GENERATED PREORDERS AND EQUIVALENCES

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Dedicated to the memory of Professor Péter Kiss

Abstract. For any relation R, we denote by R^* and $R^\bullet$ the smallest preorder and equivalence containing R, respectively. We establish some basic properties of the closures R^* and $R^\bullet$. Moreover, we provide some new characterizations of equivalences in terms of generated preorders.

The results obtained naturally supplement some former statements of Árpád Száz on preorders and equivalences. Moreover, they can be applied to relators (relational systems). Namely, each topology can be derived from a preorder relator. Moreover, equivalence relators also frequently occur in the applications.

1. Preorders, equivalences and closures

A subset R of a product set X^2 is called a relation on X. In particular, the relation $\Delta_X = \{(x, x) : \; x \in X\}$ is called the identity relation on X. Moreover, the relation $R^{-1} = \{(y, x) : \; (x, y) \in R\}$ is called the inverse of R.

Furthermore, if R and S are relations on X, then the relation $S \circ R = \big\{(x, z) : \exists \; y \in X : (x, y) \in R, \; (y, z) \in S\big\}$ is called the composition of S and R. In particular, we write $R^n = R \circ R^{n-1}$ for all $n \in N$ by agreeing that $R^0 = \Delta_X$.

A relation R on X is called reflexive, symmetric, and transitive if $\Delta_X \subset R$, $R^{-1} \subset R$ and $R^2 \subset R$, respectively. Moreover, a reflexive and transitive relation is called a preorder, and a symmetric preorder is called an equivalence.

Thus, we have $R^{-1} = R$ if R is a symmetric relation, and $R^2 = R$ if R is a preorder. Moreover, by using the above definitions, we can also easily prove the following basic characterization theorems.

Theorem 1.1. *If R is a relation on X, then the following assertions are equivalent:* (1) *R is a preorder (equivalence);* (2) *R^{-1} is a preorder (equivalence).*

Theorem 1.2. *If R is a relation on X, then the following assertions are equivalent:* (1) *R is a preorder;* (2) $\Delta_X \cup R^2 \subset R$*;* (3) $\Delta_X \cup R^2 = R$.

Remark 1.3. In [10], it was proved that R is a preorder if and only if $R = \big(R^{-1} \circ R^c\big)^c$, where $R^c = X^2 \setminus R$.

Theorem 1.4. *If R is a relation on X, then the following assertions are equivalent:* (1) *R is an equivalence;* (2) $\Delta_X \cup R \circ R^{-1} \subset R$; (3) $\Delta_X \cup R \circ R^{-1} = R$.

Remark 1.5. In the above theorem, by Theorem 1.1, we may write $R^{-1} \circ R$ in place of $R \circ R^{-1}$.

In [10], it was proved that if $R(x) \neq \emptyset$ for all $x \in X$, then R is an equivalence if and only if $R = R^{-1} \circ R$.

Definition 1.6. A function $^-$ of the power set $\mathcal{P}(X)$ into itself is called an algebraic closure [1, p. 111] on $\mathcal{P}(X)$ if

(1) $A \subset B$ implies $A^- \subset B^-$ for all $A, B \subset X$;

(2) $A \subset A^-$ for all $A \subset X$; (3) $A^- = \left(A^-\right)^-$ for all $A \subset X$.

The following characterization theorem could have already been established in [2]. However, it was later stressed only in [7] and [11].

Theorem 1.7. *If $^-$ is a function of $\mathcal{P}(X)$ into itself, then the following assertions are equivalent:*

(1) *the function $^-$ is a closure on $\mathcal{P}(X)$,*

(2) *for any $A, B \subset X$, we have $A \subset B^-$ if and only if $A^- \subset B^-$.*

Remark 1.8. If $^-$ is a closure on X, then we have $A^- \cup B^- \subset (A \cup B)^-$ for all $A, B \subset X$.

However, the corresponding equality is not, in general, true. Therefore, an algebraic closure need not be a topological closure [3, p. 43].

2. Basic properties of generated preorders

Theorem 2.1. *If $\mathcal{R}$ is a family of preorders on X, then $S = \bigcap \mathcal{R}$ is also a preorder on X.*

Hint. To prove the transitivity of S, note that $S^2 \subset R^2 \subset R$ for all $R \in \mathcal{R}$, and thus $S^2 \subset \bigcap \mathcal{R} = \mathcal{S}$.

Theorem 2.2. *If R is a relation on X, then there exists a smallest preorder R^* on X such that $R \subset R^*$.*

Proof. To prove this, denote by $\mathcal{R}$ the family of all preorders on X containing R, and define $R^* = \bigcap \mathcal{R}$.

Theorem 2.3. *If R is a relation on X, then the following assertions are equivalent:* (1) *R is a preorder;* (2) $R^* \subset R$; (3) $R^* = R$.

Proof. If the assertion (1) holds, then because of the inclusion $R \subset R$ and Theorem 2.2, the assertion (2) also holds. The implications (2)$\Longrightarrow$(3)$\Longrightarrow$(1) are even more obvious by Theorem 2.2.

Theorem 2.4. *The mapping* $R \mapsto R^*$ *is an algebraic closure on* $\mathcal{P}(X^2)$.

Proof. If $R, S \subset X^2$ such that $R \subset S^*$, then by Theorem 2.2 it is clear that $R^* \subset S^*$.

While, if $R^* \subset S^*$, then again by Theorem 2.2, it clear that $R \subset S^*$. Thus, by Theorem 1.7, the required assertion is also true.

Remark 2.5. By the above theorem, we have $R^* \cup S^* \subset (R \cup S)^*$ for all $R, S \subset X^2$.

The following example shows that the corresponding equality need not be true. Therefore, the mapping considered in Theorem 2.4 is not, in general, a topological closure.

Example 2.6. If $X = \{1, 2, 3\}$, and $R = \{(1, 2)\}$ and $S = \{(2, 3)\}$, then it can be easily seen that $(R \cup S)^* \setminus \big(R^* \cup S^*\big) = \{(1, 3)\}$.

Theorem 2.7. *If* R *is a relation on* X, *then*

$$\big(R^*\big)^{-1} = \big(R^{-1}\big)^*.$$

Proof. Since $R^{-1} \subset (R^*)^{-1}$ and $(R^*)^{-1}$ is also a preorder, we evidently have $\big(R^{-1}\big)^* \subset (R^*)^{-1}$. Hence, by writing R^{-1} in place of R, we can infer that $R^* \subset \big(\big(R^{-1}\big)^*\big)^{-1}$, and thus $(R^*)^{-1} \subset \big(R^{-1}\big)^*$ is also true.

Corollary 2.8. *If* R *is a symmetric relation on* X, *then* R^* *is an equivalence.*

Proof. In this case, by Theorem 2.7, we have $\big(R^*\big)^{-1} = \big(R^{-1}\big)^* = R^*$, and thus the preorder R^* is also symmetric.

The following example shows that the converse of the above corollary need not be true.

Example 2.9. If $X = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3), (3, 1)\}$, then $R^* = X^2$ is an equivalence despite that R is not symmetric.

Theorem 2.10. *If* R *is a relation on* X, *then*

$$R^* = \bigcup_{n=0}^{\infty} R^n.$$

Proof. By Theorem 2.2, it is clear that

$$S=\bigcup_{n=0}^{\infty} R^{\,n} \subset \bigcup_{n=0}^{\infty} \left(R^{*}\right)^{n} \subset \bigcup_{n=0}^{\infty} R^{*} = R^{*}.$$

Moreover, it can be easily seen that

$$S\circ S=\left(\bigcup_{n=0}^{\infty} R^{n}\right)\circ\left(\bigcup_{m=0}^{\infty} R^{m}\right) \subset \bigcup_{n=0}^{\infty}\bigcup_{m=0}^{\infty} R^{n+m} \subset \bigcup_{k=0}^{\infty} R^{k} = S\,.$$

Now, it is clear that S is a preorder on X containing R. Therefore, by Theorem 2.2, the inclusion $R^{*}\subset S$ is also true.

The above theorem allows an easy computation of the preorder hull R^{*} of a relation R.

Example 2.11. If $X=\{1,2,3,4\}$ and $R=\{(2,1),(3,2),(4,1),(4,3)\}$, then it can be easily seen that $R^{2}=\{(3,1),(4,2)\}$, $R^{3}=\{(4,1)\}$ and $R^{4}=\emptyset$. Therefore, by Theorem 2.10, $R^{*}=\Delta_X\cup R\cup R^{2}$.

Remark 2.12. Because of Theorem 2.10, one may naturally write R^{∞} in place of R^{*}.

3. Basic properties of generated equivalences

Analogously to the corresponding results of Section 2, we can also easily establish the following theorems.

Theorem 3.1. *If $\mathcal{R}$ is a family of equivalences on X, then $S=\bigcap\mathcal{R}$ is also an equivalence on X.*

Theorem 3.2. *If R is a relation on X, then there exists a smallest equivalence $R^{\bullet}$ on X such that $R\subset R^{\bullet}$.*

Theorem 3.3. *If R is a relation on X, then the following assertions are equivalent:*

(1) *R is an equivalence;* (2) $R^{\bullet}\subset R$; (3) $R=R^{\bullet}$.

Theorem 3.4. *The mapping $R\mapsto R^{\bullet}$ is an algebraic closure on $\mathcal{P}(X^{2})$.*

Remark 3.5. By the above theorem, we have $R^{\bullet}\cup S^{\bullet}\subset(R\cup S)^{\bullet}$ for all $R,S\subset X^{2}$.

The following example shows that the corresponding equality need not be true. Therefore, the mapping considered in Theorem 3.4 is not, in general, a topological closure.

Example 3.6. It can be easily seen that, under the notations of Example 2.6, we have $(R \cup S)^{\bullet} \setminus \left(R^{\bullet} \cup S^{\bullet}\right) = \{(1, 3), (3, 1)\}$.

Theorem 3.7. *If R is a relation on X, then*

$$R^{\bullet} = \left(R^{\bullet}\right)^{-1} = \left(R^{-1}\right)^{\bullet}.$$

Proof. By Theorem 3.2, it is clear that $R^{-1} \subset (R^{\bullet})^{-1} = R^{\bullet}$. Hence, by Theorem 3.4, it follows that $\left(R^{-1}\right)^{\bullet} \subset \left(R^{\bullet}\right)^{\bullet} = R^{\bullet}$. Now, by writing R^{-1} in place of R, we can at once see that the converse inclusion is also true.

Theorem 3.8. *If R is a relation on X, then $R^{*} \subset R^{\bullet}$.*

Proof. By Theorem 3.2, $R^{\bullet}$ is, in particular, a preorder containing R. Therefore, by Theorem 2.2, the required inclusion is also true.

Corollary 3.9. *If R is a symmetric relation on X, then $R^{*} = R^{\bullet}$.*

Proof. In this case, by Corollary 2.8 and Theorem 2.2, R^{*} is an equivalence containing R. Thus, by Theorem 3.2, $R^{\bullet} \subset R^{*}$. Moreover, by Theorem 3.8, the converse inclusion is always true.

Remark 3.10. From Example 2.9, we can see that the converse of the above corollary need not be true.

Theorem 3.11. *If R is a relation on X, then*

$$R^{\bullet} = \left(R^{\bullet}\right)^{*} = \left(R^{*}\right)^{\bullet}.$$

Proof. By Theorems 3.2 and 2.3, it is clear that $\left(R^{\bullet}\right)^{*} = R^{\bullet}$. On the other hand, since $R \subset R^{*} \subset \left(R^{*}\right)^{\bullet}$ and $\left(R^{*}\right)^{\bullet}$ is an equivalence on X, it is clear that $R^{\bullet} \subset \left(R^{*}\right)^{\bullet}$. Moreover, since $R^{*} \subset R^{\bullet}$ and $R^{\bullet}$ is an equivalence on X, it is clear that $\left(R^{*}\right)^{\bullet} \subset R^{\bullet}$ is also true.

Theorem 3.12. *If R is a relation on X, then*

$$R^{\bullet} = \left(R \cup R^{-1}\right)^{*}.$$

Proof. By Theorem 3.4 and Corollary 3.9, it is clear that $R^{\bullet} \subset \left(R \cup R^{-1}\right)^{\bullet} = \left(R \cup R^{-1}\right)^{*}$.

Moreover, by Theorem 3.2, it is clear that $R \cup R^{-1} \subset R^{\bullet} \cup \left(R^{\bullet}\right)^{-1} = R^{\bullet}$. Hence, by Theorems 3.4 and 3.11, it follows that $\left(R \cup R^{-1}\right)^{*} \subset \left(R^{\bullet}\right)^{*} = R^{\bullet}$.

Remark 3.13. Concernig the intersection $R \cap R^{-1}$, we can only state that $\big(R \cap R^{-1}\big)^* = \big(R \cap R^{-1}\big)^\bullet \subset R^\bullet$.

Namely, the following example shows that the corresponding equality need not be true.

Example 3.14. If $X = \{1, 2\}$ and $R = \{(1, 2)\}$, then $R^\bullet \setminus \big(R \cap R^{-1}\big)^* = R \cup R^{-1}$.

Now, in contrast to Example 3.6, we can also prove the following

Theorem 3.15. *If R is a relation on X, then*

$$\big(R \cup R^{-1}\big)^\bullet = R^\bullet \cup \big(R^{-1}\big)^\bullet.$$

Proof. By Corollary 3.9 and Theorems 3.12 and 3.7, we evidently have $\big(R \cup R^{-1}\big)^\bullet = \big(R \cup R^{-1}\big)^* = R^\bullet = R^\bullet \cup \big(R^{-1}\big)^\bullet$.

Remark 3.16. Note that, by Remark 2.5, $R^* \cup \big(R^{-1}\big)^* \subset \big(R \cup R^{-1}\big)^*$ is always true.

However, the following example shows that, in contrast to Theorem 3.15, the corresponding equality need not be true.

Example 3.17. If $X = \{1, 2, 3\}$ and $R = \{(1, 2), (1, 3)\}$, then $\big(R \cup R^{-1}\big)^* \setminus \big(R^* \cup \big(R^{-1}\big)^*\big) = \{(2, 3), (3, 2)\}$.

Theorem 3.18. *If R is a reflexive relation on X, then*

$$R^\bullet = \big(R \circ R^{-1}\big)^* = \big(R^{-1} \circ R\big)^*.$$

Proof. Because of the reflexivity of R and Theorem 3.2, we evidently have $R \subset R \circ \Delta_X \subset R \circ R^{-1} \subset \big(R \circ R^{-1}\big)^*$. Moreover, by using Corollary 2.8, we can at once see that $\big(R \circ R^{-1}\big)^*$ is an equivalence on X. Therefore, by Theorem 3.2, we have $R^\bullet \subset \big(R \circ R^{-1}\big)^*$.

On the other hand, by using Theorems 3.2, 3.7 and 3.11, we can easily see that $\big(R \circ R^{-1}\big)^* \subset \big(R^\bullet \circ \big(R^{-1}\big)^\bullet\big)^* = \big(R^\bullet \circ R^\bullet\big)^* = \big(R^\bullet\big)^* = R^\bullet$ also holds. Therefore, $R^\bullet = \big(R \circ R^{-1}\big)^*$. Hence, by writing R^{-1} in place of R, we can at once see that $\big(R^{-1} \circ R\big)^* = \big(R^{-1}\big)^\bullet = R^\bullet$ is also true.

Theorem 3.19. *If R is a relation on X, then*

$$R^\bullet = \Big(R^* \circ \big(R^{-1}\big)^*\Big)^* = \Big(\big(R^{-1}\big)^* \circ R^*\Big)^*.$$

Proof. By Theorems 3.11, 3.18 and 2.7, it is clear that $R^{\bullet} = \big(R^{*} \big)^{\bullet} = \big(R^{*} \circ \big(R^{*} \big)^{-1} \big)^{*} = \big(R^{*} \circ \big(R^{-1} \big)^{*} \big)^{*}$.

Hence, by writing R^{-1} in place of R, and using Theorem 3.7, we can see that the other equality is also true.

4. Further characterizations of equivalences

By Theorems 3.3 and 3.12, we evidently have the following

Theorem 4.1. *If R is a relation on X, then the following assertions are equivalent:*

(1) *R is an equivalence;* (2) $\big(R \cup R^{-1} \big)^{*} \subset R$*;* (3) $\big(R \cup R^{-1} \big)^{*} = R$.

Remark 4.2. Note that, by Corollary 3.9 and Theorem 3.15, we have $\big(R \cup R^{-1} \big)^{*} = \big(R \cup R^{-1} \big)^{\bullet} = R^{\bullet} \cup \big(R^{-1} \big)^{\bullet}$.

Therefore, it is also of some interest to prove the following

Theorem 4.3. *If R is a relation on X, then the following assertions are equivalent:*

(1) *R is an equivalence;* (2) $R^{*} \cup \big(R^{-1} \big)^{*} \subset R$*;* (3) $R^{*} \cup \big(R^{-1} \big)^{*} = R$.

Proof. If the assertion (1) holds, then by Theorem 2.3, it is clear that $R^{*} \cup \big(R^{-1} \big)^{*} = R^{*} \cup R^{*} = R^{*} = R$. That is, the assertion (3) also holds.

While, if the assertion (2) holds, then by Theorem 2.2 it is clear that $\Delta_X \subset R^{*} \subset R^{*} \cup \big(R^{-1} \big)^{*} \subset R$. Moreover, $R^{-1} \subset \big(R^{-1} \big)^{*} \subset R^{*} \cup \big(R^{-1} \big)^{*} \subset R$, and thus $R \circ R^{-1} \subset R^{2} \subset \big(R^{*} \big)^{2} \subset R^{*} \subset R^{*} \cup \big(R^{-1} \big)^{*} \subset R$. Therefore, by Theorem 1.4, the assertion (1) also holds.

Theorem 4.4. *If R is a relation on X, then the following assertions are equivalent:*

(1) *R is an equivalence;* (2) $\big(R \circ R^{-1} \big)^{*} \subset R$*;* (3) $\big(R \circ R^{-1} \big)^{*} = R$.

Proof. If the assertion (1) holds, then by Theorem 2.3, it is clear that $\big(R \circ R^{-1} \big)^{*} = \big(R \circ R \big)^{*} = R^{*} = R$. That is, the assertion (3) also holds.

While if the assertion (2) holds, then by Theorem 2.2 we have $\Delta_X \cup R \circ R^{-1} \subset \big(R \circ R^{-1} \big)^{*} \subset R$. Therefore, by Theorem 1.4, the assertion (1) also holds.

Theorem 4.5. *If R is a relation on X, then the following assertions are equivalent:*

(1) *R is an equivalence;* (2) $R^{*} \circ \big(R^{-1} \big)^{*} \subset R$*;* (3) $R^{*} \circ \big(R^{-1} \big)^{*} = R$.

Proof. If the assertion (1) holds, then by Theorems 2.2, 3.19 and 3.3, it is clear that $R = R \circ \Delta_X \subset R^* \circ \left(R^{-1}\right)^* \subset \left(R^* \circ \left(R^{-1}\right)^*\right)^* = R^\bullet = R$. Therefore, the assertion (3) also holds.

While if the assertion (2) holds, then by Theorem 2.2 we have $\Delta_X \subset \Delta_X \circ \Delta_X \subset R^* \circ \left(R^{-1}\right)^* \subset R$ and $R \circ R^{-1} \subset R^* \circ \left(R^{-1}\right)^* \subset R$. Therefore, by Theorem 1.4, the assertion (1) also holds.

Remark 4.6. In the above theorems, by Theorems 1.1 and 2.7, we may write $\left(R^{-1} \circ R\right)^*$ and $\left(R^{-1}\right)^* \circ R^*$ in place of $\left(R \circ R^{-1}\right)^*$ and $R^* \circ \left(R^{-1}\right)^*$, respectively.

Theorem 4.7. *If R is a relation on X, then the following assertions are equivalent:*

(1) *R is an equivalence;* (2) $R = \left(R \cap R^{-1}\right)^*$; (3) $R = R^* \cap \left(R^{-1}\right)^*$.

Proof. If the assertion (1) holds, then by Theorem 2.3 and the symmmetry of R it is clear that $R = R^* = \left(R \cap R^{-1}\right)^*$ and $R = R^* = R^* \cap \left(R^{-1}\right)^*$. That is, the assertions (2) and (3) also hold.

While, if the assertion (2) holds, then by Corollary 2.8 it is clear that the assertion (1) also holds. On the other hand, if the assertion (3) holds, then by Theorems 2.2 and 2.1, it is clear that R is a preorder. Moreover, by using Theorem 2.7 we can see that $R^{-1} = \left(R^* \cap \left(R^{-1}\right)^*\right)^{-1} = \left(R^*\right)^{-1} \cap \left(\left(R^{-1}\right)^*\right)^{-1} = \left(R^{-1}\right)^* \cap R^* = R$. Therefore, the assertion (1) also holds.

The following examples shows that in Theorem 4.7 we cannot write inclusions in place of the equalities.

Example 4.8. If $X = \{1, 2\}$ and $R = \Delta_X \cup \{(1, 2)\}$, then $\left(R \cap R^{-1}\right)^* \subset R$ and $R^* \cap \left(R^{-1}\right)^* \subset R$ despite that R is not an equivalence.

Example 4.9. If $X = \{1, 2\}$ and $R = \{(1, 2), (2, 1)\}$, then $R \subset \left(R \cap R^{-1}\right)^*$ and $R \subset R^* \cap \left(R^{-1}\right)^*$ despite that R is not an equivalence.

Remark 4.10. Note that, by Theorem 2.2, $\left(R \cap R^{-1}\right)^* \subset R^* \cap \left(R^{-1}\right)^*$ is always true.

However, if $X = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3), (3, 1)\}$, then $\left(R \cap R^{-1}\right)^* = \Delta_X$ and $R^* \cap \left(R^{-1}\right)^* = X^2$.

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