SECOND ORDER LINEAR RECURRENCES AND PELL'S EQUATIONS OF HIGHER DEGREE

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Dedicated to the memory of Professor Péter Kiss

Abstract. In this note solutions are given to an infinite family of Pell's equations of degree $n \geq 2$ based on second order linear recursive sequences of integers.

AMS Classification Number: 11B39

1. Introduction

Let A and B be non-zero integers. The second order linear recursive sequences $R = \{R_n\}_{n=0}^{\infty}$ and $V = \{V_n\}_{n=0}^{\infty}$ are defined by the recursions

$$R_n = AR_{n-1} + BR_{n-2} \quad \text{and} \quad V_n = AV_{n-1} + BV_{n-2}, \tag{1}$$

for $n \geq 2$, while $R_0 = 0, R_1 = 1, V_0 = 2$ and $V_1 = A$. If $A = B = 1$ then $R_n = F_n$ and $V_n = L_n$, where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

The polynomial $g(x) = x^2 - Ax - B$ is said to be the characteristic polynomial of the sequences R and V, the complex numbers α and β are the roots of $g(x) = 0$. In this note we suppose that $A^2 + 4B \neq 0$, i.e. $\alpha \neq \beta$. Then, by the well-known Binet formulae, for $n \geq 0$

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n. \tag{2}$$

The classical Pell's equation $x^2 - dy^2 = \pm 1$ ($d \in \mathbf{Z}$) can be rewritten as

$$\det \begin{pmatrix} x & dy \\ y & x \end{pmatrix} = \pm 1.$$

Research supported by the Hungarian OTKA Foundation, No. T 032898.

To generalize this Lin Dazheng [1] investigated the quasi-cyclic matrix

$$\mathbf{C}_n = \mathbf{C}_n(d; x_1, x_2, \dots, x_n) = \begin{pmatrix} x_1 & dx_n & dx_{n-1} & \dots & dx_2 \\ x_2 & x_1 & dx_n & \dots & dx_3 \\ x_3 & x_2 & x_1 & \dots & dx_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_{n-1} & x_{n-2} & \dots & x_1 \end{pmatrix}, \tag{3}$$

i.e. every entry of the upper triangular part (not including the main diagonal) of the cyclic matrix of entries $x_1, x_2, \dots, x_n$ is multiplied by d. The equation

$$\det(\mathbf{C}_n) = \pm 1 \tag{4}$$

is called Pell's equation of degree $n \geq 2$. For example, if $n = 3$ then (4) has the form

$$x_1^3 + dx_2^3 + d^2 x_3^3 - 3dx_1x_2x_3 = \pm 1.$$

Lin Dazheng [1] proved that $\det(\mathbf{C}_n(L_n; F_{2n-1}, F_{2n-2}, \dots, F_n)) = 1$, i.e. if $d = L_n$ then $(x_1, x_2, \dots, x_n) = (F_{2n-1}, F_{2n-2}, \dots, F_n)$ is a solution of (4). The aim of this paper is to extend and generalize this result for more general sequences defined by (1) with $A^2 + 4B \neq 0$. In the proofs of our theorems we'll apply the methods and algorithms developed and presented in [1] by Lin Dazheng.

2. Results

Using (1) with $A^2 + 4B \neq 0$ and (3), we can state our results.

Theorem 1. *For $n \geq 2$*

$$\det(\mathbf{C}_n(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n)) = B^{n(n-1)},$$

i.e. $(x_1, x_2, \dots, x_n) = (R_{2n-1}, R_{2n-2}, \dots, R_n)$ is a solution of the generalized Pell's equation of degree n

$$\det(\mathbf{C}_n(V_n; x_1, x_2, \dots, x_n)) = B^{n(n-1)}.$$

Corollary 1. *For $n \geq 2$*

$$\prod_{k=0}^{n-1} \left(\sum_{j=1}^{n} R_{2n-j} \left(\sqrt[n]{V_n} \right)^{j-1} \varepsilon^{k(j-1)} \right) = B^{n(n-1)},$$

where $\sqrt[n]{V_n}$ denotes a fixed n^{th} complex root of V_n and $\varepsilon = e^{2\pi i/n}$.

It is known from [3] that the inverse of a quasi-cyclic matrix is quasi-cyclic. In our case we can prove the following result, too.

Theorem 2. *For $n \geq 3$ the matrix $\mathbf{C}_n(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n)$ is invertible and its inverse matrix $\mathbf{C}_n^{-1}$ is as follows:*

$$\mathbf{C}_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n) = (-1)^{n-1}B^{-n}(B\mathbf{I}_n + A\mathbf{E}_n - \mathbf{E}_n^2),$$

where $\mathbf{I}_n$ and $\mathbf{E}_n$ denotes the identity matrix of order n and the n by n matrix

$$\mathbf{E}_n = \begin{pmatrix} 0 & 0 & \dots & 0 & V_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{5}$$

respectively.

Remark. Naturally, if $|B| \neq 1$ then the entries of the matrix

$$\mathbf{C}_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n)$$

are not integers.

Corollary 2.

$$(x_1, x_2, \dots, x_n) = \begin{cases} (1, A, -1, 0, \dots, 0), & \text{if } n \geq 3 \text{ odd and } B = 1, \\ (1, -A, 1, 0, \dots, 0), & \text{if } n \geq 3 \text{ odd and } B = -1, \\ (-1, -A, 1, 0, \dots, 0), & \text{if } n \geq 4 \text{ even and } B = 1, \\ (1, -A, 1, 0, \dots, 0), & \text{if } n \geq 4 \text{ even and } B = -1 \end{cases}$$

is an other solution of the generalized Pell's equation

$$\det\left(\mathbf{C}_n(V_n; x_1, x_2, \dots, x_n)\right) = 1. \tag{6}$$

3. Proofs

To prove our theorems we need the following

Lemma. *Let the sequences R and V be defined by (1) and we suppose that $\alpha \neq \beta$ in (2). Then*

$$R_{n+1}R_{n-1} - R_n^2 = (-1)^n B^{n-1} \quad (n \geq 1), \tag{7/1}$$

(7/2) $$R_nV_n = R_{2n} \quad (n \geq 0),$$

(7/3) $$V_nR_{n+1} = R_{2n+1} + (-B)^n \quad (n \geq 0),$$

(7/4) $$\mathbf{E}_n^n = V_n\mathbf{I}_n \quad \text{and} \quad \mathbf{E}_n^{n+1} = V_n\mathbf{E}_n \quad (n \geq 3),$$

where $\mathbf{E}_n$ *is defined by (5).*

Proof. The first three properties of the Lemma are known or, using (2), they can be proven easily. For the proof of (7/4) consider the multiplication of matrices. For example:

$$\mathbf{E}_n^2 = \mathbf{E}_n \cdot \mathbf{E}_n = \begin{pmatrix} 0 & 0 & \dots & 0 & V_n & 0 \\ 0 & 0 & \dots & 0 & 0 & V_n \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{E}_n^3 = \mathbf{E}_n^2 \cdot \mathbf{E}_n = \begin{pmatrix} 0 & 0 & \dots & 0 & V_n & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & V_n & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & V_n \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 \end{pmatrix}, \ \dots,$$

$$\mathbf{E}_n^n = \begin{pmatrix} V_n & 0 & \dots & 0 & 0 \\ 0 & V_n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & V_n & 0 \\ 0 & 0 & \dots & 0 & V_n \end{pmatrix} = V_n\mathbf{I}_n$$

and so $\mathbf{E}_n^{n+1} = \mathbf{E}_n^n \cdot \mathbf{E}_n = (V_n\mathbf{I}_n)\,\mathbf{E}_n = V_n\mathbf{E}_n$.

Proof of Theorem 1. For $n = 2$ we get that

$$\det\left(\mathbf{C}_2(V_2; R_3, R_2)\right) = \begin{vmatrix} A^2 + B & A^3 + 2AB \\ A & A^2 + B \end{vmatrix} = B^2.$$

If $n > 2$, let us consider the n by n matrices

$$\mathbf{T}_n = \begin{pmatrix} 1 & -A & -B & 0 & \dots & 0 & 0 \\ 0 & 1 & -A & -B & \dots & 0 & 0 \\ 0 & 0 & 1 & -A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -A & -B \\ 0 & 0 & 0 & 0 & \dots & 1 & -A \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{C}_n = \mathbf{C}_n(V_n, R_{2n-1}, R_{2n-2}, \dots, R_n) = \begin{pmatrix} R_{2n-1} & V_n R_n & \dots & V_n R_{2n-2} \\ R_{2n-2} & R_{2n-1} & \dots & V_n R_{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ R_n & R_{n+1} & \dots & R_{2n-1} \end{pmatrix}.$$

Then, by (1), (2) and (7/1)–(7/3), one can verify that

$$\mathbf{C}_n\mathbf{T}_n = \begin{pmatrix} R_{2n-1} & BR_{2n-2} & (-B)^n & 0 & \dots & 0 \\ R_{2n-2} & BR_{2n-3} & 0 & (-B)^n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{n+2} & BR_{n+1} & 0 & 0 & \dots & (-B)^n \\ R_{n+1} & BR_n & 0 & 0 & \dots & 0 \\ R_n & BR_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Developing the $\det(\mathbf{C}_n\mathbf{T}_n)$ we get that

$$\begin{aligned} \det(\mathbf{C}_n\mathbf{T}_n) &= (-1)^{2n+2} \det \begin{pmatrix} R_{n+1} & BR_n \\ R_n & BR_{n-1} \end{pmatrix} \det\left((-B)^n \mathbf{I}_{n-2}\right) \\ &= B(R_{n+1}R_{n-1} - R_n^2)(-B)^{n(n-2)} = B(-1)^n B^{n-1}(-B)^{n(n-2)} \\ &= (-1)^{n(n-1)} B^{n(n-1)} = B^{n(n-1)}. \end{aligned}$$

But, since $\det(\mathbf{T}_n) = 1$, $\det(\mathbf{C}_n\mathbf{T}_n) = \det(\mathbf{C}_n) \cdot \det(\mathbf{T}_n) = \det(\mathbf{C}_n)$, therefore $\det(\mathbf{C}_n) = B^{n(n-1)}$, i.e. Theorem 1 is true.

Proof of Corollary 1. In [2] it is proven that if $\mathbf{C}_n$ is as in (3) then

$$\det\left(\mathbf{C}_n(d, x_1, x_2, \dots, x_n)\right) = \prod_{k=0}^{n-1} \left(\sum_{j=1}^{n} x_j \left(\sqrt[n]{d}\right)^{j-1} \varepsilon^{k(j-1)} \right), \tag{8}$$

where $\varepsilon = e^{2\pi i/n}$. Substituting in (8)

$$d = V_n \quad \text{and} \quad (x_1, x_2, \dots, x_n) = (R_{2n-1}, R_{2n-2}, \dots, R_n),$$

by Theorem 1, the statement of Corollary 1 immediately yields.

Proof of Theorem 2. Theorem 1 implies that $\mathbf{C}_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n)$ exists. It is easily verifyable that

$$\mathbf{C}_n(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n) = R_{2n-1}\mathbf{I}_n + R_{2n-2}\mathbf{E}_n + \dots + R_n\mathbf{E}_n^{n-1},$$

therefore we have to show that

$$(9) \quad \left(R_{2n-1}\mathbf{I}_n + R_{2n-2}\mathbf{E}_n + \dots + R_n\mathbf{E}_n^{n-1}\right)(-1)^{n-1}B^{-n}(B\mathbf{I}_n + A\mathbf{E}_n - \mathbf{E}_n^2) = \mathbf{I}_n.$$

By (1), the left hand side of (9) can be written as

$$(10) \quad (-1)^{n-1}B^{-n}\left(R_{2n-1}B\mathbf{I}_n + R_{2n-2}B\mathbf{E}_n + R_{2n-1}A\mathbf{E}_n + R_nA\mathbf{E}_n^n\right.$$
$$\left. -R_{n+1}\mathbf{E}_n^n - R_n\mathbf{E}_n^{n+1} + \mathbf{O}_n + \dots + \mathbf{O}_n\right),$$

where $\mathbf{O}_n$ is the zero-matrix of order n.
Thus, applying (1), (7/1)–(7/4) and (2), the form (10) is equal to

$$\begin{aligned}
&(-1)^{n-1}B^{-n}\left(R_{2n-1}B\mathbf{I}_n + (BR_{2n-2} + AR_{2n-1})\mathbf{E}_n\right. \\
&\left. + R_nAV_n\mathbf{I}_n - R_{n+1}V_n\mathbf{I}_n - R_nV_n\mathbf{E}_n\right) \\
&= (-1)^{n-1}B^{-n}\left(R_{2n-1}B\mathbf{I}_n + (R_{2n} - R_nV_n)\mathbf{E}_n + V_n(AR_n - R_{n+1})\mathbf{I}_n\right) \\
&= (-1)^{n-1}B^{-n}\left(R_{2n-1}B\mathbf{I}_n + \mathbf{O}_n - V_nBR_{n-1}\mathbf{I}_n\right) \\
&= (-1)^{n-1}B^{-n+1}\left(R_{2n-1} - V_nR_{n-1}\right)\mathbf{I}_n \\
&= (-1)^{n-1}B^{-n+1}(-B)^{n-1}\mathbf{I}_n = (-1)^{2n-2}B^0\mathbf{I}_n = \mathbf{I}_n,
\end{aligned}$$

which completes the proof of Theorem 2.

Proof of Corollary 2. By Theorem 2

$$\det\left(\mathbf{C}_n(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n)\right) \cdot \det\left(\mathbf{C}_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n)\right) = 1$$

thus, if $|B| = 1$ then, by Theorem 1,

$$\det\left(\mathbf{C}_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n)\right) = 1.$$

E.g. let $n \geq 3$ be an odd integer and $B = 1$. Then, by Theorem 2,

$$\mathbf{C}_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n) = \mathbf{I}_n + A\mathbf{E}_n - \mathbf{E}_n^2$$

$$= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & -V_n & AV_n \\ A & 1 & 0 & \dots & 0 & 0 & -V_n \\ -1 & A & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & A & 1 \end{pmatrix},$$

i.e. $(x_1, x_2, \dots, x_n) = (1, A, -1, 0, \dots, 0)$ is a solution of (6).

The proof is similar when $n \geq 3$ odd and $B = -1$, or $n \geq 4$ even and $|B| = 1$.

References

[1] Lin Dazheng, Fibonacci–Lucas Quasi-Cyclic Matrices, *The Fibonacci Quarterly* **40.2** (2002), 280–286.

[2] Shen Guangxing, On Eigenvalues of Some Cyclic Matrices, *Math. Application* **4.3** (1991), 76–82.

[3] Tang Taiming and Lin Dazheng, Diophantine Approximation, Circulant Matrix and Pell Equation, *J. Shaansi Normal Univ. (Natural Science Ed.)*, **28.14** (2000), 6–11.

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