LINEAR RECURRENCES AND ROOTFINDING METHODS

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Abstract. Let A,B,G_0 and G_1 be fixed complex numbers, where $AB(|G_0|+|G_1|)\neq 0$. Denote by α and β the roots of the equation $\lambda^2-A\lambda+B=0$ and suppose that $|\alpha|>|\beta|$. The sequence $\left\{W_{n,d}^{(k)}\right\}_{n=0}^{\infty}$ is defined by $W_{n,d}^{(k)}=\left(a^k\alpha^{nk+d}-b^k\beta^{nk+d}\right)/(\alpha-\beta)$, where $k\geq 1$ and $d\geq 0$ are fixed integers, $a=G_1-\beta G_0\neq 0$ and $b=G_1-\alpha G_0$. In this paper, using new identities of the sequence $\left\{W_{n,d}^{(k)}\right\}_{n=0}^{\infty}$, an other proof is presented for the Newton–Raphson and Halley transformations (accelerations) of the sequence $\left\{W_{n,d}^{(k)}/W_{n,0}^{(k)}\right\}_{n=0}^{\infty}$. It is also shown that the (transformed) sequences obtained by the secant, Newton–Raphson, Halley and Aitken transformations of the sequence $\left\{W_{n,d}^{(k)}/W_{n,0}^{(k)}\right\}_{n=0}^{\infty}$ tend to α^d in order of $\mathrm{o}\left(W_{n,d}^{(k)}/W_{n,0}^{(k)}-\alpha^d\right)$.

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1. Introduction

Let the n^{th} $(n\geq 2)$ term of the sequence $\{G_n\}_{n=0}^{\infty}$ be defined by the recursion

$$G_n = AG_{n-1} - BG_{n-2},$$

where A, B, G_0 and G_1 are fixed complex numbers and $AB(|G_0|+|G_1|)\neq 0$. If it is needed then the notation $G_n(A,B,G_0,G_1)$ is also used. For example, the n^{th} term of the Fibonacci sequence is $F_n = G_n(1,-1,0,1)$. The abbreviations $U_n = G_n(A,B,0,1)$ and $V_n = (A,B,2,A)$ will also be very useful for us.

Let α and β be the roots of the equation $\lambda^2 - A\lambda + B = 0$ $(\alpha+\beta = A,\ \alpha\beta = B)$ and suppose that $|\alpha| > |\beta|$. By the well known Binet formula we get that the explicit form of the term $G_n(A,B,G_0,G_1)$ is

$$G_n(A,B,G_0,G_1) = \frac{a\alpha^n - b\beta^n}{\alpha-\beta} \quad (n\geq 0), \tag{1}$$

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where $a = G_1 - \beta G_0$, $b = G_1 - \alpha G_0$ and suppose that $a \neq 0$. For example, $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $V_n = \alpha^n + \beta^n$ if $\alpha, \beta = \left(A \pm \sqrt{A^2 - 4B}\right)/2$.

Z. Zhang [7] has defined the sequence $\left\{W_{n,d}^{(k)}(A, B, G_0, G_1)\right\}_{n=0}^{\infty}$ in the following manner.

$$W_{n,d}^{(k)}(A, B, G_0, G_1) = \left(\alpha^k + \beta^k\right) W_{n-1,d}^{(k)} - \alpha^k \beta^k W_{n-2,d}^{(k)} \quad (n \geq 2), \tag{2}$$

where $k \geq 1$ and $d \geq 0$ are fixed integers, while

$$W_{0,d}^{(k)}(A, B, G_0, G_1) = \frac{a^k \alpha^d - b^k \beta^d}{\alpha - \beta}, \quad W_{1,d}^{(k)}(A, B, G_0, G_1) = \frac{a^k \alpha^{k+d} - b^k \beta^{k+d}}{\alpha - \beta}.$$

For brevity, we write $W_{n,d}^{(k)}$ instead of $W_{n,d}^{(k)}(A, B, G_0, G_1)$.

It is obvious that α^k and β^k are the roots of the equation

$$\lambda^2 - (\alpha^k + \beta^k)\lambda + \alpha^k \beta^k = \lambda^2 - V_k \lambda + B^k = 0$$

and $|\alpha| > |\beta|$ implies $|\alpha^k| > |\beta|^k$. Using the Binet formula for (2) we get that

$$W_{n,d}^{(k)} = \frac{\left(W_{1,d}^{(k)} - \beta^k W_{0,d}^{(k)}\right) \alpha^{nk} - \left(W_{1,d}^{(k)} - \alpha^k W_{0,d}^{(k)}\right) \beta^{nk}}{\alpha^k - \beta^k},$$

from which

$$W_{n,d}^{(k)} = \frac{a^k \alpha^{nk+d} - b^k \beta^{nk+d}}{\alpha - \beta} \tag{3}$$

yields for $n \geq 0$. It can be seen that $W_{n,d}^{(k)}$ is a generalization of G_n because e. g.

$$G_n = G_n(A, B, G_0, G_1) = W_{n,0}^{(1)}(A, B, G_0, G_1).$$

If $W_{n,0}^{(k)} \neq 0$ then let

$$R_{n,d}^{(k)} = \frac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}}. \tag{4}$$

By (3), $a \neq 0$ and $|\alpha| > |\beta|$, one can easily prove that

$$\lim_{n \to \infty} R_{n,d}^{(k)} = \alpha^d,$$

i. e. the sequence $\left\{R_{n,d}^{(k)}\right\}_{n=0}^{\infty}$ tends to the root α^d of the polynomial

$$f(\lambda) = \lambda^2 - (\alpha^d + \beta^d)\lambda + \alpha^d\beta^d = \lambda^2 - V_d\lambda + B^d. \tag{5}$$

Recently, many authors have studied the connection between recurrences and iterative transformations. The main idea is to consider such sequence transformations T of the convergent sequence $\{X_n\}_{n=0}^{\infty}$ into the sequence $\{T_n\}_{n=0}^{\infty}$, where $\{T_n\}_{n=0}^{\infty}$ converges more quickly to the same limit X. Thus, one can investigate the properties of these transformations or the accelerations of the convergence. We say that $\{T_n\}_{n=0}^{\infty}$ converges more quickly to X than $\{X_n\}_{n=0}^{\infty}$ if $T_n - X = \mathrm{o}(X_n - X)$, i. e. if $\lim\limits_{n\to\infty} ((T_n - X) / (X_n - X)) = 0$.

The most known four sequence transformations to accelerate the convergence of a sequence are the secant $S(X_n, X_m)$, Newton–Raphson $N(X_n)$, Halley $H(X_n)$ and Aitken transformation $A(X_n, X_m, X_t)$, namely if $\{X_n\}_{n=0}^{\infty} = \left\{R_{n,d}^{(k)}\right\}_{n=0}^{\infty}$ and $X = \alpha^d$ (i. e. the root of $f(\lambda) = 0$ in (5)), then

$$S(X_n, X_m) = \frac{X_n X_m - B^d}{X_n + X_m - V_d}, \tag{6}$$

$$N(X_n) = \frac{X_n^2 - B^d}{2X_n - V_d}, \tag{7}$$

$$H(X_n) = \frac{X_n^3 - 3B^d X_n + V_d B^d}{3X_n^2 - 3V_d X_n + V_d^2 - B^d}, \tag{8}$$

$$A(X_n, X_m, X_t) = \frac{X_n X_t - X_m^2}{X_n - 2X_m + X_t}, \tag{9}$$

where we assume that division by zero does not occur. (The formulae (6)-(9) can be obtained from (5) using the known forms of the transformations S, N, H and A, or they can be found in [4] p. 366 and p. 369.)

Some results from the recent past: G. M. Phillips [5] proved that if $r_n^{'} = \frac{F_{n+1}}{F_n}$ then $A(r_{n-t}^{'}, r_n^{'}, r_{n+t}^{'}) = r_{2n}^{'}$. J. H. McCabe and G. M. Phillips [3] generalized this for $r_n^{''} = \frac{U_{n+1}}{U_n}$, and they also proved that $S\left(r_n^{''}, r_m^{''}\right) = r_{n+m}^{''}$ and $N(r_n^{''}) = r_{2n}^{''}$. M. J. Jamieson [1] investigated the case $r_n^{''} = \frac{F_{n+d}}{F_n}$ for $d > 1$. J. B. Muskat [4], using the notations $r_n = \frac{U_{n+d}}{U_n}$ and $R_n = \frac{V_{n+d}}{V_n}$ $(d > 1)$, proved that

(10)
(a) $S(r_n, r_m) = r_{n+m}$, $\quad S(R_n, R_m) = r_{n+m}$,
(b) $N(r_n) = r_{2n}$, $\quad N(R_n) = r_{2n}$,

$(c)\ H(r_n) = r_{3n}, \qquad H(R_n) = R_{3n},$
$(d)\ A(r_{n-t}, r_n, r_{n+t}) = r_{2n}, \qquad A(R_{n-t}, R_n, R_{n+t}) = r_{2n}.$

Similar results were obtained for special second order linear recurrences in [2] by F. Mátyás, while Z. Zhang ([7],[8]) stated and partially proved that

$$\begin{aligned}
&(a)\ S\left(R^{(k)}_{n,d}, R^{(k)}_{m,d}\right) = R^{(2k)}_{(n+m)/2,d}, \quad (2|n+m),\\
(11)\qquad &(b)\ N\left(R^{(k)}_{n,d}\right) = R^{(2k)}_{n,d},\\
&(c)\ H\left(R^{(k)}_{n,d}\right) = R^{(3k)}_{n,d},\\
&(d)\ A\left(R^{(k)}_{n-t,d}, R^{(k)}_{n,d}, R^{(k)}_{n+t,d}\right) = R^{(2k)}_{n,d}.
\end{aligned}$$

It is easy to see that (11) implies (10) if $k = 1, G_0 = 0, G_1 = 1$ or $k = 1, G_0 = 2, G_1 = A$. We mention that R. B. Taher and M. Rachidi [6] investigated the so-called ε–algorithm to the ratio of the terms of linear recurrences of order $r \geq 2$.

The purpose of this paper is to present some new properties of the sequence $\left\{W^{(k)}_{n,d}\right\}_{n=0}^{\infty}$ (see Lemma 1 and Lemma 2) and, using them, to give new proofs for (11)/(b) and (c), since Z. Zhang, using some other properties proven by him, presented the proof for only the cases (11)/(a) and (d) in [7] and [8]. We also show that the transformations S, N, H and A creat such sequences from $\left\{R^{(k)}_{n,d}\right\}_{n=0}^{\infty}$ which tend to α^d in order of $\mathrm{o}\left(R^{(k)}_{n,d} - \alpha^d\right)$.

2. Results

Applying the notations introduced in this paper, assume that $k \geq 1$ and $d \geq 0$ are fixed integers, in (1) $AB\left(|G_0| + |G_1|\right) \neq 0, a \neq 0$ and $|\alpha| > |\beta|$. We always assume that division by zero does not occur. First we formulate two lemmas.

Lemma 1. *Let n and m be non-negative integers with the same parity. Then*

$$(a)\quad W^{(k)}_{n,d}W^{(k)}_{m,d} - W^{(k)}_{n,0}W^{(k)}_{m,0}B^d = W^{(2k)}_{\frac{n+m}{2},d}U_d,$$

$$(b)\quad W^{(k)}_{n,d}W^{(k)}_{m,0} + W^{(k)}_{m,d}W^{(k)}_{n,0} - W^{(k)}_{n,0}W^{(k)}_{m,0}V_d = W^{(2k)}_{\frac{n+m}{2},0}U_d.$$

Lemma 2. *Let n be a non-negative integer. Then*

$$(a)\ W^{(k)}_{n,d}W^{(2k)}_{n,d} - W^{(k)}_{n,0}W^{(2k)}_{n,0}B^d = W^{(3k)}_{n,d}U_d,$$

$$(b)\ W^{(2k)}_{n,d}W^{(k)}_{n,0} - W^{(2k)}_{n,0}W^{(k)}_{n,0} + W^{(k)}_{n,d}W^{(2k)}_{n,0} = W^{(3k)}_{n,0}U_d.$$

Theorem 1. *Let n be a non-negative integer. Then*

(a) $N\left(R_{n,d}^{(k)}\right) = R_{n,d}^{(2k)}$,

(b) $H\left(R_{n,d}^{(k)}\right) = R_{n,d}^{(3k)}$.

The following theorem implies that the transformations S, N, H and A produce such sequences from the sequence $\left\{R_{n,d}^{(k)}\right\}_{n=0}^{\infty}$ which tend very quickly to α^d.

Theorem 2. *Let $l > k \geq 1$ be fixed integers. Then*

$$R_{n,d}^{(l)} - \alpha^d = o\left(R_{n,d}^{(k)} - \alpha^d\right).$$

Corollary. *Theorem 1 and (11) show that the transformations S, N, A and H transform $R_{n,d}^{(k)}$ into $R_{n,d}^{(2k)}$ and into $R_{n,d}^{(3k)}$, respectively, thus Theorem 2 implies that all of the mentioned transformations give accelerations of the convergence.*

3. Proofs of Lemmas and Theorems

Proof of Lemma 1. Because of the similarity of the proofs we present only the proof of part (a). Using the explicit form (3) of $W_{n,d}^{(k)}$, we write

$$W_{n,d}^{(k)} W_{m,d}^{(k)} - W_{n,0}^{(k)} W_{m,0}^{(k)} B^d = \frac{(a^k\alpha^{nk+d} - b^k\beta^{nk+d})(a^k\alpha^{mk+d} - b^k\beta^{mk+d})}{(\alpha-\beta)^2}$$

$$-\frac{(a^k\alpha^{nk} - b^k\beta^{nk})(a^k\alpha^{mk} - b^k\beta^{mk})\alpha^d\beta^d}{(\alpha-\beta)^2} = \cdots = \frac{\alpha^d - \beta^d}{\alpha-\beta}$$

$$\cdot \frac{a^{2k}\alpha^{\frac{n+m}{2}2k+d} - b^{2k}\beta^{\frac{n+m}{2}2k+d}}{\alpha-\beta} = U_d W_{\frac{n+m}{2},d}^{(2k)}.$$

Proof of Lemma 2. Here we also give only the proof of part (a). By (3)

$$W_{n,d}^{(k)} W_{n,d}^{(2k)} - W_{n,0}^{(k)} W_{n,0}^{(2k)} B^d = \frac{(a^k\alpha^{nk+d} - b^k\beta^{nk+d})(a^{2k}\alpha^{2nk+d} - b^{2k}\beta^{2nk+d})}{(\alpha-\beta)^2}$$

$$-\frac{(a^k\alpha^{nk} - b^k\beta^{nk})(a^{2k}\alpha^{2nk} - b^{2k}\beta^{2nk})\alpha^d\beta^d}{(\alpha-\beta)^2} = \cdots = \frac{\alpha^d - \beta^d}{\alpha-\beta}$$

$$\cdot \frac{a^{3k}\alpha^{3nk+d} - b^{3k}\beta^{3nk+d}}{\alpha-\beta} = U_d W_{n,d}^{(3k)}.$$

Proof of Theorem 1. (a) By (7) and (4)

$$N\left(R_{n,d}^{(k)}\right) = \frac{\left(\frac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}}\right)^2 - B^d}{\frac{2W_{n,d}^{(k)}}{W_{n,0}^{(k)}} - V_d} = \frac{\left(W_{n,d}^{(k)}\right)^2 - \left(W_{n,0}^{(k)}\right)^2 B^d}{2W_{n,d}^{(k)} \cdot W_{n,0}^{(k)} - \left(W_{n,0}^{(k)}\right)^2 V_d}.$$

Applying Lemma 1 in the case $n = m$, we have

$$N\left(R_{n,d}^{(k)}\right) = \frac{U_d \cdot W_{n,d}^{(2k)}}{U_d \cdot W_{n,0}^{(2k)}} = R_{n,d}^{(2k)}.$$

(b) By the Halley transformation (8) and (4)

$$H\left(R_{n,d}^{(k)}\right) = \frac{\left(R_{n,d}^{(k)}\right)^3 - 3B^d R_{n,d}^{(k)} + V_d B^d}{3\left(R_{n,d}^{(k)}\right)^2 - 3V_d R_{n,d}^{(k)} + V_d^2 - B^d}$$

$$= \frac{\left(W_{n,d}^{(k)}\right)^3 - 3B^d W_{n,d}^{(k)}\left(W_{n,0}^{(k)}\right)^2 + V_d B^d \left(W_{n,0}^{(k)}\right)^3}{3\left(W_{n,d}^{(k)}\right)^2 W_{n,0}^{(k)} - 3V_d W_{n,d}^{(k)}\left(W_{n,0}^{(k)}\right)^2 + (V_d^2 - B^d)\left(W_{n,0}^{(k)}\right)^3}$$

$$= \frac{W_{n,d}^{(k)}\left(\left(W_{n,d}^{(k)}\right)^2 - B^d\left(W_{n,0}^{(k)}\right)^2\right) - B^d W_{n,0}^{(k)}\left(2W_{n,d}^{(k)}W_{n,0}^{(k)} - V_d\left(W_{n,0}^{(k)}\right)^2\right)}{W_{n,0}^{(k)}\left(\left(W_{n,d}^{(k)}\right)^2 - B^d\left(W_{n,0}^{(k)}\right)^2\right) + \left(W_{n,d}^{(k)} - V_d W_{n,0}^{(k)}\right)\left(2W_{n,d}^{(k)}\cdot W_{n,0}^{(k)} - V_d\left(W_{n,0}^{(k)}\right)^2\right)}.$$

The numerator and the denominator of the last fraction, by Lemma 1, can be rewritten as

$$U_d\left(W_{n,d}^{(2k)} - B^d W_{n,0}^{(k)} W_{n,0}^{(2k)}\right)$$

and

$$U_d\left(W_{n,0}^{(k)} W_{n,d}^{(2k)} + \left(W_{n,d}^{(k)} - V_d W_{n,0}^{(k)}\right) W_{n,0}^{(2k)}\right),$$

respectively. From these, by Lemma 2,

$$H\left(R_{n,d}^{(k)}\right) = \frac{U_d^2 W_{n,d}^{(3k)}}{U_d^2 W_{n,0}^{(3k)}} = R_{n,d}^{(3k)}$$

follows.

Proof of Theorem 2. To prove the theorem we have to show that

$$\lim_{n\to\infty} \frac{R_{n,d}^{(l)} - \alpha^d}{R_{n,d}^{(k)} - \alpha^d} = 0.$$

Applying (4) and (3), we get that

$$\frac{R_{n,d}^{(l)} - \alpha^d}{R_{n,d}^{(k)} - \alpha^d} = \frac{W_{n,d}^{(l)} - \alpha^d W_{n,0}^{(l)}}{W_{n,d}^{(k)} - \alpha^d W_{n,0}^{(k)}} \, \frac{W_{n,0}^{(k)}}{W_{n,0}^{(l)}} = \cdots$$

$$= \left(\frac{b}{a}\right)^{l-k} \left(\frac{\beta}{\alpha}\right)^{n(l-k)} \frac{1 - \left(\frac{b}{a}\right)^k \left(\frac{\beta}{\alpha}\right)^{nk}}{1 - \left(\frac{b}{a}\right)^k \left(\frac{\beta}{\alpha}\right)^{nl}},$$

from which, by $|\alpha| > |\beta|$ and $l > k \geq 1$,

$$\lim_{n\to\infty} \frac{R_{n,d}^{(l)} - \alpha^d}{R_{n,d}^{(k)} - \alpha^d} = 0$$

follows.

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