CONVERGENCE OF HOMOGENEOUS MATRIX-VALUED Λ-MARTINGALES

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Abstract. I. Fazekas in [3] studied the classical martingale convergence theorem of Doob for one-parameter Λ-martingales. The theme of this paper is similar but for two-parameters homogeneous Λ-martingales.

1. Preliminary result

Let $(\Omega, \mathcal{F}, P)$ be a probability space in which $(\xi_i\colon i = 1, 2, \ldots)$ is a sequence of random variables. Let $(\mathcal{F}_i\colon i = 1, 2, \ldots)$ be a sequence of σ-subalgebras of $\mathcal{F}$. We call the process $(\xi_i, \mathcal{F}_i)$ $i = 1, 2, \ldots$ a linear martingale if ξ_i are $\mathcal{F}_i$-measurable and integrable for every $i = 1, 2, \ldots$ furthermore

$$\mathbb{E}(\xi_i \mid \mathcal{F}_{i-1}) = a_1(i)\xi_{i-1} + \cdots + a_m(i)\xi_{i-m}$$

for every $i > m$ integers where m is a fixed integer. This process satisfies equation $\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) = \Lambda(t)X_{t-1}$ for every $t \geq m$ where

$$X_t = \begin{pmatrix} \xi_t \\ \vdots \\ \xi_{t-m+1} \end{pmatrix} \quad \text{and} \quad \Lambda(t) = \begin{pmatrix} a_1(t) & \ldots & & a_m(t) \\ 1 & & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

Generalized we call an m-dimensional process $(X_t, \mathcal{F}_t)$ $t = 1, 2, \ldots$ Λ-martingale if X_t integrable, $\Lambda(t)$ are given non-random matrices for every t positive integers and $\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) = \Lambda(t)X_{t-1}$ $(t = 1, 2, \ldots)$. If $\Lambda(t)$ does not depend on t then X_t is called a homogeneous martingale. Let $\Delta_t = X_t - \Lambda(t)X_{t-1}$, $A(s, s) = I$ the identity matrix and

$$A(t, s) = \Lambda(t)A(t-1, s)$$

for every $t > s$ furthermore we assume that the limit $A(s) = \lim\limits_{t\to\infty} A(t, s)$ exists for every $s = 1, 2, \ldots$. Let $Y_t = \sum\limits_{s=1}^{t} A(s)\Delta_s$ which is called the accompanying martingale of X_t. I. Fazekas proved in [3] the following theorem:

If $\|A(t, s) - A(s)\| \leq c_{t-s}$ $(t \geq s)$, $\sum\limits_{n=0}^{\infty} c_n < \infty$, there exists a positive function $f(\omega)$ for which $f(\omega)\|\Delta_s(\omega)\| \leq \|A(s)\Delta_s(\omega)\|$ for every $s \geq 1$ and $\omega \in \Omega$ and

$\sup_t \mathbb{E}\|X_t\| < \infty$ then $\lim_{t\to\infty} X_t = X_\infty$ almost surely. ($\|.\|$ denotes the norm of matrix.) In this paper this result is extended to two-parameter version.

2. Main result

Let $\mathbb{N}$ denote the set of positive integers and let m be a fixed positive integer. Let $(\Omega, \mathcal{F}, P)$ be a probability space in which $(\xi_{ij}\colon i,j \in \mathbb{N})$ is a sequence of real-valued random variables. Let $(\mathcal{F}_{ij}\colon i,j \in \mathbb{N})$ be a sequence of σ-subalgebras of $\mathcal{F}$ which satisfies the so-called condition (F4) introduced by Cairoli and Walsh [2]:

$$\mathbb{E}(\xi \mid \mathcal{F}_{ij}) = \mathbb{E}\big(\mathbb{E}(\xi \mid \mathcal{F}_{i\infty}) \mid \mathcal{F}_{\infty j}\big) = \mathbb{E}\big(\mathbb{E}(\xi \mid \mathcal{F}_{\infty j}) \mid \mathcal{F}_{i\infty}\big), \tag{F4}$$

for every $i,j \in \mathbb{N}$ where $\mathcal{F}_{i\infty} = \sigma\{\mathcal{F}_{ij}\colon j \in \mathbb{N}\}$ and $\mathcal{F}_{\infty j} = \sigma\{\mathcal{F}_{ij}\colon i \in \mathbb{N}\}$ ($\sigma\{.\}$ means generated σ-algebra).

In order to study a convergence property of ξ_{ij} we introduce the following matrix:

$$X_{ij} = \begin{pmatrix} \xi_{i,j-m+1} & \cdots & \xi_{i,j} \\ \vdots & \ddots & \vdots \\ \xi_{i-m+1,j-m+1} & \cdots & \xi_{i-m+1,j} \end{pmatrix}$$

Definition 1. Let Λ_{kl} be given non-random real matrices (their types are $m \times m$). Suppose that $\Lambda_{0,0} = I$ (the identity matrix),

$$\Lambda_{ij}\Lambda_{kl} = \Lambda_{i+k,j+l} \qquad \forall i,j,k,l \in \mathbb{N} \cup \{0\} \tag{1}$$

X_{ij} is $\mathcal{F}_{ij}$-measurable and integrable for every $i,j \in \mathbb{N}$. If

$$\mathbb{E}(X_{i+k,j+l} \mid \mathcal{F}_{ij}) = \Lambda_{kl} X_{ij}$$

for every $k,l \in \mathbb{N} \cup \{0\}$ and $i,j > m$ integers then the process $(X_{ij}, \mathcal{F}_{ij})$ $i,j \in \mathbb{N}$ is called a homogeneous matrix-valued Λ-martingale.

Let us introduce the martingale difference

$$\begin{aligned}\Delta_{ij} &= X_{ij} - \mathbb{E}(X_{i,j} \mid \mathcal{F}_{i-1,j}) - \mathbb{E}(X_{i,j} \mid \mathcal{F}_{i,j-1}) + \mathbb{E}(X_{i,j} \mid \mathcal{F}_{i-1,j-1}) \\ &= X_{ij} - \Lambda_{1,0} X_{i-1,j} - \Lambda_{0,1} X_{i,j-1} + \Lambda_{1,1} X_{i-1,j-1}\end{aligned}$$

for $i,j > 1$ integers, $\Delta_{1,1} = X_{1,1}$, $\Delta_{i,1} = X_{i,1} - \Lambda_{1,0} X_{i-1,1}$ for $i > 1$ integers and $\Delta_{1,j} = X_{1,j} - \Lambda_{0,1} X_{1,j-1}$ for $j > 1$ integers.

Lemma 1. *With the previous notations and conditions* $X_{ij} = \sum_{k=1}^{i} \sum_{l=1}^{j} \Lambda_{i-k,j-l} \Delta_{k,l}$ *for every* $i,j \in \mathbb{N}$.

Proof. Using (1) we have this lemma by induction.

Definition 2. We assume that Λ_{kl} is convergent and $\Lambda = \lim\limits_{\substack{k\to\infty \\ l\to\infty}} \Lambda_{kl}$. Then

$$Y_{ij} = \sum_{k=1}^{i} \sum_{l=1}^{j} \Lambda\Delta_{k,l}$$

is called the accompanying martingale of X_{ij}.

Lemma 2. *If $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a convex non-decreasing function and*

$$\sup_{i,j} \mathbb{E} f(\|X_{ij}\|) < c < \infty$$

then $\sup\limits_{i,j} \mathbb{E} f(\|Y_{ij}\|) < c$ as well. (In this paper $\|.\|$ denotes the norm of a matrix.)

Proof. Let r, s be fixed integers, $1 \le i \le r$, $1 \le j \le s$ and

$$Y_{ij}^{(rs)} = \sum_{k=1}^{i} \sum_{l=1}^{j} \Lambda_{r-k,s-l}\Delta_{k,l}.$$

Then it is easy to see that $(f(\|Y_{ij}^{(rs)}\|, \mathcal{F}_{ij})$ $1 \le i \le r$, $1 \le j \le s$ is a real submartingale, so we get by Lemma 1

$$\mathbb{E} f(\|Y_{ij}^{(rs)}\|) \le \mathbb{E} f(\|Y_{rs}^{(rs)}\|) = \mathbb{E} f(\|X_{rs}\|) < c$$

for every $1 \le i \le r$, $1 \le j \le s$ integers. On the other hand $\lim\limits_{\substack{r\to\infty \\ s\to\infty}} Y_{ij}^{(rs)} = Y_{ij}$ thus by Fatou's lemma we have Lemma 2.

Theorem. *Let the process $(X_{ij}, \mathcal{F}_{ij})$ $i, j \in \mathbb{N}$ is a homogeneous matrix-valued Λ-martingale which is satisfies (F4). Let us suppose that Λ_{kl} is convergent, $\Lambda = \lim\limits_{\substack{k\to\infty \\ l\to\infty}} \Lambda_{kl}$ and there exist constants c_{kl} such that*

$$\|\Lambda_{kl} - \Lambda\| < c_{kl} \quad \text{and} \quad \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} c_{kl} < \infty \tag{2}$$

for every $k, l \in \mathbb{N}$. If

$$\|\Delta_{kl}\| \le q^{k+l} \tag{3}$$

for every $k, l \in \mathbb{N}$ where $0 < q < 1$ is a fixed real number and

$$\sup_{k,l} \mathbb{E}\Big(\|X_{kl}\| \log^+(\|X_{kl}\|)\Big) < \infty \tag{4}$$

then X_{ij} converges almost surely.

Proof. We get by Lemma 1 and (2)

$$\|X_{ij} - Y_{ij}\| = \left\| \sum_{k=1}^{i} \sum_{l=1}^{j} (\Lambda_{i-k,j-l}\Delta_{kl} - \Lambda\Delta_{kl}) \right\| \leq$$

$$\leq \sum_{k=1}^{i} \sum_{l=1}^{j} \|\Lambda_{i-k,j-l} - \Lambda\| \cdot \|\Delta_{kl}\| \leq \sum_{k=1}^{i} \sum_{l=1}^{j} c_{i-k,j-l} \|\Delta_{kl}\|.$$

Let $r = i - k$ and $s = j - l$ thus we have by (3)

$$\|X_{ij} - Y_{ij}\| = \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} c_{rs} \|\Delta_{i-r,j-s}\| \leq \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} c_{rs} q^{i-r+j-s} =$$

$$= \frac{1}{q^{-(i+j)}} \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} c_{rs} q^{-(r+s)}$$

So we get by Kronecker's lemma (see it for example [4]) that $\lim\limits_{\substack{i\to\infty \\ j\to\infty}} \|X_{ij} - Y_{ij}\| = 0$. By (4), Lemma 2 and Cairoli's theorem (see in [1]) there exists $\lim\limits_{\substack{i\to\infty \\ j\to\infty}} Y_{ij}$ thus the Theorem is proved.

References

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