of constrained modification of the curve which first outlined in [1] and discussed in a detailed form in [4] and [7]. Further special shape control techniques discussed in [8]. In terms of surfaces the theoretical generalization of these theorems can be found in [5].

In this paper I extend the possibilities of a shape control method described in [4] and [7] by letting not necessarily neighboring knots to change.

## 2. Modifying a knot

Definition 1. The curve $\mathbf{s}(u)$ defined by

$$
\mathbf{s}(u)=\sum_{l=0}^{n} N_{l}^{k}(u) \mathbf{d}_{l}, u \in\left[u_{k-1}, u_{n+1}\right]
$$

is called B-spline curve of order $k$ (degree $k-1$ ), $(1<k \leq n+1)$, where $N_{l}^{k}(u)$ is the $l$ th normalized B-spline basis function of order $k$, for the evaluation of which the knots $u_{0}, u_{1}, \ldots, u_{n+k}$ are necessary. Points $\mathbf{d}_{l}$ are called control points or de Boor points, while the polygon formed by these points is called control polygon.

The $j$ th arc of the B-spline curve of Definition 1 is of the form

$$
\mathbf{s}_{j}(u)=\sum_{l=j-k+1}^{j} \mathbf{d}_{l} N_{l}^{k}(u), u \in\left[u_{j}, u_{j+1}\right),(j=k-1, \ldots, n)
$$

The modification of the knot value $u_{i}$ alters the shape of the $\operatorname{arcs} \mathbf{s}_{\mathbf{j}}(\mathbf{u}), j=$ $i-k+1, i-k+2, \ldots, i+k-2$. The point of such an arc that corresponds to an arbitrarily chosen parameter value $\tilde{u} \in\left[u_{j}, u_{j+1}\right)$ describes the curve

$$
\mathbf{s}_{j}\left(\tilde{u}, u_{i}\right)=\sum_{l=j-k+1}^{j} \mathbf{d}_{l} N_{l}^{k}\left(\tilde{u}, u_{i}\right), u_{i} \in\left[u_{i-1}, u_{i+1}\right] .
$$

In [2] Juhász and Hoffmann proved the following property.
Theorem 1. Altering a knot value $u_{i} \in\left[u_{i-1}, u_{i+1}\right)$ of a $B$-spline curve $\mathbf{s}(u)$ of order $k(k>2)$, the one-parameter family of $B$-spline curves

$$
\mathbf{s}\left(u, u_{i}\right)=\sum_{l=0}^{n} \mathbf{d}_{l} N_{l}^{k}\left(u, u_{i}\right), u \in\left[u_{k-1}, u_{n+1}\right]
$$

has an envelope which is a B-spline curve of order $(k-1)$ and can be written in the form

$$
\mathbf{h}(v)=\sum_{l=i-k+1}^{i-1} \mathbf{d}_{l} N_{l}^{k-1}(v), v \in\left[v_{i-1}, v_{i}\right]
$$

where the knot values

$$
v_{j}= \begin{cases}u_{j}, & \text { if } j<i \\ u_{j+1}, & \text { otherwise }\end{cases}
$$

i.e., from the knot values $\left\{u_{j}\right\}$ we have to leave out the $i$ th one, where the multiplicity of $u_{i}$ is one. Their points of contact are $\mathbf{h}\left(u_{i}\right)=\mathbf{s}\left(u_{i}, u_{i}\right)$.

For $k=4$ by the modification of the knot value $u_{i+1}$, we obtain a oneparameter family of cubic B-spline curves of the form

$$
\mathbf{s}\left(u, u_{j+1}\right)=\sum_{l=0}^{n} \mathbf{d}_{l} N_{l}^{4}\left(u, u_{j+1}\right), u \in\left[u_{3}, u_{n+1}\right], u_{j+1} \in\left[u_{j}, u_{j+2}\right)
$$

with knots $u_{0}, u_{1}, \ldots u_{n+4}$, and the envelope is the parabolic arc

$$
\begin{equation*}
\mathbf{h}_{j}(v)=\sum_{l=j-2}^{j} N_{l}^{3}(v) \mathbf{d}_{l}, v \in\left[v_{j}, v_{j+1}\right) \tag{1}
\end{equation*}
$$

with knots $v_{j-2}=u_{j-2}, v_{j-1}=u_{j-1}, v_{j}=u_{j}, v_{j+1}=u_{j+2}, v_{j+2}=u_{j+3}, v_{j+3}=$ $u_{j+4}$.

## 3. Move a point of the curve to a specified location

A generally accepted shape modification method is, when the user picks a point of the curve, then species a new location where the picked point has to be moved. Furthermore, let's assume that for the parameter of the picked point $\mathbf{s}(\tilde{u}), \tilde{u} \in$ $\left[u_{j}, u_{j+2}\right)$ holds. The new location will be denoted by $\mathbf{p}$, and its coordinates in the coordinate system $\left\{\mathbf{d}_{j-1} ; \mathbf{d}_{j-2}-\mathbf{d}_{j-1} ; \mathbf{d}_{j}-\mathbf{d}_{j-1}\right\}$ by $x$ and $y$. It is known (c.f. [1], [4], [7]) that the $\mathbf{s}(\tilde{u}) \rightarrow \mathbf{p}$ shape modification can be performed by the alteration of three consecutive knots of the curve $\mathbf{s}(u)$.

For the determination of the permissible positions of $\mathbf{p}$ the following has to be taken into account: in the Bézier representation of the envelope the value $\tilde{t}$ which corresponds to $\tilde{v}=\tilde{u}$ varies with the variation of the knots $v_{j}$ and $v_{j+1}$, since $\tilde{t}=\left(\tilde{v}-v_{j}\right) /\left(v_{j+1}-v_{j}\right)$. Therefore, the B-spline representation of the envelope can be used.

Utilizing that $N_{j-2}^{3}(v)+N_{j-1}^{3}(v)+N_{j}^{3}(v)=1, \forall v \in\left[v_{j}, v_{j+1}\right)$ Eq. (1) can be written in the form

$$
\mathbf{h}_{j}(v)=\mathbf{d}_{j-1}+N_{j-2}^{3}(v)\left(\mathbf{d}_{j-2}-\mathbf{d}_{j-1}\right)+N_{j}^{3}(v)\left(\mathbf{d}_{j}-\mathbf{d}_{j-1}\right)
$$

where

$$
\begin{aligned}
N_{j-2}^{3}(v) & =\frac{\left(v_{j+1}-v\right)^{2}}{\left(v_{j+1}-v_{j-1}\right)\left(v_{j+1}-v_{j}\right)} \\
N_{j}^{3}(v) & =\frac{\left(v-v_{j}\right)^{2}}{\left(v_{j+2}-v_{j}\right)\left(v_{j+1}-v_{j}\right)} .
\end{aligned}
$$

### 3.1. Examined areas so far

In [1], [4] and [7] three pairs of knots are allowed to change. These are $\left(v_{j-1}, v_{j}\right)$, $\left(v_{j}, v_{j+1}\right)$ and $\left(v_{j+1}, v_{j+2}\right)$. The corresponding permissible regions of $\mathbf{p}$ will be denoted by $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$ respectively. (In this case the aim is to minimize the number of altering arcs of $\mathbf{s}(u)$, so only the change of consecutive knots are allowed.) The boundary of sub regions $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$ are formed by paths that belong to different extreme positions of the point $\mathbf{h}(\tilde{v})$.
$\Omega_{1}$ is bounded by three paths. The first path is determined by letting $v_{j-2}=$ $v_{j-1}$ and varying $v_{j}$; the second by letting $v_{j}=\tilde{v}_{j}$ and varying $v_{j-1}$ and the third path is determined by letting $v_{j-1}=v_{j}$ and varying them simultaneously.
$\Omega_{2}$ is bounded by four paths. The first path is determined by letting $v_{j+1}=$ $v_{j+2}$ and varying $v_{j}$; the second by letting $v_{j+1}=\tilde{v}_{j}$ and varying $v_{j}$ the third by letting $v_{j}=v_{j-1}$ and varying $v_{j+1}$ and the fourth path is determined by letting $v_{j}=\tilde{v}_{j}$ and varying $v_{j+1}$.
$\Omega_{3}$ is bounded by three paths. The first path is determined by letting $v_{j+2}=$ $v_{j+3}$ and varying $v_{j+1}$; the second by letting $v_{j+1}=\tilde{v}_{j}$ and varying $v_{j+2}$ and the third path is determined by letting $v_{j+1}=v_{j+2}$ and varying them simultaneously.

These three overlapping regions are shown in Fig. 1. a), d), e).
Thus, if the point $\mathbf{p}$ is in the union of these three regions above, then the solution to the shape modification problem $\mathbf{s}(\tilde{u}) \rightarrow \mathbf{p}$ is guaranteed. In such a case, the number of solutions can be 1,2 or 3 depending on the position of $\mathbf{p}$ with respect to the regions $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$. In order to obtain the solutions, we have to solve the system of equations

$$
\begin{align*}
& x=\frac{\left(v_{j+1}-\tilde{v}\right)^{2}}{\left(v_{j+1}-v_{j-1}\right)\left(v_{j+1}-v_{j}\right)} \\
& y=\frac{\left(\tilde{v}-v_{j}\right)^{2}}{\left(v_{j+2}-v_{j}\right)\left(v_{j+1}-v_{j}\right)} \tag{2}
\end{align*}
$$

either for the pair of unknowns $\left(v_{j-1}, v_{j}\right)$ or for $\left(v_{j}, v_{j+1}\right)$ or for $\left(v_{j+1}, v_{j+2}\right)$. Only those solutions of Eq. (2) provide solutions to the shape modification problem which fulfills the monotonicity condition

$$
v_{j-1} \leq v_{j} \leq \tilde{v} \leq v_{j+1} \leq v_{j+2}
$$

as well. Such a solution always exists, when $\mathbf{p}$ is in the region that corresponds to the pair of unknowns, for which the system is solved.

### 3.2. New areas, that extend the possibilites

What is more interesting, we can choose not consecutive knots of the curve. This way we can reach points of three other regions. In this case three pairs of knots are allowed to change also. These are $\left(v_{j-1}, v_{j+1}\right),\left(v_{j-1}, v_{j+2}\right)$ and $\left(v_{j}, v_{j+2}\right)$. The corresponding permissible regions of $\mathbf{p}$ will be denoted by $\Omega_{4}, \Omega_{5}$ and $\Omega_{6}$ respectively. The boundary of sub regions are formed by paths that belong to different extreme positions of the point $\mathbf{h}(\tilde{v})$. These new three regions will overlap each other and unfortunately they mean only a little region compared to the union of $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$. Another disadvantage is that the union of $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$ overlaps mainly the union of $\Omega_{4}, \Omega_{5}$ and $\Omega_{6}$. In spite of all of these facts, these new solutions can be useful. They let greater freedom for the designer to modify the shape of a curve. Here we shall discuss these three regions. The detailed discussion of the permissible positions of the point, the parameter values and the unknowns can be found in [7].

### 3.2.1. $\Omega_{4}$ : the unknowns are $v_{j-1}$ and $v_{j+1}$

The boundaries of the permissible positions of the point $\mathbf{p}$ in this case are the paths connecting the following four extreme positions of a point of the quadratic B-spline curve $\operatorname{arc} \mathbf{b}_{j}(v),\left(v_{j-1} \in\left[v_{j-2}, v_{j}\right]\right.$ and $\left.v_{j+1} \in\left[\widetilde{u}, v_{j+2}\right]\right)$ :
(1) $v_{j-2}=v_{j-1}<v_{j}<\tilde{u}=v_{j+1}<v_{j+2}$
(2) $v_{j-2}=v_{j-1}<v_{j}<\widetilde{u}<v_{j+1}=v_{j+2}$
(3) $v_{j-2}<v_{j-1}=v_{j}<\widetilde{u}=v_{j+1}<v_{j+2}$
(4) $v_{j-2}<v_{j-1}=v_{j}<\widetilde{u}<v_{j+1}=v_{j+2}$.

The paths can be described similarly to the preceding case, but only three of them are actual boundaries, the other three paths run inside the region. The boundaries can be seen in Fig. 1. b).

### 3.2.2. $\Omega_{5}$ : the unknowns are $v_{j-1}$ and $v_{j+2}$

The boundaries in this case are straight line segments. The paths connect the following extreme positions:
(1) $v_{j-2}=v_{j-1}<v_{j}<\widetilde{u}<v_{j+1}=v_{j+2}<v_{j+3}$
(2) $v_{j-2}=v_{j-1}<v_{j}<\widetilde{u}<v_{j+1}<v_{j+2}=v_{j+3}$
(3) $v_{j-2}<v_{j-1}=v_{j}<\widetilde{u}<v_{j+1}=v_{j+2}<v_{j+3}$
(4) $v_{j-2}<v_{j-1}=v_{j}<\widetilde{u}<v_{j+1}<v_{j+2}=v_{j+3}$

In this case only four of the six paths form the boundary of the area that can be seen in Fig. 1. c).


Figure 1.

### 3.2.3. $\Omega_{6}$ : the unknowns are $v_{j}$ and $v_{j+2}$

Due to the symmetry this final case is similar to the one with the unknowns $v_{j-1}, v_{j+1}$. The four extreme cases can be described as follows $\left(v_{j} \in\left[v_{j-1}, \tilde{u}\right], v_{j+2} \in\right.$ $\left.\left[v_{j+1}, v_{j+3}\right]\right):$
(1) $v_{j-1}=v_{j}<\widetilde{u}<v_{j+1}=v_{j+2}<v_{j+3}$
(2) $v_{j-1}<v_{j}=\widetilde{u}<v_{j+1}=v_{j+2}<v_{j+3}$
(3) $v_{j-1}=v_{j}<\widetilde{u}<v_{j+1}<v_{j+2}=v_{j+3}$
(4) $v_{j-1}<v_{j}=\widetilde{u}<v_{j+1}<v_{j+2}=v_{j+3}$

The resulted region can be seen in Fig. 1. f).

## 4. Results

By fixing one parameter and choosing two parameters for unknown, we got a system of two equations having two unknown parameters. (So it has a solution.) These three parameters shall not be necessarily neighbours. The resulted new areas will overlap partly. However points can be chosen from these areas, where from up to now could not.

## References

[1] Hoffmann M., Juhász I., Shape control of cubic B-spline and NURBS curves by knot modifications, in: Banissi, E. et al (eds.): Proc. of the 5th International Conference on Information Visualisation, London, IEEE CSPress, 63-68, 2001.
[2] Juhász I., Hoffmann M., The effect of knot modifications on the shape of B-spline curves, Journal for Geometry and Graphics 5 (2001), 111-119.
[3] Hoffmann M., On the derivatives of a special family of B-spline curves, Acta Acad. Paed. Agriensis 28 (2001), 61-68.
[4] Juhász I., Hoffmann M., Knot modification of B-spline curves, in: SzirmayKalos, L, Renner, G. (eds.): I. Magyar Számítógépes Grafika és Geometria Konferencia, Budapest, (2002), 38-44.
[5] Hoffmann M., Juhász I., Geometric aspects of knot modification of B-spline surfaces, Journal for Geometry and Graphics 6 (2002), 141-149.
[6] Juhász I., Hoffmann M., Modifying a knot of B-spline curves, Computer Aided Geometric Design 20 (2003), 243-245.
[7] Juhász I., Hoffmann M., Constrained shape modification of cubic B-spline curves by means of knots, Computer-Aided Design 36 (2004), 437-445.
[8] Juhász I., A shape modifiaction of B-spline curves by symmetric translation of two knots, Acta Acad. Paed. Agriensis 28 (2001), 69-77.
[9] Piegl, L., Tiller, W., The NURBS book, Springer-Verlag, 1995.
[10] Fowler, B., Bartels, R., Constrained-based curve manipulation, IEEE Computer Graphics and Applications 13 (1993), 43-49.
[11] Zheng, J.M., Chan, K. W., Gibson, I., A new approach for direct manipulation of free-form curve, Computer Graphics Forum 17 (1998), 327-334.
[12] Lyche, T., Morken, K., The sensitivity of a spline function to perturbations of the knots, BIT 39 (1999), 305-322.
[13] Piegl, L., Tiller, W., Surface approximation to scanned data, The visual computer 16 (2000), 386-395.

## Róbert Tornai

Institute of Informatics
University of Debrecen
Egyetem tér 1.
H-4010 Debrecen, Hungary

