On a special interpolation problem of functional B-spline curves

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Dedicated to the memory of Professor Péter Kiss

Abstract. A multistep interpolation method is presented in this paper by which one can compute a planar functional B-spline curve from its gradient function with special endconditions. The method is applied in turbine blade section curve design where the curve consists of two interpolating B-spline arcs connecting two predefined circular arcs.

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1.Introduction and problem statement

Interpolation or approximation of a set of data by B-spline curves is a well-studied area of computer aided design and manufacture. One can find numerous methods solving the problem for different types of data, see e.g. [3] and references therein. All of these classical methods, however, require direct geometrical data (points and/or tangent lines) of the curve.

In the problem presented here, the curve has to be created from the *angles* of its tangent lines, while the exact position of the curve is determined by endpoints and tangential information in them. A special, multistep method will be described in the following section to solve this problem. Application of this method for turbine blade section curve design can be found in Section 3. Finally conclusion and directions of future research close the paper.

To clarify the notations here we present the definition of the B-spline curve.

Definition. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} N_l^k(u)\mathbf{d}_l, \quad u \in [u_{k-1}, u_{n+1}]$$

is called B-spline curve of order k (degree $k-1$), where $N_l^k(u)$ is the lth normalized B-spline basis function, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are

necessary. The points $\mathbf{d}_l$ are called control points, while the polygon formed by these points is called control polygon.

The turbine blade section curve (or simply profile curve) consists of two circular arcs and two B-spline arcs connecting them. In profile curve design the connecting B-spline curves are smooth, almost linear pieces along the x direction, hence throughout this paper one can consider them as functions of x, which is certainly not the general case.

2. Multistep interpolation

Let a continuously differentiable real function $\alpha = f(x), x \in [a, b]$ be given. We would like to create an arc (function) over $[a, b]$ with the following property: at every $x_0 \in [a, b]$ the angle of the x axis and the tangent line of the arc at this point is $\alpha_0 = f(x_0)$. Thus the function $\alpha = f(x)$ can be considered as the gradient function of the arc we are looking for. Moreover, the arcs have to be defined as a B-spline curves, because this type of curves is the standard description method in CAGD. Thus let this arc be denoted by $\mathbf{s}(u), u \in [a, b]$.

At first a sample set of pairs $(x_i, \alpha_i), i = 1, ..., n$ is chosen, where n depends on the desired error bound. The x_i-s can be uniformly distributed over $[a, b]$. We will create a cubic B-spline curve $\mathbf{s}(u)$ in a way that at every point of the curve cooresponding to the coordinate values x_i the angle between the tangent line of the curve and the x axis is α_i. Denote these corresponding points of the curve by $\mathbf{q}_i(x_i, y_i)$. Thus

$$(1) \qquad \mathbf{s}(u_i) = \mathbf{q}_i(x_i, y_i), \quad i = 1, ..., n$$

where the parameters u_i and the coordinates y_i are the unknowns. Let $\mathbf{e}_i(\cos \alpha_i, \sin \alpha_i)$ denote the unit vector at the direction of the ith tangent line. This yields the equations

$$(2) \qquad \dot{\mathbf{s}}(u_i) = \lambda_i \mathbf{e}_i, \quad i = 1, ..., n$$

where the values λ_i are unknowns.

Denote the coordinates of the future control points $\mathbf{d_j}$ by $(x_j^{\mathbf{d}}, y_j^{\mathbf{d}}), j = 1, ..., n$ and the coordinate functions of the B-spline curve by $\mathbf{s}(u) = (x^{\mathbf{s}}(u), y^{\mathbf{s}}(u))$. Note, that the number of control points is the same as the number of samples. Thus from (1) and (2) we obtain the following system of equations:

$$\begin{aligned} x^{\mathbf{s}}(u_i) &= x_i, \\ \dot{x}^{\mathbf{s}}(u_i) &= \lambda_i \cos \alpha_i, \qquad i = 1, ..., n. \\ \dot{y}^{\mathbf{s}}(u_i) &= \lambda_i \sin \alpha_i, \end{aligned}$$

By the definition of the B-spline curve this system can be written as

(3)
$$\sum_{j=0}^{n} N_j^4(u_i) x_j^{\mathbf{d}} = x_i,$$

$$\sum_{j=0}^{n} \dot{N}_j^4(u_i) x_j^{\mathbf{d}} = \lambda_i \cos \alpha_i, \qquad i = 1, ..., n.$$

$$\sum_{j=0}^{n} \dot{N}_j^4(u_i) y_j^{\mathbf{d}} = \lambda_i \sin \alpha_i,$$

Here each row represents n equations and all subsystems are linear. If we fix the parameter values u_i then we can compute the B-spline basis functions $N_j^4(u_i)$ and their derivatives $\dot{N}_j^4(u_i)$. There are several methods for the choice of u_i (uniform, cumulative chord length, centripetal model, c.f. [1]), but our future curve is quite simple, smooth, and the samples x_i are uniformly distributed, thus we can apply the uniform model as

$$u_i = \frac{i-1}{n-1}.$$

Now the first subsystem of (3) —containing the first n equations— can be solved for $x_j^{\mathbf{d}}$, then the second subsystem for λ_i, finally the third n equations for $y_j^{\mathbf{d}}$. The first and the third systems of equations have a banded matrix where the bandwidth is 4, while the solution of the second one is even more straightforward, containing one unknown per equation. After this process we will obtain the control points $\mathbf{d_j}$ $(x_i^{\mathbf{d}}, y_i^{\mathbf{d}})$, with the help of which one can compute the desired B-spline curve. In case of non-uniform curve the knot vector can be the same as the sequence of the parameter values.

2. An application: turbine blade section curve design

The design process of turbine blades generally contains several steps. At first designers create planar curves which will serve as section curves of the future surface. A typical section curve can be seen in Fig.1.
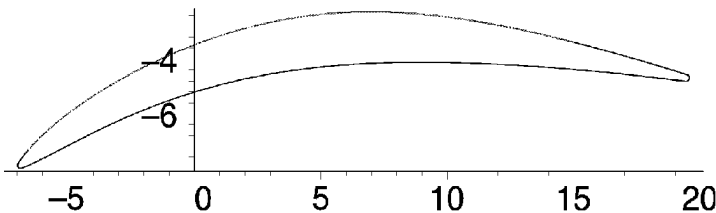

Figure 1

Focusing on this problem a designer may prefer the following scenario: he can
define two circular arcs by their centres, radii and endpoints. The system connects
them by two B-spline arcs which can be modified by direct manipulation of their
predefined gradient functions. The gradient functions, of course, always satisfy the
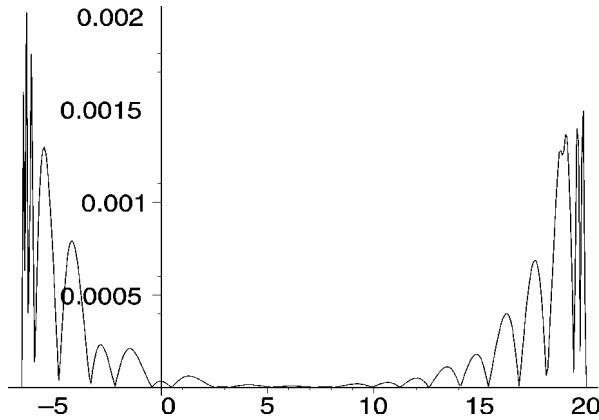endconditions defined by the circular arcs.

Figure 2

These two B-spline arcs can be calculated by the interpolation methods
described in the previous section. To estimate the correctness of this technique
we considered an arc of an existing section curve and picked 350 points of them as
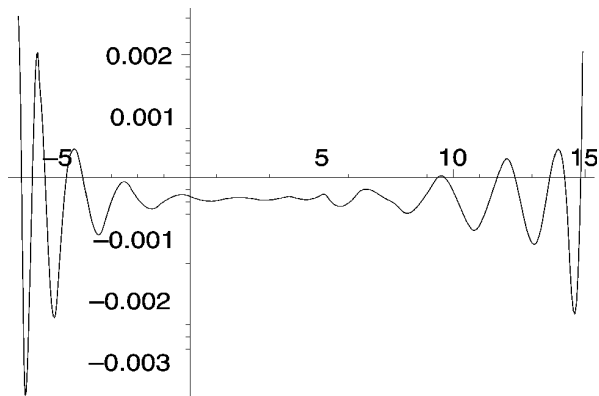samples.

Figure 3

Creating the new B-spline arc two types of error measured at these sample
points: the distance of the two curve and the difference between the tangential
angles of the two curve. The distance is measured by the following way: at each
sample point **p** of the original curve a perpendicular straight line is dropped. This

line intersects the computed B-spline curve at $\hat{\mathbf{p}}$ and the error at this point is defined as the length of $\hat{\mathbf{p}}\mathbf{p}$. The graph of this error function (connecting the discrete points for better visualization) can be seen in Fig.2.

The other type of error is measured by the difference between the angles of the tangent lines of the original and the computed curve at the sample points. This graph can be seen in Fig.3.

4. Conclusion

A special interpolation method has been presented in this paper with the help of which one can create and modify a planar B-spline curve based on its gradient function. Using this method a turbine blade section curve design technique has been developed: two circular arcs are defined which are connected by B-spline arcs computed by the new method.

References

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