Acta Academiae Paedagogicae Agriensis, Sectio Mathematicae, 25. (1998) pp. 15–20

Bounds for the zeros of Fibonacci-like polynomials

FERENC MÁTYÁS

Abstract. The Fibonacci-like polynomials $G_n(x)$ are defined by the recursive formula $G_n(x)=xG_{n-1}(x)+G_{n-2}(x)$ for $n\geq 2$, where $G_0(x)$ and $G_1(x)$ are given seed-polynomials. The notation $G_n(x)=G_n(G_0(x),G_1(x),x)$ is also used. In this paper we determine the location of the zeros of polynomials $G_n(a,x+b,x)$ and give some bounds for the absolute values of complex roots of these polynomials if $a,b\in\mathbf{R}$ and $a\neq 0$. Our result generalizes the result of P. E. Ricci who investigated this problem in the case $a=b=1$.

Introduction

Let $G_0(x)$ and $G_1(x)$ be polynomials with real coefficients. For any $n\in\mathbf{N}\setminus\{0,1\}$ the polynomial $G_n(x)$ is defined by the recurrence relation

$$G_n(x)=xG_{n-1}(x)+G_{n-2}(x) \tag{1}$$

and these polynomials are called Fibonacci-like polynomials. If it is necessary then the initial or seed polynomials $G_0(x)$ and $G_1(x)$ can also be detected and in this case we use the form $G_n(x)=G_n(G_0(x),G_1(x),x)$. Note that $G_n(0,1,1)=F_n$ where F_n is the n^{th} Fibonacci number.

In some earlier papers the Fibonacci-like polynomials and other polynomials, defined by similar recursions, were studied. G. A. Moore [5] and H. Prodinger [6] investigated the maximal real roots (zeros) of the polynomials $G_n(-1,x-1,x)\quad(n\geq 1)$. Hongquan Yu, Yi Wang and Mingfeng He [2] studied the limit of maximal real roots of the polynomials $G_n(-a,x-a,x)$ if $a\in\mathbf{R}_+$ as n tends to infinity.

Under some restrictions in [3] we proved a necessary and sufficient condition for seed-polynomials when the set of the real roots of polynomials $G_n(G_0(x),G_1(x),x)\quad(n=0,1,2,\dots)$ has nonzero accumulation points. These accumulation points can be effectively determined. In [4], using this result, we proved the following

Theorem A. *If $a<0$ or $2<a$ then, apart from 0, the single accumulation point of the set of real roots of polynomials $G_n(a,x\pm a,x)\quad(n=1,2,\dots)$ is $\pm\frac{a(2-a)}{a-1}$, while in the case $0<a\leq 2$ the above set has no nonzero accumulation point.*

Research supported by the Hungarian OTKA Foundation, No. T 020295.

According to Theorem A, apart from finitely many real roots, all of the real roots of polynomials $G_n(a, x \pm a, x)$ ($a \in \mathbf{R} \setminus \{0\}$, $n = 1, 2, \dots$) can be found in the open intervals

$$\left(\pm\frac{a(2-a)}{a-1} - \varepsilon, \pm\frac{a(2-a)}{a-1} + \varepsilon\right) \quad \text{or} \quad (-\varepsilon, \varepsilon)\,,$$

where ε is an arbitrary positive real number.

Investigating the complex zeros of Fibonacci-like polynomials V. E. Hogatt, Jr. and M. Bicknell [1] proved that the roots of the equation $G_n(0, 1, x) = 0$ are $x_k = 2i\cos\frac{k\pi}{n}$ ($k = 1, 2, \dots, n-1$), i.e. apart from 0 if n is even, all of the roots are purely imaginary and their absolute values are less than 2. P. E. Ricci [7] among others studied the location of zeros of polynomials $G_n(1, x+1, x)$ and proved the following result.

Theorem B. *All of the complex zeros of polynomials* $G_n(1, x+1, x)$ ($n = 1, 2, \dots$) *are in or on the circle with midpoint* $(0, 0)$ *and radius 2 in the Gaussian plane.*

The purpose of this paper is to generalize the result of P. E. Ricci for the polynomials $G_n(a, x+b, x)$ where $a, b \in \mathbf{R}$ and $a \neq 0$, i.e. to give bounds for the absolute values of zeros. To prove our results we are going to use linear algebraic methods as it was applied by P. E. Ricci [7], too.

At the end of this part we list some terms of the polynomial sequence $G_n(x) = G_n(a, x+b, x) \quad (n = 2, 3, \dots)$. We have

$$\begin{aligned}
G_2(x) &= x^2 + bx + a,\\
G_3(x) &= x^3 + bx^2 + (a+1)x + b,\\
G_4(x) &= x^4 + bx^3 + (a+2)x^2 + 2bx + a,\\
G_5(x) &= x^5 + bx^4 + (a+3)x^3 + 3bx^2 + (2a+1)x + b,\\
G_6(x) &= x^6 + bx^5 + (a+4)x^4 + 4bx^3 + (3a+3)x^2 + 3bx + a.
\end{aligned}$$

Known facts from linear algebra

To estimate the absolute values of zeros of polynomials $G_n(a, x+b, x) \quad (n \geq 1)$ we need the following notations and theorem. Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix with complex entries, λ_i ($i = 1, 2, \dots, n$) and $f(x)$ denote the eigenvalues and the charecteristic polynomial of $\mathbf{A}$, respectively. It is known that

$$f(\lambda_i) = 0 \tag{2}$$

and

$$\max|\lambda_i| \le \|\mathbf{A}\|, \tag{3}$$

where $\|\mathbf{A}\|$ denotes a norm of the matrix $\mathbf{A}$. In this paper we apply the norms

$$\|\mathbf{A}\|_1 = n \max|a_{ij}| \tag{4}$$

and

$$\|\mathbf{A}\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}. \tag{5}$$

Using the so called Gershgorin's theorem we can get a better estimation for the absolute values of the roots of $f(x) = 0$ and it gives the location of zeros of $f(x)$, too. Let us consider the sets C_i of complex numbers z defined by

$$C_i = \{z : \ |z - a_{ii}| \le r_i\}, \tag{6}$$

where $i = 1, 2, \dots, n$ and

$$r_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \quad (n \ge 2). \tag{7}$$

So C_i is the set of complex numbers z which are inside the circle or on the circle with midpoint a_{ii} and radius r_i in the complex plane. These sets (circles) are called to be Gershgorin-circles. Using these notations we formulate the following well-known theorem.

Gershgorin's theorem. *Let $n \ge 2$. For every $i \quad (1 \le i \le n)$ there exists a $j \quad (1 \le j \le n)$ such that*

$$\lambda_i \in C_j \tag{8}$$

and so

$$\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset C_1 \cup C_2 \cup \dots \cup C_n. \tag{9}$$

Theorems and the Main Result

Let us consider the $n \times n$ matrix

$$\mathbf{A}_n = \begin{pmatrix} -b & -ai & 0 & \cdots & 0 & 0 & 0 \\ -i & 0 & -i & \cdots & 0 & 0 & 0 \\ 0 & -i & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -i & 0 & -i \\ 0 & 0 & 0 & \cdots & 0 & -i & 0 \end{pmatrix},$$

where $b \in \mathbf{R}$ and $a \in \mathbf{R} \setminus \{0\}$.

Further on we prove the following

Theorem 1. *Let $n \geq 1$ and $a, b \in \mathbf{R}$ $(a \neq 0)$. The characteristic polynomial of matrix $\mathbf{A}_n$ is the polynomial $G_n(a, x+b, x)$.*

Let $n \geq 2$ and $a, b \in \mathbf{R}$ $(a \neq 0)$. If $\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nn}$ denote the zeros of the polynomial $G_n(a, x+b, x)$ then, using the norms defined by (4) and (5) for the matrix $\mathbf{A}_n$, one can get the following estimations by (2),(3) and Theorem 1.

$$\max_{1 \leq i \leq n} |\lambda_{ni}| \leq n \max(|a|, |b|, 1) \tag{10}$$

and

$$\max_{1 \leq i \leq n} |\lambda_{ni}| \leq \sqrt{a^2 + b^2 + 2n - 3}. \tag{11}$$

From (10) and (11) it can be seen that these bounds depend on a, b and n but using the Gershgorin-circles we can get a more precise bound for $|\lambda_{ni}|$ and this bound depends only on a and b.

We shall prove

Theorem 2. *Let $n \geq 2$ and $a, b \in \mathbf{R}$ $(a \neq 0)$ and let us denote by K_1 the set $K_1 = \{z\colon\ |z + b| \leq |a|\}$ and by K_2 the set $K_2 = \{z\colon\ |z| \leq 2\}$ in the Gaussian plane. Then*

$$\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nn} \in K_1 \cup K_2. \tag{12}$$

Now we are able to formulate our main result.

Main Result. *For any* $n \geq 1$ *and* $a, b \in \mathbf{R}$ $(a \neq 0)$ *if* $G_n(a, x+b, x) = 0$, *then*

$$|x| \leq \max(|a| + |b|, 2), \tag{13}$$

i.e. the absolute values of all zeros of all polynomial terms of polynomial sequence $G_n(a, x+b, x)$ $(n = 1, 2, 3, \ldots)$ *have a common upper bound, and by (13) this bound depends only on* a *and* b *in explicit way.*

We mention that Theorem B can be obtained as a special case ($a = b = 1$) of our Main Result.

Proofs

Proof of Theorem 1. It is known that the characteristic polynomial $f_n(x)$ of matrix $\mathbf{A}_n$ can be obtained by the determinant of matrix $x\mathbf{I}_n - \mathbf{A}_n$, where $\mathbf{I}_n$ is the $n \times n$ unit matrix. So

$$f_n(x) = \det(x\mathbf{I}_n - \mathbf{A}_n) = \det \begin{pmatrix} x+b & ai & 0 & \cdots & 0 & 0 & 0 \\ i & x & i & \cdots & 0 & 0 & 0 \\ 0 & i & x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & i & x & i \\ 0 & 0 & 0 & \cdots & 0 & i & x \end{pmatrix}. \tag{14}$$

We prove the theorem by induction on n. It can be seen directly that $f_1(x) = x + b = G_1(a, x+b, x)$ and $f_2(x) = x^2 + bx + a = G_2(a, x+b, x)$. Let us suppose that $f_{n-2}(x) = G_{n-2}(a, x+b, x)$ and $f_{n-1}(x) = G_{n-1}(a, x+b, x)$ hold for an integer $n \geq 3$. Then developing (14) with respect to the last column and the resulting determinant with respect to the last row, we get

$$f_n(x) = xf_{n-1}(x) - iif_{n-2}(x) = xf_{n-1}(x) + f_{n-2}(x),$$

i.e. by our induction hipothesis

$$f_n(x) = xG_{n-1}(a, x+b, x) + G_{n-2}(a, x+b, x)$$

and so by (1)

$$f_n(x) = G_n(a, x+b, x)$$

holds for every integer $n \geq 1$.

Proof of the Theorem 2. From the matrix $\mathbf{A}_n$ we determine the so-called Gershgorin-circles. By the definition of $\mathbf{A}_n$ and (6) now there are only two distinct Gershgorin-circles. The midpoints of these circles are $-b$ and 0 in the Gaussian plane, while by (7) their radii are $|a|$ and 2, respectively, i.e. they are the sets (circles) K_1 and K_2. (We omitted the circle with midpoint 0 and radius 1, because this circle is contained by one of the above circles.)

Since $G_n(a, x+b, x)$ is the characteristic polynomial of the matrix $\mathbf{A}_n$, and $\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nn}$ are the zeros of it so from (8) and (9) we get that

$$\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nn} \in K_1 \cup K_2.$$

This completes the proof.

Proof of the Main Result. We have seen in the proof of Theorem 2 that the Gershgorin-circles K_1 and K_2 don't depend on n if $n \geq 2$, therefore for any $n \geq 2$ the zeros of the polynomials $G_n(a, x+b, x)$ belong to the sets (circles) K_1 and K_2. I.e. if $G_n(a, x+b, x) = 0$ for a complex x, then

$$|x| \leq \max(|a| + |b|, 2). \tag{15}$$

Since $G_1(a, x+b, x) = 0$ if $x = -b$ therefore (15) also holds if $n = 1$. This completes our proof for every integer $n \geq 1$.

References

[1] V. E. Hoggat, Jr. and M. Bicknell, Roots of Fibonacci Polynomials, *The Fibonacci Quarterly* **11.3** (1973), 271–274.

[2] Hongquan Yu, Yi Wang and Mingfeng He, On the Limit of Generalized Golden Numbers, *The Fibonacci Quarterly* **34.4** (1996), 320–322.

[3] F. Mátyás, Real Roots of Fibonacci-like Polynomials, *Proceedings of Number Theory Conference, Eger* (1996) (to appear)

[4] F. Mátyás, The Asymptotic Behavior of Real Roots of Fibonacci-like Polynomials, *Acta Acad. Paed. Agriensis, Sec. Mat.*, **24** (1997), 55–61.

[5] G. A. Moore, The Limit of the Golden Numbers is 3/2, *The Fibonacci Quarterly* **32.3** (1994), 211–217.

[6] H. Prodinger, The Asymptotic Behavior of the Golden Numbers, *The Fibonacci Quarterly* **35.3** (1996), 224–225.

[7] P. E. Ricci, Generalized Lucas Polynomials and Fibonacci Polynomials, *Riv. Mat. Univ. Parma* **(5)** 4 (1995), 137–146.

Ferenc Mátyás
Károly Eszterházy Teachers' Training College
Department of Mathematics
H–3301 Eger, Pf. 43
Hungary
E-mail: matyas@ektf.hu