

An application of the continued fractions for \sqrt{D} in solving some types of Pell's equations

BÉLA ZAY

Abstract. In this paper we study the positive solutions of the Diophantine equation $x^2 - Dy^2 = N$, where D and $|N|$ are natural numbers, $|N| < \sqrt{D}$ and D is not the square of a natural number. Let $\sqrt{D} = (a_0, \overline{a_1, \dots, a_s})$ be the representation of \sqrt{D} as a simple continued fraction expansion. We prove that if the n -th convergent to \sqrt{D} is $\frac{H_n}{K_n} = (a_0, \dots, a_n)$, then

$$H_{(n+2)s+r} = 2H_{s-1}H_{(n+1)s+r} + (-1)^{s+1}H_{ns+r}$$

and

$$H_{(n+2)s+r} = 2H_{s-1}K_{(n+1)s+r} + (-1)^{s+1}K_{ns+r}.$$

In cases of $D = (2k+1)^2 - 4$ (for any $k \geq 2$), $D = (2k)^2 - 4$ (for any $k \geq 3$), $D = k^2 - 1$ (for any $k \geq 2$) and $D = k^2 + 1$ (for any $k \geq 1$) we give all positive solutions of $x^2 - Dy^2 = N$ ($|N| < \sqrt{D}$) with the help of Binet formulae of the sequences (H_{ns+r}) and (K_{ns+r}) (for any $r = 1, 2, \dots, s$).

Introduction

In this paper we consider the equation

$$(1) \quad x^2 - Dy^2 = N$$

and its solutions in natural numbers, provided D and N are rational integers, $D > 0$, furthermore D is not the square of a natural number. Many authors studied these Diophantine equations. Among others D. E. FERGUSON [1] solved the equations $x^2 - 5y^2 = \pm 4$, V. E. HOGATT, JR. and M. BICKNELL-JOHNSON [2] solved the equations

$$(2) \quad x^2 - (A^2 \pm 4)y^2 = \pm 4$$

where A is a fixed natural number. K. LIPTAI [4] proved that if there is a solution to (1) then all solutions can be given with the help of finitely many, well determined second order linear recurrences.

Auxiliary results

The purpose of this paper is to give such second order linear recurrences in case of $|N| < \sqrt{D}$ and in some special cases.

We shall use a lemma of P. KISS [3] and some theorems from [5] and [6].

Let γ be a real quadratic irrational number and let

$$(3) \quad \gamma = (a_0, a_1, a_2, \dots) = (a_0, a_1, \dots, a_{t-1}, \overline{a_t, \dots, a_{t+s-1}})$$

be the representation of γ as a simple periodic continued fraction, where s is the minimal period length of (3). P. KISS proved:

If the n -th convergent to γ is $\frac{H_n}{K_n} = (a_0, a_1, \dots, a_n)$ and the n -th convergent to $\gamma_0 = (\overline{a_t, \dots, a_{t+s-1}})$ is $\frac{h_n}{k_n} = (a_t, a_{t+1}, \dots, a_{t+n})$, then (as it was proved by P. Kiss [3])

$$(4) \quad H_{(n+2)s+r} = (h_{s-1} + k_{s-2})H_{(n+1)s+r} + (-1)^{s+1}H_{ns+r},$$

and

$$(5) \quad K_{(n+2)s+r} = (h_{s-1} + k_{s-2})K_{(n+1)s+r} + (-1)^{s+1}K_{ns+r},$$

where $n \geq 0, r = 0, 1, \dots, s-1$ and we assume, that $k_{-1} = 0$.

In the special case of $\gamma = \sqrt{D}$ we prove the following lemma.

Lemma 1. *Let D be a positive integer which is not a square of a natural number and let*

$$(6) \quad \sqrt{D} = (a_0, \overline{a_1, \dots, a_s})$$

be the representation of \sqrt{D} as a simple continued fraction expansion, where s is the period length of (6). If the n -th convergent to \sqrt{D} is

$$\frac{H_n}{K_n} = (a_0, a_1, \dots, a_n)$$

then

$$(7) \quad H_{(n+2)s+r} = 2H_{s-1}H_{(n+1)s+r} + (-1)^{s+1}H_{ns+r}$$

and

$$(8) \quad K_{(n+2)s+r} = 2H_{s-1}K_{(n+1)s+r} + (-1)^{s+1}H_{ns+r}$$

for every integer $n \geq 0$ and r ($0 \leq r \leq s-1$).

The first $2s$ terms of sequences (H_n) and (K_n) can be got from the following well known relations

$$(9) \quad H_m = a_m H_{m-1} + H_{m-2}, \quad H_{-1} = 1, \quad H_0 = a_0,$$

and

$$(10) \quad K_m = a_m K_{m-1} + K_{m-1}, \quad K_{-1} = 0, \quad K_0 = 1,$$

for any $m \geq 0$.

The following algorithm for representing the number \sqrt{D} as a simple continued fraction is well known (see in [6], p. 319): We set $a_0 = \left[\sqrt{D} \right]$, $b_1 = a_0$, $c_1 = D - a_0^2$ and we find the numbers a_{n-1} , b_n and c_n successively using the formulae

$$a_{n-1} = \left[\frac{a_0 + b_{n-1}}{c_{n-1}} \right], \quad b_n = a_{n-1} c_{n-1} - b_{n-1}, \quad c_n = \frac{D - b_n^2}{c_{n-1}}.$$

Now consider the sequence

$$(b_2, c_2), \quad (b_3, c_3), \quad (b_4, c_4), \dots$$

and find the smallest index s for which $b_{s+1} = b_1$ and $c_{s+1} = c_1$. Then the representation of \sqrt{D} as a simple continued fraction is

$$\sqrt{D} = (a_0, \overline{a_1, a_2, \dots, a_s}).$$

We shall use two other results from [5] (pp 158–159).

Lemma 2. *If D is a positive integer, not a perfect square, then $H_n^2 - DK_n^2 = (-1)^{n-1} c_{n+1}$ for all integer $n \geq -1$.*

Lemma 3. *Let D be a positive integer not a perfect square, and let the convergents to the continued fraction expansion of \sqrt{D} be H_n/K_n . Let N be an integer for which $|N| < D$. Then any positive solution $x = u, y = t$ of $x^2 - Dy^2 = N$ with $(u, t) = 1$ satisfies $u = H_n, t = K_n$, for some positive integer n .*

Recalling that $c_n = c_{n+s}$ in the Lemma 2., we can formulate Lemma 4. which is a consequence of the first three lemmas.

Lemma 4. *Let D be a positive integer not a perfect square, and let*

$$\sqrt{D} = (a_0, \overline{a_1, \dots, a_s})$$

be the representation of \sqrt{D} as a simple continued fraction. Suppose that N is a non-zero integer with $|N| < \sqrt{D}$, and let

$$(11) \quad H_{-1} = 1, \quad H_0 = a_0, \quad H_m = a_m H_{m-1} + H_{m-2}, \quad 1 \leq m \leq 2s,$$

$$(12) \quad K_{-1} = 0, \quad K_0 = 1, \quad K_m = a_m K_{m-1} + K_{m-2}, \quad 1 \leq m \leq 2s,$$

$$(13) \quad H_{(n+2)s+r} = 2H_{s-1}H_{(n+1)s+r} + (-1)^{s+1}H_{ns+r}, \quad 1 \leq r \leq s, \quad n \geq 0,$$

$$(14) \quad K_{(n+2)s+r} = 2H_{s-1}K_{(n+1)s+r} + (-1)^{s+1}K_{ns+r}, \quad 1 \leq r \leq s, \quad n \geq 0,$$

and

$$(15) \quad c_{ns+r+1} = (-1)^{ns+r-1}(H_r^2 - DK_r^2), \quad 1 \leq r \leq s.$$

If $1 \leq r \leq s, c_{r+1} \neq 0$ and $\sqrt{\frac{(-1)^{n-1}N}{c_{r+1}}}$ is a natural number then let $d_r = \sqrt{\frac{(-1)^{r-1}N}{c_{r+1}}}$. Denote by M the set of positive solutions (x, y) of $x^2 - Dy^2 = N$. Then

$$(16) \quad M = \{(x, y) : x = d_r H_{ns+r}, \quad y = d_r K_{ns+r}, \quad n \geq 0, \quad 1 \leq r \leq s\}.$$

This also means that: If there exists no natural numbers d_r ($1 \leq r \leq s$) which satisfy the above conditions then there isn't integer solution $x = u, y = t$ of $x^2 - Dy^2 = N$ ($|N| < D$), that is M is the empty set.

Theorems

Applying Lemma 4. for some special equations we obtain the following results.

Theorem 1. Let k ($k \geq 2$) be a natural number with $D = (2k+1)^2 - 4$. Let α and β denote the zeros of $f_1(x) = x^2 - (2k+1)x + 1$ and let $\alpha > \beta$. Denote by M the set of positive (x, y) solutions of $x^2 - Dy^2 = N$.

(a) If $N = 4l^2$ and l ($1 \leq l \leq \sqrt{\frac{k}{2}}$) is a natural number, then

$$M = \left\{ (x, y) : x = l(\alpha^m + \beta^m), \quad y = l \frac{\alpha^m - \beta^m}{\alpha - \beta}, \quad m \geq 1 \right\}.$$

(b) If $N = (2l-1)^2$ and $1 \leq l \leq \frac{1}{2} + \sqrt{\frac{k}{2}}$ then

$$M = \left\{ (x, y) : x = \left(l - \frac{1}{2} \right) (\alpha^{3m+3} + \beta^{3m+3}), \right. \\ \left. y = \left(l - \frac{1}{2} \right) \frac{\alpha^{3m+3} - \beta^{3m+3}}{\alpha - \beta}, \quad m \geq 1 \right\}.$$

(c) If $N = 1 - 2k$ then

$$M = \left\{ (x, y) : x = \frac{(\alpha - 1)\alpha^{3m+1} + (\beta - 1)\beta^{3m+1}}{2}, \right. \\ \left. y = \frac{(\alpha - 1)\alpha^{3m+1} - (\beta - 1)\beta^{3m+1}}{2(\alpha - \beta)}, m \geq 1 \right\}.$$

(d) If $1 \leq |N| \leq 2k$, $N \neq 1 - 2k$ and N isn't a square of a natural number then $M = \emptyset$ (empty set).

Theorem 2. Let k ($k \geq 3$) be a natural number and $D = (2k)^2 - 4$. Let α and β denote the zeros of $f_2(x) = x^2 - 2kx + 1$ with $\alpha > \beta$. Denote by M the set of positive (x, y) solutions of $x^2 - Dy^2 = N$.

(a) If $N = 4l^2$ and l ($1 \leq l \leq \sqrt{\frac{k-1}{2}}$) is a natural number then

$$M = \left\{ (x, y) : x = l(\alpha^m + \beta^m), y = l \frac{\alpha^m - \beta^m}{\alpha - \beta}, m \geq 1 \right\}.$$

(b) If $N = (2l - 1)^2$ and l is a natural number ($1 \leq l < \frac{1}{2} + \sqrt{k^2 - 1}$) then

$$M = \left\{ (x, y) : x = \left(l - \frac{1}{2}\right) (\alpha^{2m} + \beta^{2m}), \right. \\ \left. y = \left(l - \frac{1}{2}\right) \frac{\alpha^{2m} - \beta^{2m}}{\alpha - \beta}, m \geq 1 \right\}.$$

(c) If $1 \leq |N| < 2k$ and N isn't a square of a natural number then $M = \emptyset$.

Theorem 3. Let k ($k \geq 2$) be a natural number and $D = k^2 - 1$. Let α and β denote the zeros of $f_3(x) = x^2 - 2kx + 1$ where $\alpha > \beta$. Denote by M the set of positive solutions of $x^2 - Dy^2 = N$.

(a) If $N = l^2$ and $1 \leq l \leq \sqrt{k-1}$ then

$$M = \left\{ (x, y) : x = \frac{l}{2}(\alpha^{n+1} + \beta^{n+1}), y = \frac{l(\alpha^{n+1} - \beta^{n+1})}{2(\alpha - \beta)}, m \geq 1 \right\}.$$

(b) If $1 \leq |N| < 2k - 1$ and N isn't a square of a natural number then $M = \emptyset$.

Theorem 4. Let k ($k \geq 1$) be a natural number and $D = k^2 + 1$. Let α and β denote the zeros of $f_4(x) = x^2 - 2kx - 1$ with $\alpha > \beta$. Denote by M the set of positive solutions of $x^2 - Dy^2 = N$.

(a) If $N = l^2$ and $1 \leq l \leq \sqrt{k}$ then

$$M \left\{ (x, y) : x = \frac{l}{2} (\alpha^{2n+1} + \beta^{2n+1}), y = \frac{l(\alpha^{2n+1} - \beta^{2n+1})}{\alpha - \beta}, m \geq 1 \right\}.$$

(b) If $N = -l^2$ and $1 \leq l \leq \sqrt{k}$ then

$$M = \left\{ (x, y) : x = \frac{l}{2} (\alpha^{2m} + \beta^{2m}), y = \frac{l(\alpha^{2m} - \beta^{2m})}{\alpha - \beta}, m \geq 1 \right\}.$$

(c) If $1 \leq |N| \leq k$ and $|N|$ isn't a square of a natural number then $M = \emptyset$.

Proofs

To prove Lemma 1. we need the following two lemmas.

Lemma 5. Let $f_{n+2}(x_1, x_2, \dots, x_n)$ and $g_{n+2}(x_1, x_2, \dots, x_n)$ be the polynomials which are defined by recurring relations

$$f_{n+2}(x_1, \dots, x_n) = x_n f_{n+1}(x_1, \dots, x_{n-1}) + f_n(x_1, \dots, x_{n-2}), \quad n \geq 1$$

and

$$g_{n+2}(x_1, \dots, x_n) = x_1 g_{n+1}(x_2, \dots, x_n) + g_n(x_3, \dots, x_n), \quad n \geq 1$$

respectively, where $f_1 = g_1 = 0$ and $f_2 = g_2 = 1$. Then

$$f_{n+2}(x_1, \dots, x_n) = g_{n+2}(x_1, \dots, x_n), \quad n \geq -1$$

also holds.

Proof. We can easily verify that

$$f_1 = g_1, \quad f_2 = g_2, \quad f_3(x_1) = x_1 = g_3(x_1)$$

and

$$f_4(x_1, x_2) = x_2 f_3(x_1) + f_2 = g_3(x_2) x_1 + g_2 = g_4(x_1, x_2).$$

Assume that $n \geq 3$ and

$$f_{n+2-i}(x_1, \dots, x_{n-1}) = g_{n+2-i}(x_1, \dots, x_{n-1})$$

holds for $i = 1, 2, 3, 4$.

Using the definitions and the last assumptions we can finish the proof by induction for n :

$$\begin{aligned}
f_{n+2}(x_1, \dots, x_n) &= x_n f_{n+1}(x_1, \dots, x_{n-1}) + f_n(x_1, \dots, x_{n-2}) \\
&= x_n g_{n+1}(x_1, \dots, x_{n-1}) + g_n(x_1, \dots, x_{n-2}) \\
&= x_n x_1 g_n(x_2, \dots, x_{n-1}) + x_n g_{n-1}(x_3, \dots, x_{n-1}) \\
&\quad + x_1 g_{n-1}(x_2, \dots, x_{n-2}) + g_{n-2}(x_3, \dots, x_{n-2}) \\
&= x_1 (x_n f_n(x_2, \dots, x_{n-1}) + f_{n-1}(x_2, \dots, x_{n-2})) \\
&\quad + (x_n f_{n-1}(x_3, \dots, x_{n-1}) + f_{n-2}(x_3, \dots, x_{n-2})) \\
&= x_1 f_{n+1}(x_2, \dots, x_{n-1}, x_n) + f_n(x_3, \dots, x_{n-1}, x_n) \\
&\quad + x_1 g_{n+1}(x_2, \dots, x_n) + g_n(x_3, \dots, x_n) = g_{n+2}(x_1, \dots, x_n).
\end{aligned}$$

Lemma 6. *If $x_i = x_{n+2-i}$ holds for every i ($1 \leq i \leq n+1$) then*

$$f_{n+2}(x_1, \dots, x_n) = f_{n+2}(x_2, \dots, x_{n+1})$$

is also valid for every integer n ($n \geq -1$).

Proof. This is evident for $-1 \leq n \leq 2$, because

$$f_1 = 0, f_2 = 1, f_3(x_1) = x_1 = x_2 = f_3(x_2)$$

and

$$f_4(x_1, x_2) = x_2 x_1 + 1 = x_3 x_2 + 1 = f_4(x_2, x_3) \text{ (since } x_1 = x_3\text{)}.$$

Let $n > 2$. Assume that if $y_i = y_{n-i}$ holds for every i ($1 \leq i \leq n-1$) then

$$f_n(y_1, \dots, y_{n-2}) = f_n(y_2, \dots, y_{n-1})$$

is also valid. Let $y_i = x_{i+1}$ for every i ($1 \leq i \leq n-1$).

Then

$$y_i = x_{i+1} = x_{n+2-(i+1)} = x_{n-i+1} = y_{n-i}$$

and so

$$f_n(x_2, \dots, x_{n-1}) = f_n(x_3, \dots, x_n).$$

Using this equation, Lemma 5. and the relation $x_1 = x_{n+1}$ we obtain, that

$$\begin{aligned}
&f_{n+2}(x_1, \dots, x_n) \\
&= g_{n+2}(x_1, \dots, x_n) = x_1 g_{n+1}(x_2, \dots, x_n) + g_n(x_3, \dots, x_n) \\
&= x_{n+1} f_{n+1}(x_2, \dots, x_n) + f_n(x_3, \dots, x_n) \\
&= x_{n+1} f_{n+1}(x_2, \dots, x_n) + f_n(x_2, \dots, x_{n-1}) \\
&= f_{n+2}(x_2, \dots, x_{n+1})
\end{aligned}$$

which completes the proof of the lemma.

Proof of Lemma 1. It is well known [see in [6] p. 317] that in the representation of \sqrt{D} as a simple continued fraction, the sequence a_1, a_2, \dots, a_{s-1} is symmetric, that is $a_i = a_{s-i}$, for every i ($1 \leq i \leq s-1$) and

$$(17) \quad a_s = 2a_0.$$

If $\frac{h_n}{k_n}$ is the n^{th} convergent of $(a_1, a_2, \dots) = (\overline{a_1, a_2, \dots, a_s})$ then

$$h_{-1} = 1, \quad h_0 = a_1, \quad h_n = a_{n+1}h_{n-1} + h_{n-2}, \quad n \geq 1$$

and

$$k_{-1} = 0, \quad k_0 = 1, \quad k_n = a_{n+1}k_{n-1} + k_{n-2}, \quad n \geq 1.$$

Using this last definition and (10) by Lemma 6. we obtain, that

$$k_{s-2} = f_s(a_2, \dots, a_{s-1}) = f_s(a_1, \dots, a_{s-2}) = K_{s-2}.$$

It is known (and it is easy to see by induction for n) that

$$(19) \quad K_n = h_{n-1}, \quad n \geq 0$$

and

$$(20) \quad H_n = a_0h_{n-1} + k_{n-1}, \quad n \geq 0.$$

By (19), (12), (17), (18) and (20)

$$\begin{aligned} h_{s-1} + k_{s-2} &= K_s + k_{s-2} = a_s K_{s-1} + K_{s-2} + k_{s-2} \\ &= 2a_0 K_{s-1} + 2k_{s-2} = 2(a_0 h_{s-2} + k_{s-2}) = 2H_s. \end{aligned}$$

Using this equation we obtain (7) and (8) from (4) and (5) respectively. Thus the theorem is proved.

To proofs of the Theorem 1., Theorem 2., Theorem 3. and Theorem 4. we use the Lemma 4. and the representation of \sqrt{D} as a simple continued fraction:

Lemma 7. *Let k be a rational integer. Then*

$$(21) \quad \sqrt{(2k+1)^2 - 4} = (2k, \overline{1, k-1, 2, k-1, 1, 4k}) \quad \text{for } k \geq 2.$$

$$(21) \quad \sqrt{(2k)^2 - 4} = (2k-1, \overline{1, k-2, 1, 4k-2}) \quad \text{for } k \geq 3,$$

$$(23) \quad \sqrt{k^2 - 1} = (k-1, \overline{1, 2k-2}) \quad \text{for } k \geq 2,$$

$$(24) \quad \sqrt{k^2 + 1} = (k, \overline{2k}) \quad \text{for } k \geq 1.$$

Proof. If $x = (\overline{1, k-1, 2, k-1, 1, 4k})$ then $x > 1$,

$$x = 1 + \frac{1}{k-1 + \frac{1}{2 + \frac{1}{k-1 + \frac{1}{1 + \frac{1}{4k + \frac{1}{x}}}}}}$$

and so

$$\left(\frac{1}{x}\right)^2 + 4k\frac{1}{x} - (4k-3) = 0$$

from which (using $\frac{1}{x} > 0$) we can see that

$$\sqrt{(2k+1)^2 - 4} = 2k + \frac{1}{x}$$

It is known that $\sqrt{D} = \sqrt{(2k+1)^2 - 4} = (a_0, \overline{a_1, \dots, a_s})$ where $a_0 = \lfloor \sqrt{D} \rfloor = 2k$, and

$$\sqrt{D} = a_0 + \frac{1}{x}, \quad \text{where } x = (\overline{a_1, \dots, a_s}).$$

Every irrational number can be expressed in exactly one way as an infinite simple continued fraction. Thus the first part of the lemma is proved. The proof of other three parts is carried out analogously. We can see these formulae in [6] p. 321., too.

Proof of Theorem 1. We have only to apply the Lemma 4. and Lemma 7. By (22)

$$\sqrt{D} = \sqrt{(2k+1)^2 - 4} = (2k, \overline{1, k-1, 2, k-1, 1, 4k}), \quad k \geq 2$$

and so the representation of \sqrt{D} as a simple continued fraction has a period consisting of $s = 6$ terms. This terms are

$$a_0 = 2k, \quad a_1 = 1, \quad a_2 = k-1, \quad a_3 = 2, \quad a_n = k-1, \quad a_5 = 1, \quad a_6 = 4k.$$

By the formulas (9), (10), (13) and (14) we can verify that

$$\begin{aligned}
 H_{6n+1} &= \alpha^{3n+1} + \beta^{3n+1}, & K_{6n+1} &= \frac{\alpha^{3n+1} - \beta^{3n+1}}{\alpha - \beta}, \\
 H_{6n+2} &= \frac{(\alpha - 1)\alpha^{3n+1} + (\beta - 1)\beta^{3n+1}}{2}, \\
 K_{6n+2} &= \frac{(\alpha - 1)\alpha^{3n+1} - (\beta - 1)\beta^{3n+1}}{2(\alpha - \beta)}, \\
 H_{6n+3} &= \alpha^{3n+2} + \beta^{3n+3}, & K_{6n+3} &= \frac{\alpha^{3n+2} - \beta^{3n+2}}{\alpha - \beta}, \\
 H_{6n+4} &= \frac{(\alpha - 2)\alpha^{3n+2} + (\beta - 2)\beta^{3n+2}}{2}, \\
 K_{6n+4} &= \frac{(\alpha - 2)\alpha^{3n+2} - (\beta - 2)\beta^{3n+2}}{2(\alpha - \beta)},
 \end{aligned}$$

for $n \geq 0$ and

$$\begin{aligned}
 H_{6n+5} &= \frac{\alpha^{3n+3} + \beta^{3n+3}}{2}, & K_{6n+5} &= \frac{\alpha^{3n+3} - \beta^{3n+3}}{2}, \\
 H_{6n+6} &= \frac{(2\alpha - 1)\alpha^{3n+3} + (2\beta - 1)\beta^{3n+3}}{2}, \\
 K_{6n+6} &= \frac{(2\alpha - 1)\alpha^{3n+3} - (2\beta - 1)\beta^{3n+3}}{2(\alpha - \beta)},
 \end{aligned}$$

for $n \geq -1$. From these equations we obtain, that

$$H_{6n+r}^2 - DK_{6n+r}^2 = \left\{ \begin{array}{ll} 1, & \text{for } r = 5 \\ 4, & \text{for } r = 1 \text{ or } 3 \\ 1 - 2k, & \text{for } r = 2 \\ 3 - 4k, & \text{for } r = 4 \text{ } 6 \end{array} \right\} = (-1)^{r-1} c_{r+1}$$

for any $n \geq 0$. From (25) and (16) we can easily verify that the statements of Theorem 1. are valid.

The proofs of the Theorem 2., Theorem 3. and Theorem 4. are carried out analogously to the proof of the preceding theorem. For brevity we write only few formulas (without details) in this proofs.

Proof of Theorem 2.

$$\begin{aligned}
 \sqrt{D} &= \sqrt{(2k)^2 - 4} = (2k - 1, \overline{1, k - 2, 1, k - 2}), \text{ for } k \geq 3 \\
 H_{4n+1} &= \alpha^{2n+1} + \beta^{2n+1}, & K_{4n+1} &= \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta},
 \end{aligned}$$

$$\begin{aligned}
H_{4n+2} &= \frac{(\alpha - 2)\alpha^{2n+1} + (\beta - 2)\beta^{2n+1}}{2}, \\
K_{4n+2} &= \frac{(\alpha - 2)\alpha^{2n+1} - (\beta - 2)\beta^{2n+1}}{2(\alpha - \beta)}, \\
H_{4n+3} &= \frac{\alpha^{2n+2} + \beta^{2n+2}}{2}, \quad K_{4n+3} = \frac{\alpha^{2n+2} - \beta^{2n+2}}{2(\alpha - \beta)},
\end{aligned}$$

for $n \geq 0$ and

$$\begin{aligned}
H_{4n+4} &= \frac{(2\alpha - 1)\alpha^{2n+2} + (2\beta - 1)\beta^{2n+2}}{2}, \\
K_{4n+4} &= \frac{(2\alpha - 1)\alpha^{2n+2} - (2\beta - 1)\beta^{2n+2}}{2(\alpha - \beta)},
\end{aligned}$$

for $n \geq -1$ and

$$H_n^2 - DK_n^2 = \left\{ \begin{array}{ll} 1 & \text{for } r = 3 \\ 4, & \text{for } r = 1 \\ 5 - 4k, & \text{for } r = 2 \text{ or } 4 \end{array} \right\} = (-1)^{r-1}c_{r+1}, \quad n \geq 0.$$

Proof of Theorem 3.

$$\begin{aligned}
\sqrt{D} &= \sqrt{k^2 - 1} = (k - 1, \overline{1, 2k - 2}), \quad \text{for } k \geq 2 \\
H_{2n+1} &= \frac{\alpha^{n+1} + \beta^{n+1}}{2}, \quad K_{2n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, \\
H_{2n+2} &= \frac{(\alpha - 1)\alpha^{n+2} + (\beta - 1)\beta^{n+2}}{2}, \\
K_{2n+2} &= \frac{(\alpha - 1)\alpha^{n+2} - (\beta - 1)\beta^{n+2}}{\alpha - \beta},
\end{aligned}$$

for $n \geq -1$ and

$$H_{2n+r} - DK_{2n+r}^2 = \left\{ \begin{array}{ll} 1, & \text{for } r = 1 \\ 2 - 2k, & \text{for } r = 0 \end{array} \right\} = (-1)^{r-1}c_{r+1}, \quad n \geq 0.$$

Proof of Theorem 4.

$$\begin{aligned}
\sqrt{D} &= \sqrt{k^2 + 1} = (k, \overline{2k}), \quad \text{for } k \geq 1 \\
H_n &= \frac{\alpha^{n+1} + \beta^{n+1}}{2}, \quad K_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}
\end{aligned}$$

$$H_n^2 - DK_n^2 = (-1)^{n+1} = (-1)^{n-1}c_{n+1}, \text{ (that is } c_{n+1} = 1 \text{ for any } n).$$

References

- [1] D. E. FERGUSON, Letter to the editor, *Fibonacci Quart.*, **8** (1970), 88–89.
- [2] V. E. HOGATT JR., and M. BICKNELL-JOHNSON, A primer for the Fibonacci numbers XVII: Generalized Fibonacci numbers satisfying $U_{n+1}U_{n-1} - U_n^2 = \pm 1$, *Fibonacci Quart.*, **2** (1978), 130–137.
- [3] P. KISS, On second order recurrences and continued fractions, *Bull. Malaysian Math. Soc.* (**2**) 5 (1982) 33–41.
- [4] K. LIPTAI, On a Diophantine problem, *Discuss. Math.*, (to appear).
- [5] I. NIVEN, H. S. ZUCKERMAN, *An introduction to the theory of numbers*, John Wiley and Sons, London · New York, 1960.
- [6] W. SIERPINSKI, *Elementary Theory of Numbers*, PWN-Polish Scientific Publishers, Warszawa, 1987.

BÉLA ZAY

KÁROLY ESZTERHÁZY TEACHERS' TRAINING COLLEGE

DEPARTMENT OF MATHEMATICS

H-3301 EGER, PF. 43

HUNGARY