# GENERALIZATIONS OF BOTTEMA'S THEOREM ON PEDAL POINTS

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**Abstract.** Given a polygon and one of its inner points $P$, the orthogonal projections of $P$ onto the sides of the polygon are called pedal points of $P$. Here we prove different results concerning configurations by attaching different types of polygons to the segments of the sides defined by the pedals. These theorems can be considered as the generalizations of Bottema's classical theorem.

## 1. Introduction

Consider a triangle $ABC$ and one of its inner points $P$. Let the orthogonal projection of $P$ onto the sides $AB, BC, CA$ be $P_1, P_2$ and $P_3$, respectively. These are the pedal points of $P$. If we build squares on the segments of the sides defined by the pedals (outside of the triangle), we obtain six different squares. In [1] Bottema proved the following theorem about the areas of these squares:

**Theorem 1.** *The sum of the areas of the squares erected on the segments $AP_1, BP_2$ and $CP_3$ equals the sum of the squares erected on the segments $P_1B, P_2C$ and $P_3A$.*

More recently van Lamoen and other studied similar configurations ([2], [3]) and showed the following in [3]:

**Theorem 2.** *Let $A_1B_1C_1$ be the triangle bounded by the lines containing the sides of the squares opposite to $AP_1, BP_2$ and $CP_3$. Similarly let $A_2B_2C_2$ be the triangle bounded by the lines containing the sides of the squares opposite to $P_1B, P_2C$ and $P_3A$. These two triangles are each homothetic to $ABC$ and the ratio of homothety is*

$$\lambda = 1 + \frac{a^2 + b^2 + c^2}{4t},$$

*where $a, b, c$ are the sides and $t$ is the area of $ABC$.*

To simplify the equation we use the following notations:

**Definition.** The *Brocard point* $\Omega$ and the *Brocard angle* $\omega$ of $ABC$ is the point and angle for which

$$\angle AB\Omega = \angle BC\Omega = \angle CA\Omega = \omega.$$

Since for the Brocard angle

$$\cot\omega = \frac{a^2+b^2+c^2}{4t} \tag{1}$$

holds (c.f. [4]), the ratio of the homothety in Theorem 2 can simply be written as

$$\lambda = 1 + \cot\omega.$$

Throughout the paper we use the phrases "left" and "right" to distinguish the two families of squares or other builded polygons.

## 2. New results on triangles

At first we prove that Bottema's statement holds not only for squares but for any rectangles similar for each other and also for regular triangles. Then we examine the ratio of homothety of Theorem 2 in the case when the squares are erected onto the inner side of the triangle and show that it equals $\cot\omega - 1$.

**Theorem 3.** *Consider the triangle $ABC$ and one of its inner points $P$. Let the pedals of $P$ on the sides $AB, BC, CA$ be $P_1, P_2$ and $P_3$, respectively. If we build similar rectangles on the segments of the sides defined by the pedals, then the sum of the areas of the rectangles erected on the segments $AP_1, BP_2$ and $CP_3$ (i.e. the "left" rectangles) equals the sum of the rectangles erected on the segments $P_1B, P_2C$ and $P_3A$ (i.e. the "right" rectangles).*

**Proof.** Here we use the basic idea of [3]. Let us denote the sides of the triangle by $a, b, c$ and the segments defined by the pedals by the following: $c_l = AP_1$; $c_r = P_1B$; $a_l = BP_2$; $a_r = P_2C$; $b_l = CP_3$; $b_r = P_3A$. From Theorem 1 it is follows, that

$$a_l^2 + b_l^2 + c_l^2 = a_r^2 + b_r^2 + c_r^2. \tag{2}$$

Let us denote the other side of the rectangle erected onto $a_l$ by $s$ and let $\rho = \frac{s}{a_l}$. Thus the area of this rectangle can be written as $a_l s = a_l \rho a_l = a_l^2 \rho$. Since the rectangles are similar to each other, $\rho$ is the ratio of their sides for all rectangles. Thus the sum of the areas of the "left" rectangles is

$$a_l^2\rho + b_l^2\rho + c_l^2\rho = \rho(a_l^2 + b_l^2 + c_l^2).$$

Similarly for the "right" rectangles

$$a_r^2\rho + b_r^2\rho + c_r^2\rho = \rho(a_r^2 + b_r^2 + c_r^2)$$

holds, which, together with (2) proves the statement.

**Corollary.** *Let $A_1B_1C_1$ be the triangle bounded by the lines containing the sides of the rectangles opposite to $AP_1, BP_2$ and $CP_3$. Similarly let $A_2B_2C_2$ be the triangle bounded by the lines containing the sides of the rectangles opposite to $P_1B, P_2C$ and $P_3A$. These two triangles are each homothetic to $ABC$ and the ratio of homothety is $\lambda = 1 + \rho \cot \omega$.*

Back to the original situation, building the squares to the inner side of the segments of the side of the triangle, Theorem 1 naturally remains valid (see Fig. 1). The ratio of the homotethy, however will be changed as follows.

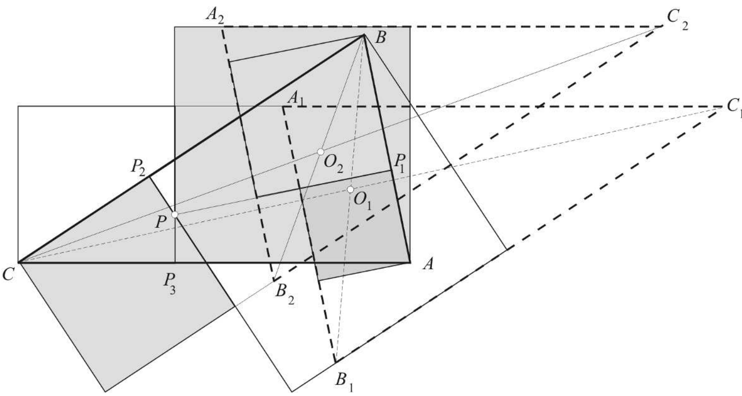


Figure 1.

**Theorem 4.** *Consider the triangle $ABC$ and one of its inner points $P$. Let the pedals of $P$ on the sides $AB, BC, CA$ be $P_1, P_2$ and $P_3$, respectively. If we build squares onto the inner side of the segments of the sides defined by the pedals, as in Fig.1., then the ratio of the homothety between the triangle $ABC$ and $A_1B_1C_1$ as well as between $ABC$ and $A_2B_2C_2$ is $\lambda = \cot \omega - 1$.*

**Proof.** Denote the center of homothety between $ABC$ and $A_1B_1C_1$ by $O_1$ and the segments $BP_2, CP_3, AP_1$ by $a_l, b_l$ and $c_l$. Let the distances of the sides $BC, CA, AB$ from $O_1$ be $f, g, h$, respectively. Obviously the distances of the sides $B_1C_1, C_1A_1, A_1B_1$ from $O_1$ are $(a_l - f), (b_l - g)$ and $(c_l - h)$. Due to the homothety $f : g : h = (a_l - f) : (b_l - g) : (c_l - h)$ holds. From equation (2)

$$a_l^2 + b_l^2 + c_l^2 = (a - a_l)^2 + (b - b_l)^2 + (c - c_l)^2.$$

Applying equation (1) this can be written as

$$aa_l + bb_l + cc_l = \frac{a^2 + b^2 + c^2}{2} = 2t \cot \omega,$$

where $t$ is the area of the triangle $ABC$. Summarizing the area of the subtriangles $O_1BC$, $O_1AC$ and $O_1AB$ we find

$$af + bg + ch = 2t,$$

which, together with the previous equation yields

$$\frac{a_l}{f} = \frac{b_l}{g} = \frac{c_l}{h} = \cot\omega.$$

Thus the ratio of homothety is

$$\lambda = \frac{a_l - f}{f} = \frac{b_l - g}{g} = \frac{c_l - h}{h} = \cot\omega - 1,$$

which completes the proof.

By applying this method one can prove several similar theorems and compute the ratios of homothety. Here we mention only one more example (see Fig. 2).

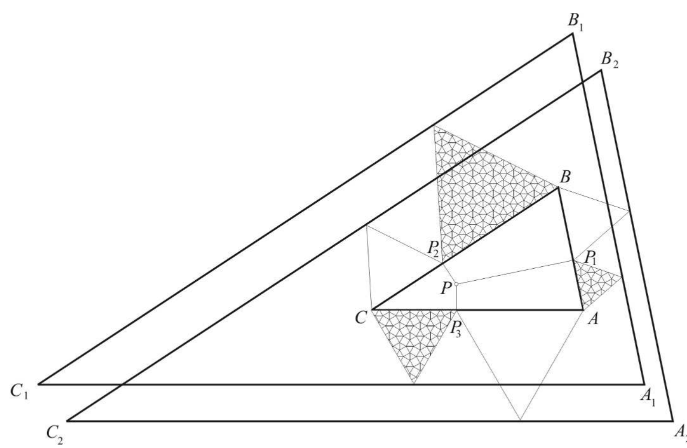


Figure 2.

**Theorem 5.** *Consider the triangle $ABC$ and one of its inner points $P$. Let the pedals of $P$ on the sides $AB, BC, CA$ be $P_1, P_2$ and $P_3$, respectively. If we build regular triangles on the segments of the sides defined by the pedals, then the sum of the areas of the triangles erected on the segments $AP_1, BP_2$ and $CP_3$ equals the sum of the triangles erected on the segments $P_1B, P_2C$ and $P_3A$. Moreover, if we consider those vertices of the "left" triangles which are not on the sides of $ABC$ and draw parallel lines to the sides of the original triangle through of them, then the triangle bounded by these lines is homothetic to $ABC$ and the ratio of homothety is*

$$\lambda = 1 + \frac{\sqrt{3}}{2}\cot\omega.$$

*Similar homothety holds for the triangle constructed from the "right" builded triangles.*

## 3. New results on polygons

In this section we generalize Theorem 1 for convex polygons and prove some further results about quadrilaterals.

**Theorem 6.** *Consider the convex polygon $A_1A_2\dots A_n$ and one of its inner points $P$. Let the pedals of $P$ on the sides $A_1A_2, A_2A_3, \dots, A_{n-1}A_n, A_nA_1$ be $P_1, P_2, \dots, P_{n-1}, P_n$, respectively. If we build "left" squares onto the segments $A_iP_i$, $(i = 1, \dots, n)$ and "right" squares onto the segments $P_iA_{i+1}$, $(i = 1, \dots, n-1)$ and $P_nA_1$, then the sum of the areas of "left" squares equals the sum of the area of "right" squares.*

**Proof.** Applying the phytagorean theorem for the triangles $PA_iP_i$ one can write

$$A_iP_i{}^2 = PA_i{}^2 - PP_i{}^2,\ i = 1, \dots, n.$$

Similarly

$$\begin{aligned} P_iA_{i+1}{}^2 &= PA_{i+1}{}^2 - PP_i{}^2,\ i = 1, \dots, n-1 \\ P_nA_1{}^2 &= PA_1{}^2 - PP_n{}^2. \end{aligned}$$

This yields

$$\begin{aligned} \sum_{i=1}^{n} A_iP_i{}^2 &= \sum_{i=1}^{n}(PA_i{}^2 - PP_i{}^2) \\ &= \sum_{i=1}^{n-1}(PA_{i+1}{}^2 - PP_i{}^2) + PA_1{}^2 - PP_n{}^2 \\ &= \sum_{i=1}^{n-1} P_iA_{i+1}{}^2 + P_nA_1{}^2, \end{aligned}$$

which completes the proof.

The statement remains valid if the builded quadrilaterals are not squares but rectangles similar to each other as it was in the triangle case (c.f. the proof of Theorem 3).

The statement of Theorem 6 can be seen for pentagons in Fig. 3. We have to remark, that if we consider the pentagons bounded by the lines containing the sides of the squares parallel to the sides of the original pentagon, the two pentagons are not homothetic to each other. Generally speaking this property is valid only for triangles. For special cases, however, homothety still holds for quadrilaterals, as we will see in the next theorems.

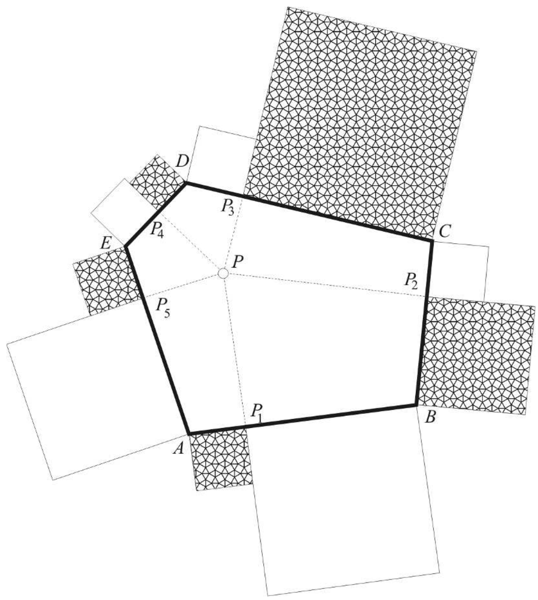


Figure 3.

**Theorem 7.** *Consider the rectangle $ABCD$ and one of its inner points $P$. Let the pedals of $P$ on the sides $AB, BC, CD$ and $DA$ be $P_1, P_2, P_3$ and $P_4$, respectively. If we build similar rectangles on the segments of the sides defined by the pedals in a way, that the larger sides of the rectangles are all parallel to the larger side of the original one, then the sum of the areas of the rectangles erected on the segments $AP_1, BP_2, CP_3$ and $DP_4$ equals the sum of the rectangles erected on the segments $P_1B, P_2C, P_3D$ and $P_4A$. Moreover, the rectangle bounded by the lines containing the outer sides of the "left" rectangles is homothetic to the original one and the ratio of homothety is $\lambda = 2$. Similar statement holds for the rights rectangles.*

**Proof.** The first part of the statement can be proved analogously to Theorem 3 and 6. For the ratio of homothety let us denote the ratio of the two sides of the rectangle by $\rho = \frac{AB}{BC}$. Consider the "left" rectangles. The sides of these rectangles parallel to $AB$ are $AP_1$, $\rho BP_2$, $CP_3$ and $\rho DP_4$ (c.f. Fig. 4).

The side $A'B'$ of the large rectangle parallel to $AB$ is the sum of these sides:

$$A'B' = AP_1 + \rho BP_2 + CP_3 + \rho DP_4,$$

but $AP_1 + CP_3 = AB$, while $\rho BP_2 + \rho DP_4 = \rho BC = AB$, thus $A'B' = 2AB$. Similarly $B'C' = 2BC$ and this was to be proved.

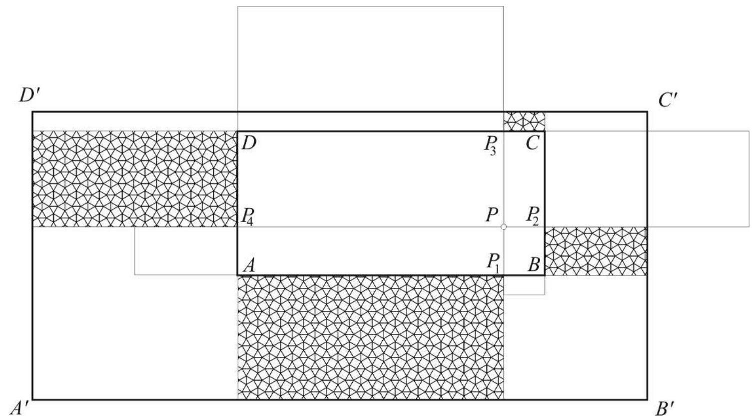


Figure 4.

Finally we remark, that the orientation of the builded rectangles in Theorem 7 is important only in terms of homothety. If the rectangles are builded in a way that always their longer sides coincide to the segments defined by the pedals, then the sum of the areas of the "left" rectangles remains equal to the "right" one, but the large rectangle is no longer similar to the original one: the ratio of its sides is

$$\frac{A'B'}{B'C'} = \frac{a^2 + b^2}{2ab},$$

where $a$ and $b$ are the sides of the original rectangle.

## References

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