Bounds for the zeros of Fibonacci-like polynomials

FERENC MÁTYÁS

Abstract. The Fibonacci-like polynomials $G_n(x)$ are defined by the recursive formula $G_n(x)=xG_{n-1}(x)+G_{n-2}(x)$ for $n\geq 2$, where $G_0(x)$ and $G_1(x)$ are given seed-polynomials. The notation $G_n(x)=G_n(G_0(x),G_1(x),x)$ is also used. In this paper we determine the location of the zeros of polynomials $G_n(a,x\pm b,x)$ and give some bounds for the absolute values of complex roots of these polynomials if $a,b\in \mathbb{R}$ and $a\neq 0$. Our result generalizes the result of P. E. Ricci who investigated this problem in the case $a=b=1$.

Introduction

Let $G_0(x)$ and $G_1(x)$ be polynomials with real coefficients. For any $n \in \mathbb{N} \setminus \{0,1\}$ the polynomial $G_n(x)$ is defined by the recurrence relation

$$G_n(x) = xG_{n-1}(x) + G_{n-2}(x)$$

and these polynomials are called Fibonacci-like polynomials. If it is necessary then the initial or seed polynomials $G_0(x)$ and $G_1(x)$ can also be detected and in this case we use the form $G_n(x) = G_n(G_0(x),G_1(x),x)$. Note that $G_n(0,1,1) = F_n$ where $F_n$ is the $n$th Fibonacci number.

In some earlier papers the Fibonacci-like polynomials and other polynomials, defined by similar recursions, were studied. G. A. Moore [5] and H. Prodinger [6] investigated the maximal real roots (zeros) of the polynomials $G_n(-1,x-1,x)$ ($n \geq 1$). Hongquan Yu, Yi Wang and Mingfeng He [2] studied the limit of maximal real roots of the polynomials $G_n(-a,x-a,x)$ if $a \in \mathbb{R}_+$ as $n$ tends to infinity.

Under some restrictions in [3] we proved a necessary and sufficient condition for seed-polynomials when the set of the real roots of polynomials $G_n(G_0(x),G_1(x),x)$ ($n = 0,1,2,\ldots$) has nonzero accumulation points. These accumulation points can be effectively determined. In [4], using this result, we proved the following

**Theorem A.** If $a < 0$ or $2 < a$ then, apart from 0, the single accumulation point of the set of real roots of polynomials $G_n(a,x \pm a,x)$ ($n = 1,2,\ldots$) is $\pm \frac{a(2-a)}{a-1}$, while in the case $0 < a \leq 2$ the above set has no nonzero accumulation point.

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According to Theorem A, apart from finitely many real roots, all of the real roots of polynomials $G_n(a, x \pm a, x)$ ($a \in \mathbb{R} \setminus \{0\}, \ n = 1, 2, \ldots$) can be found in the open intervals

$$\left(\pm \frac{a(2 - a)}{a - 1} - \varepsilon, \pm \frac{a(2 - a)}{a - 1} + \varepsilon\right) \text{ or } (-\varepsilon, \varepsilon),$$

where $\varepsilon$ is an arbitrary positive real number.

Investigating the complex zeros of Fibonacci-like polynomials V. E. Hogatt, Jr. and M. Bicknell [1] proved that the roots of the equation $G_n(0, 1, x) = 0$ are $x_k = 2i \cos \frac{k\pi}{n}$ ($k = 1, 2, \ldots, n - 1$), i.e. apart from 0 if $n$ is even, all of the roots are purely imaginary and their absolute values are less than 2. P. E. Ricci [7] among others studied the location of zeros of polynomials $G_n(1, x + 1, x)$ and proved the following result.

**Theorem B.** All of the complex zeros of polynomials $G_n(1, x + 1, x)$ ($n = 1, 2, \ldots$) are in or on the circle with midpoint $(0, 0)$ and radius 2 in the Gaussian plane.

The purpose of this paper is to generalize the result of P. E. Ricci for the polynomials $G_n(a, x + b, x)$ where $a, b \in \mathbb{R}$ and $a \neq 0$, i.e. to give bounds for the absolute values of zeros. To prove our results we are going to use linear algebraic methods as it was applied by P. E. Ricci [7], too.

At the end of this part we list some terms of the polynomial sequence $G_n(x) = G_n(a, x + b, x)$ ($n = 2, 3, \ldots$). We have

\[
\begin{align*}
G_2(x) &= x^2 + bx + a, \\
G_3(x) &= x^3 + bx^2 + (a + 1)x + b, \\
G_4(x) &= x^4 + bx^3 + (a + 2)x^2 + 2bx + a, \\
G_5(x) &= x^5 + bx^4 + (a + 3)x^3 + 3bx^2 + (2a + 1)x + b, \\
G_6(x) &= x^6 + bx^5 + (a + 4)x^4 + 4bx^3 + (3a + 3)x^2 + 3bx + a.
\end{align*}
\]

**Known facts from linear algebra**

To estimate the absolute values of zeros of polynomials $G_n(a, x + b, x)$ ($n \geq 1$) we need the following notations and theorem. Let $A = (a_{ij})$ be an $n \times n$ matrix with complex entries, $\lambda_i$ ($i = 1, 2, \ldots, n$) and $f(x)$ denote the eigenvalues and the characteristic polynomial of $A$, respectively. It is known that

\[f(\lambda_i) = 0\]
and

\[ \max |\lambda_i| \leq \|A\|, \]

where \( \|A\| \) denotes a norm of the matrix \( A \). In this paper we apply the norms

\[ \|A\|_1 = n \max |a_{ij}| \]

and

\[ \|A\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}. \]

Using the so called Gershgorin’s theorem we can get a better estimation for the absolute values of the roots of \( f(x) = 0 \) and it gives the location of zeros of \( f(x) \), too. Let us consider the sets \( C_i \) of complex numbers \( z \) defined by

\[ C_i = \{ z : |z - a_{ii}| \leq r_i \}, \]

where \( i = 1, 2, \ldots, n \) and

\[ r_i = \sum_{\substack{j=1 \atop j \neq i}}^n |a_{ij}| \quad (n \geq 2). \]

So \( C_i \) is the set of complex numbers \( z \) which are inside the circle or on the circle with midpoint \( a_{ii} \) and radius \( r_i \) in the complex plane. These sets (circles) are called to be Gershgorin-circles. Using these notations we formulate the following well-known theorem.

**Gershgorin’s theorem.** Let \( n \geq 2 \). For every \( i \) \( (1 \leq i \leq n) \) there exists a \( j \) \( (1 \leq j \leq n) \) such that

\[ \lambda_i \in C_j \]

and so

\[ \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \subset C_1 \cup C_2 \cup \cdots \cup C_n. \]
Theorems and the Main Result

Let us consider the $n \times n$ matrix

$$A_n = \begin{pmatrix}
-b & -ai & 0 & \cdots & 0 & 0 & 0 \\
-i & 0 & -i & \cdots & 0 & 0 & 0 \\
0 & -i & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -i & 0 & -i \\
0 & 0 & 0 & \cdots & 0 & -i & 0 \\
\end{pmatrix},$$

where $b \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$.

Further on we prove the following

**Theorem 1.** Let $n \geq 1$ and $a, b \in \mathbb{R}$ ($a \neq 0$). The characteristic polynomial of matrix $A_n$ is the polynomial $G_n(a, x + b, x)$.

Let $n \geq 2$ and $a, b \in \mathbb{R}$ ($a \neq 0$). If $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn}$ denote the zeros of the polynomial $G_n(a, x + b, x)$ then, using the norms defined by (4) and (5) for the matrix $A_n$, one can get the following estimations by (2),(3) and Theorem 1.

\begin{align*}
\max_{1 \leq i \leq n} |\lambda_{ni}| &\leq n \max(|a|, |b|, 1) \\
\max_{1 \leq i \leq n} |\lambda_{ni}| &\leq \sqrt{a^2 + b^2 + 2n - 3}.
\end{align*}

From (10) and (11) it can be seen that these bounds depend on $a, b$ and $n$ but using the Gershgorin-circles we can get a more precise bound for $|\lambda_{ni}|$ and this bound depends only on $a$ and $b$.

We shall prove

**Theorem 2.** Let $n \geq 2$ and $a, b \in \mathbb{R}$ ($a \neq 0$) and let us denote by $K_1$ the set $K_1 = \{z: |z + b| \leq |a|\}$ and by $K_2$ the set $K_2 = \{z: |z| \leq 2\}$ in the Gaussian plane. Then

\begin{align*}
\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn} &\in K_1 \cup K_2.
\end{align*}

Now we are able to formulate our main result.
**Main Result.** For any \( n \geq 1 \) and \( a, b \in \mathbb{R} \) (\( a \neq 0 \)) if \( G_n(a, x+b, x) = 0 \), then

(13) \[ |x| \leq \max(|a| + |b|, 2), \]

i.e. the absolute values of all zeros of all polynomial terms of polynomial sequence \( G_n(a, x+b, x) \) \( (n = 1, 2, 3, \ldots) \) have a common upper bound, and by (13) this bound depends only on \( a \) and \( b \) in explicit way.

We mention that Theorem B can be obtained as a special case \( (a = b = 1) \) of our Main Result.

**Proofs**

**Proof of Theorem 1.** It is known that the characteristic polynomial \( f_n(x) \) of matrix \( A_n \) can be obtained by the determinant of matrix \( xI_n - A_n \), where \( I_n \) is the \( n \times n \) unit matrix. So

(14) \[ f_n(x) = \det (xI_n - A_n) = \det \begin{pmatrix} x+b & ai & 0 & \cdots & 0 & 0 & 0 \\ i & x & i & \cdots & 0 & 0 & 0 \\ 0 & i & x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & i & x & i \\ 0 & 0 & 0 & \cdots & 0 & i & x \end{pmatrix}. \]

We prove the theorem by induction on \( n \). It can be seen directly that \( f_1(x) = x + b = G_1(a, x + b, x) \) and \( f_2(x) = x^2 + bx + a = G_2(a, x + b, x) \). Let us suppose that \( f_{n-2}(x) = G_{n-2}(a, x + b, x) \) and \( f_{n-1}(x) = G_{n-1}(a, x + b, x) \) hold for an integer \( n \geq 3 \). Then developing (14) with respect to the last column and the resulting determinant with respect to the last row, we get

\[ f_n(x) = xf_{n-1}(x) - if_{n-2}(x) = xf_{n-1}(x) + f_{n-2}(x), \]

i.e. by our induction hypothesis

\[ f_n(x) = xG_{n-1}(a, x + b, x) + G_{n-2}(a, x + b, x) \]

and so by (1)

\[ f_n(x) = G_n(a, x + b, x) \]

holds for every integer \( n \geq 1 \).

**Proof of the Theorem 2.** From the matrix \( A_n \) we determine the so-called Gershgorin-circles. By the definition of \( A_n \) and (6) now there are only
two distinct Gershgorin-circles. The midpoints of these circles are $-b$ and 0 in the Gaussian plane, while by (7) their radii are $|a|$ and 2, respectively, i.e. they are the sets (circles) $K_1$ and $K_2$. (We omitted the circle with midpoint 0 and radius 1, because this circle is contained by one of the above circles.)

Since $G_n(a, x + b, x)$ is the characteristic polynomial of the matrix $A_n$, and $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn}$ are the zeros of it so from (8) and (9) we get that

$$\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn} \in K_1 \cup K_2.$$ 

This completes the proof.

**Proof of the Main Result.** We have seen in the proof of Theorem 2 that the Gershgorin-circles $K_1$ and $K_2$ don’t depend on $n$ if $n \geq 2$, therefore for any $n \geq 2$ the zeros of the polynomials $G_n(a, x + b, x)$ belong to the sets (circles) $K_1$ and $K_2$. I.e. if $G_n(a, x + b, x) = 0$ for a complex $x$, then

$$|x| \leq \max(|a| + |b|, 2) .$$

Since $G_1(a, x + b, x) = 0$ if $x = -b$ therefore (15) also holds if $n = 1$. This completes our proof for every integer $n \geq 1$.

**References**


