REMARKS ON UNIFORM DENSITY OF SETS OF INTEGERS

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Dedicated to the memory of Professor Péter Kiss

Abstract. The concept of the uniform density is introduced in papers [1], [2]. Some properties of this concept are studied in this paper. It is proved here that the uniform density has the Darboux property.

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Introduction

Let \( A \subseteq \mathbb{N} = \{1,2,3,\ldots\} \) and \( m,n \in \mathbb{N} \), \( m < n \). Denote by \( A(m,n) \) the cardinality of the set \( A \cap [m,n] \). The numbers

\[
\underline{d}(A) = \lim_{n \to \infty} \frac{A(1,n)}{n}, \quad \overline{d}(A) = \lim_{n \to \infty} \frac{A(1,n)}{n}
\]

are called the lower and the upper asymptotic density of the set \( A \). If there exists

\[
d(A) = \lim_{n \to \infty} \frac{A(1,n)}{n}
\]

then it is called the asymptotic density of \( A \).

According to [1], [2] we set

\[
\alpha_s = \min_{t \geq 0} A(t+1,t+s), \quad \alpha^s = \max_{t \geq 0} A(t+1,t+s).
\]

Then there exist

\[
\underline{u}(A) = \lim_{s \to \infty} \frac{\alpha_s}{s}, \quad \overline{u}(A) = \lim_{s \to \infty} \frac{\alpha^s}{s}
\]

and they are called the lower and the upper uniform density of \( A \), respectively.
It is obvious that for every $A \subseteq N$

$$u(A) \leq d(A) \leq \bar{d}(A) \leq \bar{u}(A).$$

Hence if $u(A)$ exists then $d(A)$ exists as well and $u(A) = d(A)$. The converse is not true. For example put

$$A = \bigcup_{k=1}^{\infty} \{10^k + 1, 10^k + 2, \ldots, 10^k + k\}.$$ 

Then $d(A) = 0$, but $u(A) = 0$, $\bar{u}(A) = 1$.

Note that the numbers $\alpha_s$ and $\alpha^s$ can be replaced by the numbers $\beta_s$ and $\beta^s$, respectively, where

$$\beta_s = \lim_{t \to \infty} A(t + 1, t + s), \quad \beta^s = \lim_{t \to \infty} A(t + 1, t + s)$$

(cf. [1], [2]).

In this paper we introduce some elementary remarks, observations on the concept of the uniform density and prove that this density has the Darboux property.

1. Uniform density $u(A)$ and $\lim_{s \to \infty} \frac{A(t+1,t+s)}{s}$ (uniformly with respect to $t \geq 0$)

We introduce the following observation.

**Theorem 1.1.** If there exists

(1) $$\lim_{s \to \infty} \frac{A(t + 1, t + s)}{s} = L$$

uniformly with respect to $t \geq 0$, then there exists $u(A)$ and $u(A) = L$.

**Proof.** Let $\varepsilon > 0$. By the assumption there exists an $s_0 = s_0(\varepsilon) \in N$ such that for each $s > s_0$ and each $t \geq 0$ we have

$$(L - \varepsilon)s < A(t + 1, t + s) < (L + \varepsilon)s.$$

By the definition of the numbers $\beta_s, \beta^s$ we get from this for $s > s_0$

$$L - \varepsilon \leq \frac{\beta_s}{s} \leq \frac{\beta^s}{s} \leq L + \varepsilon.$$
If $s \to \infty$ we get

$$L - \varepsilon \leq \underline{u}(A) \leq \bar{u}(A) \leq L + \varepsilon.$$ 

Since $\varepsilon > 0$ is an arbitrary positive number, we get $u(A) = L$.

The foregoing theorem can be conversed.

**Theorem 1.2.** If there exists $u(A)$ then

$$\lim_{s \to \infty} \frac{A(t + 1, t + s)}{s} = u(A)$$

uniformly with respect to $t \geq 0$.

**Proof.** Put $u(A) = L$. Since

$$L = \lim_{p \to \infty} \frac{\alpha_p}{p} = \lim_{p \to \infty} \frac{\alpha^p}{p}$$

for every $\varepsilon > 0$, there exists a $p_0$ such that for each $p > p_0$ we have

$$(L - \varepsilon)p < \alpha_p \leq \alpha^p < (L + \varepsilon)p.$$ 

So we get

$$(L - \varepsilon)p < \min_{t \geq 0} A(t + 1, t + p) \leq \max_{t \geq 0} A(t + 1, t + p) < (L + \varepsilon)p.$$ 

By the definition of $A(t + 1, t + p)$ we get from this

$$\left| \frac{A(t + 1, t + p)}{p} - L \right| \leq \varepsilon$$

for each $p > p_0$ and each $t \geq 0$. Hence

$$\lim_{p \to \infty} \frac{A(t + 1, t + p)}{p} = L \ (= u(A))$$

uniformly with respect to $t \geq 0$.

2. **Uniform density and almost convergence**

The concept of almost convergence was introduced in [5] (see also [10], p. 60).

A sequence $(x_n)^\infty_{n=1}$ of real numbers almost converges to $L$ if

$$\lim_{p \to \infty} \frac{x_{n+1} + x_{n+2} + \cdots + x_{n+p}}{p} = L.$$
uniformly with respect to \( n \geq 0 \). If \((x_n)_{n=1}^{\infty}\) almost converges to \( L \), we write
\[
F - \lim x_n = L.
\]

One can conjecture that there is a relationship between the uniform density of a set \( A \subseteq \mathbb{N} \) and the characteristic function \( \chi_A \) of this set (\( \chi_A(n) = 1 \) if \( n \in A \), \( \chi_A(n) = 0 \) if \( n \in \mathbb{N} \setminus A \)).

**Theorem 2.1.** Let \( A \subseteq \mathbb{N} \). Then \( u(A) = v \) if and only if \( F - \lim \chi_A(n) = v \).

**Proof.** Let \( t \geq 0, s \in \mathbb{N} \). By the definition of the sequence \((\chi_A(n))_{n=1}^{\infty}\) we see that
\[
\frac{A(t+1, t+s)}{s} = \frac{\chi_A(t+1) + \chi_A(t+2) + \cdots + \chi_A(t+s) - t}{s}.
\]
The assertion follows from this equality by Theorem 1.1 and 1.2.

3. Another way for defining the uniform density of sets

If \( A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subseteq \mathbb{N} \) is an infinite set then it is well–known that
\[
d(A) = \lim_{n \to \infty} \frac{n}{a_n}, \quad \bar{d}(A) = \lim_{n \to \infty} \frac{n}{a_n}
\]
and
\[
d(A) = \lim_{n \to \infty} \frac{n}{a_n}
\]
(if \( d(A) \) exists) (cf. [8], p. 247). A similar result can be stated also for the uniform density.

**Theorem 3.1.** Let \( A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subseteq \mathbb{N} \) be an infinite set. Then \( u(A) = L \) if and only if
\[
\lim_{p \to \infty} \frac{p}{a_{k+p} - a_{k+1}} = L
\]
uniformly with respect to \( k \geq 0 \).

**Proof.** 1. Let \( u(A) = L \). Consider that for \( p \geq 2 \)
\[
\frac{p}{a_{k+p} - a_{k+1}} = \frac{A(a_{k+1}, a_{k+p})}{a_{k+p} - a_{k+1}}.
\]
By Theorem 1.2 (see (1)) the right-hand side converges by \( p \to \infty \) (uniformly with respect to \( k \geq 0 \)) to \( u(A) = L \). Hence (2) holds.

2. Suppose that (2) holds (uniformly with respect to \( k \geq 0 \)). By Theorem 1.1 it suffices to prove that
\[
\lim_{p \to \infty} \frac{A(t+1, t+p)}{p} = L
\]
uniformly with respect to \( t \geq 0 \).

We shall show it. Suppose in the first place that \( t \geq a_1 \). Then there exist \( k, s \in \mathbb{N} \) such that
\[
a_k < t + 1 < a_{k+1} < \cdots < a_{k+s} \leq t + p < a_{k+s+1}.
\]
Then \( A(t + 1, t + p) \) equals to \( s \) and so
\[
\frac{A(t + 1, t + p)}{p} = \frac{s}{p}.
\]

Further on the basis of choice of the numbers \( k, s \) we get
\[
a_{k+s} - a_{k+1} \leq p - 1 < a_{k+s+1} - a_k.
\]
Therefore
\[
\frac{s}{a_{k+s+1} - a_k + 1} < \frac{A(t + 1, t + p)}{p} < \frac{s}{a_{k+s} - a_{k+1}}.
\]
But \(-a_k + 1 \leq -a_{k-1}\), so that
\[
\frac{s}{a_{k+s+1} - a_k + 1} \geq \frac{s}{a_{k+s+1} - a_{k-1}} = \frac{s + 3}{a_{k+s+1} - a_{k-1}} \left( 1 - \frac{3}{s + 3} \right).
\]

So we get wholly
\[
(3) \quad \frac{s + 3}{a_{k+s+1} - a_{k-1}} \left( 1 - \frac{3}{s + 3} \right) < \frac{A(t + 1, t + p)}{p} < \frac{s}{a_{k+s} - a_{k+1}}.
\]
Let \( \gamma > 0 \). Then by assumption (see (2)) there exists a \( v_0 \) such that for each \( v > v_0 \) we have
\[
(4) \quad -\gamma < \frac{v}{a_{k+v} - a_{k+1}} - L < \gamma
\]
for all \( k \geq 0 \).

Using (4) we get from (3)
\[
(5) \quad \frac{s + 3}{a_{k+s+1} - a_{k-1}} - L - \frac{3}{a_{k+s+1} - a_{k-1}} < \frac{A(t + 1, t + p)}{p} - L < \frac{s}{a_{k+s} - a_{k+1}} - L.
\]
Let \( s > v_0 \). Then by (4) the right-hand side of (5) is less than \( \gamma \). On the left-hand side we get
\[
\frac{s + 3}{a_{k+s+1} - a_{k-1}} - L > -\gamma.
\]
Further
\[
\frac{-3}{a_{k+s+1} - a_{k-1}} \geq \frac{-3}{s+2},
\]
since
\[
a_{k+s+1} - a_{k-1} = (a_k - a_{k-1}) + (a_{k+1} - a_k) + \cdots + (a_{k+s+1} - a_{k+s})
\]
and each summand on the right-hand side is \( \geq 1 \).

Hence for every \( t \geq a_1 \) we get from (5) \((s > v_0)\)
\[
(6) \quad -\gamma - \frac{3}{s + 2} < \frac{A(t + 1, t + p)}{p} - L < \gamma
\]

From this
\[
\lim_{p \to \infty} \frac{A(t + 1, t + p)}{p} = L
\]
uniformly with respect to \( t \geq a_1 \).

It remains the case if \( 0 \leq t < a_1 \). Since there is only a finite number of such \( t \)’s, it suffices to show that for each fixed \( t, 0 \leq t < a_1 \), we have
\[
(7) \quad \lim_{p \to \infty} \frac{A(t + 1, t + p)}{p} = L.
\]

If \( t \) is fixed, \( 0 \leq t < a_1 \) and \( p \) is sufficiently large we can determine a \( k \) such that \( a_k \leq t + p < a_{k+1} \). Then
\[
0 \leq t < a_1 < a_2 < \cdots < a_k \leq t + p < a_{k+1}
\]
and
\[
(8) \quad A(t + 1, t + p) = A(t + 1, a_1) + A(a_2, a_k).
\]

From this
\[
(8') \quad p < a_{k+1}, \quad p > a_k - a_1
\]
and so from (8), (8’) we obtain
\[
\frac{A(t + 1, a_1)}{p} + \frac{A(a_2, a_{k+1}) - 1}{a_{k+1}} \leq \frac{A(t + 1, t + p)}{p} \leq \frac{A(t + 1, a_1)}{p} + \frac{k - 1}{a_k - a_1}.
\]
Obviously we have \( A(t + 1, a_1) \leq a_1 \) and so

\[
\frac{A(t + 1, a_1)}{p} = o(1) \quad (p \to \infty).
\]

We arrange the left-hand side of (9). We get

\[
\frac{A(a_2, a_{k+1}) - 1}{a_{k+1}} = \frac{k}{a_{k+1} - a_2} \frac{a_{k+1} - a_2}{a_{k+1}} = o(1) + \frac{k}{a_{k+1} - a_2}
\]

(if \( p \to \infty \) then \( k \to \infty \), as well).

Wholly we have

\[
\frac{k}{a_{k+1} - a_2} + o(1) \leq \frac{A(t + 1, t + p)}{p} \leq \frac{k - 1}{a_k - a_1} + o(1).
\]

If \( p \to \infty \), then \( k \to \infty \) and by assumption (cf (2)) the terms

\[
\frac{k - 1}{a_k - a_1} = L, \quad \frac{k}{a_{k+1} - a_2} = L
\]

converge to zero. But then (9) yields

\[
\lim_{p \to \infty} \frac{A(t + 1, t + p)}{p} = L
\]

uniformly with respect to \( t \geq 0 \). So \( u(A) = L \).

The following theorem is a simple consequence of Theorem 3.1

**Theorem 3.2.** Let \( A = \{a_1 < a_2 < \cdots \} \subseteq N \) be a lacunary set, i.e.

\[
(10) \quad \lim_{n \to \infty} (a_{n+1} - a_n) = +\infty.
\]

Then \( u(A) = 0 \).

**Proof.** Let \( \varepsilon > 0 \). Choose \( M \in N \) such that \( M^{-1} < \varepsilon \). By the assumption there exists an \( n_0 \) such that for each \( n > n_0 \) we get \( a_{n+1} - a_n > M \).

Let \( k > n_0, s \in N, s > 1 \). Then

\[
a_{k+s} - a_{k+1} = (a_{k+2} - a_{k+1}) + (a_{k+3} - a_{k+2}) + \cdots + (a_{k+s} - a_{k+s-1}) > (s - 1)M
\]

and so

\[
\frac{s}{a_{k+s} - a_{k+1}} < \frac{s}{(s - 1)M} < 2\varepsilon.
\]
Hence for each $k > n_0$ and $s \geq 2$ we have

$$\frac{s}{a_{k+s} - a_{k+1}} < 2\epsilon.$$  

If $0 \leq k \leq n_0$, $k$ is fixed, then

$$\lim_{s \to \infty} \frac{s}{a_{k+s} - a_{k+1}} = 0,$$

since, for sufficiently large $s$

$$a_{k+s} - a_{k+1} = [(a_{k+2} - a_{k+1}) + \cdots + (a_{n_0+1} - a_{n_0})] + [(a_{n_0+2} - a_{n_0+1}) + \cdots + (a_{k+s} - a_{k+s-1})] > M(k + s - n_0 - 1) \geq M(s - (n_0 + 1)).$$

There exists only a finite number of $k$’s with $0 \leq k \leq n_0$, so we see that (11) holds uniformly with respect to $k$, $0 \leq k \leq n_0$. So we get wholly

$$\lim_{s \to \infty} \frac{s}{a_{k+s} - a_{k+1}} = 0$$

uniformly with respect to $k \geq 0$. So according to Theorem 3.1, $u(A) = 0$.

**Remark.** The assumption (10) in Theorem 3.2 cannot be replaced by the weaker assumption

$$(10') \quad \lim_{n \to \infty} (a_{n+1} - a_n) = +\infty.$$  

This can be shown by the following example:

$$A = \bigcup_{k=1}^{\infty} \{k! + 1, k! + 2, \ldots, k! + k\} = \{a_1 < a_2 < \cdots < a_n < \cdots\}.$$  

Here we have $u(A) = 0$, $\bar{u}(A) = 1$ and $(10')$ is satisfied.

**Example 3.1** Let $\alpha \in R$, $\alpha > 1$. Put $a_k = [k\alpha]$, ($k = 1, 2, \ldots$), where $[v]$ denotes the integer part of $v$. We show that the uniform density of the set $A$ is $\frac{1}{\alpha}$. This follows from Theorem 3.1, since

$$\lim_{p \to \infty} \frac{p}{a_{k+p} - a_{k+1}} = \frac{1}{\alpha}.$$
uniformly with respect to $k \geq 0$. This uniform convergence can be shown by a simple calculation which gives the estimates ($p \geq 2$)

$$\frac{p}{(p-1)\alpha + 1} \leq \frac{p}{a_{k+p} - a_{k+1}} \leq \frac{p}{(p-1)\alpha - 1}.$$ 

4. Darboux property of the uniform density

For every $A \subseteq N$ having the uniform density the number $u(A)$ belongs to $[0,1]$. The natural question arises whether also conversely for every $t \in [0,1]$ there is a set $A \subseteq N$ such that $u(A) = t$. The answer to this question is positive.

**Theorem 4.1.**

If $t \in [0,1]$ then there is a set $A \subseteq N$ with $u(A) = t$.

**Proof.** We can already suppose that $0 < t < 1$. Construct the set

$$A = \left\{ \left\lfloor \frac{1}{t} \right\rfloor, \left\lfloor \frac{2}{t} \right\rfloor, \ldots, \left\lfloor \frac{k}{t} \right\rfloor, \ldots \right\} = \{a_1 < a_2 < \cdots\}.$$

Put $a_k = \left\lfloor \frac{k}{t} \right\rfloor$ ($k = 1, 2, \ldots$) and set in Example 3.1 $\alpha = \frac{1}{t} > 1$. So we get

$$\lim_{p \to \infty} \frac{p}{a_{k+p} - a_{k+1}} = \frac{1}{\alpha} = t$$

uniformly with respect to $k \geq 0$. The assertion follows by Theorem 3.1.

Let $v$ be a non-negative set function defined on a class $S \subseteq 2^N$. The function $v$ is said to have the Darboux property provided that if $v(A) > 0$ for $A \in S$ and $0 < t < v(A)$, then there is a set $B \subseteq A, B \in S$ such that $v(B) = t$ (cf. [6], [7], [9]).

**Theorem 4.2.** The uniform density has the Darboux property.

**Proof.** Let $u(A) = \delta > 0$,

$$A = \{a_1 < a_2 < \cdots < a_k < \cdots\}$$

and $0 < t < \delta$. Construct the set

$$B = \{b_1 < b_2 < \cdots < b_k < \cdots\}$$

in such a way that we set

$$b_k = a_{\left\lfloor \frac{k}{\delta} \right\rfloor} \ (k = 1, 2, \ldots).$$
Put \( n_k = \lfloor k \delta \rfloor \) \( (k = 1, 2, \ldots) \). Then \( n_1 < n_2 < \cdots < n_k < \cdots \), \( B = \{ a_{n_1} < a_{n_2} < \cdots < a_{n_k} < \cdots \} \), \( B \subseteq A \).

We prove that \( u(B) = t \).

By Theorem 3.1 it suffices to show that

\[
\lim_{p \to \infty} \frac{p}{b_{m+p} - b_{m+1}} = t \tag{12}
\]

uniformly with respect to \( m \geq 0 \).

We have \( (p > 1) \)
\[
\frac{p}{b_{m+p} - b_{m+1}} = \frac{p}{a_{n_{m+p}} - a_{n_{m+1}}} \cdot \frac{b_{m+p} - b_{m+1}}{a_{n_{m+p}} - a_{n_{m+1}}}.
\]

By a simple arrangement we get

\[
\frac{p}{b_{m+p} - b_{m+1}} = \frac{n_{m+p} - n_{m+1} + 1}{a_{n_{m+p}} - a_{n_{m+1}}} \cdot \frac{p}{n_{m+p} - n_{m+1} + 1} \cdot \frac{b_{m+p} - b_{m+1}}{a_{n_{m+p}} - a_{n_{m+1}}} \tag{13}
\]

A simple estimation gives

\[
(p - 1)\frac{\delta}{t} - 1 < n_{m+p} - n_{m+1} < (p - 1)\frac{\delta}{t} + 1.
\]

Using this in (13) we get

\[
\lim_{p \to \infty} \frac{p}{n_{m+p} - n_{m+1} + 1} = \frac{t}{\delta} \tag{14}
\]

uniformly with respect to \( m \geq 0 \).

Further by assumption

\[
\lim_{p \to \infty} \frac{p}{a_{s+p} - a_{s+1}} = \delta
\]

uniformly with respect to \( s \geq 0 \) (Theorem 3.1).

So we get

\[
\lim_{p \to \infty} \frac{n_{m+p} - n_{m+1} + 1}{a_{n_{m+p}} - a_{n_{m+1}}} = \delta \tag{15}
\]

uniformly with respect to \( m \geq 0 \) since the sequence

\[
\left( \frac{n_{m+p} - n_{m+1} + 1}{a_{n_{m+p}} - a_{n_{m+1}}} \right)_{p=2}^{\infty}
\]
is a subsequence of the sequence

\[ \left( \frac{p}{a_{s+p} - a_{s+1}} \right)_{p=1}^{\infty} \]

By (13), (14), (15) we get (12) uniformly with respect to \( m \geq 0 \).

References


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