Residual Lie nilpotence of the augmentation ideal

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Abstract. In this paper we give necessary and sufficient conditions for the residual Lie nilpotence of the augmentation ideal for an arbitrary group ring $RG$ except for the case when the derived group of $G$ is with no generalized torsion elements with respect to the lower central series of $G$ and the torsion subgroup of the additive group of $R$ contains a non-trivial element of infinite height. From this results we get the residual Lie nilpotence of the augmentation ideal of the $p$-adic integer group rings.

1. Introduction

Let $R$ be a commutative ring with identity, $G$ a group and $RG$ its group ring. The group ring $RG$ may be considered as a Lie algebra, with the usual bracket operation. The study of this Lie algebra was initiated by I. B. S. Passi, D. S. Passman and S. K. Sehgal [5]. Additional results on the Lie structure of $RG$ may be found in [4] and [6].

Let $A(RG)$ denote the augmentation ideal of $RG$, that is the kernel of the homomorphism $RG$ onto $R$ which sends each group element to 1. It is easy to see that as $R$-module $A(RG)$ is a free module with elements $g - 1$ ($g \in G$) as a basis.

There are many problems and results relating to $A(RG)$ ([4], [6]). In particular, it is an interesting problem to characterize the group rings whose augmentation ideal satisfy some conditions. In this paper, we treat the Lie property.

The Lie powers $A^{[\lambda]}(RG)$ of $A(RG)$ are defined inductively: $A^{[1]}(RG) = A(RG)$, $A^{[\lambda + 1]}(RG) = [A^{[\lambda]}(RG), A(RG)]RG$, if $\lambda$ is not a limit ordinal, and for the limit ordinal $\lambda$, $A^{[\lambda]}(RG) = \cap_{\nu < \lambda} A^{[\nu]}(RG)$, where $[K, M]$ denotes the $R$-submodule of $RG$ generated by $[k, m] = km - mk$ ($k \in K \subseteq RG$, $m \in M \subseteq RG$), and for $K \cdot RG$ denotes the right ideal generated by $K$ in $RG$.

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For the first limit ordinal $\omega$ we adopt the notation:

$$A^{[\omega]}(RG) = \bigcap_{i=1}^{\infty} A^{[i]}(RG).$$

The ideal $A(RG)$ of the group ring $RG$ is said to be residually Lie nilpotent if $A^{[\omega]}(RG) = 0$.

In this paper we give necessary and sufficient conditions for the residual Lie nilpotence of the augmentation ideal for an arbitrary group ring $RG$ except for the case when the derived group of $G$ is with no generalized torsion elements with respect to the lower central series of $G$ and the torsion subgroup of the additive group of $R$ contains a non-trivial element of infinite height.

Our main results are given in section 3. These results (Theorem A, B and C) are rather technical so they are not stated in the introduction.

2. Notations and some known facts

If $H$ is a normal subgroup of $G$, then $I(RH)$ (or $I(H)$ for short) denotes the ideal of $RG$ generated by elements of the form $h - 1$, ($h \in H$). It is well known that $I(RH)$ is the kernel of the natural epimorphism $\phi: RG \to RG/H$ induced by the group homomorphism $\phi$ of $G$ onto $G/H$. It is clear that $I(RG) = A(RG)$.

Let $F$ be a free group on the free generators $x_i$ ($i \in I$) and $ZF$ be its integral group ring ($Z$ denotes the ring of rational integers). Then every homomorphism $\phi: F \to G$ induces a ring homomorphism $\phi: ZF \to RG$ by letting $\phi(\sum n_y y) = \sum n_y \phi(y)$. If $f \in ZF$, we denote by $A_f(RG)$ the two-sided ideal of $RG$ generated by the elements $\phi(f)$, $\phi \in \text{Hom}(F, G)$, the set of homomorphism from $F$ to $G$. In other words $A_f(RG)$ is the ideal generated by the values of $f$ in $RG$ as the elements of $G$ are substituted for the free generators $x_i$-s.

An ideal $J$ of $RG$ is called a polynomial ideal if $J = A_f(RG)$ for some $f \in ZF$. It is easy to see that the augmentation ideal $A(RG)$ is a polynomial ideal. Really, $A(RG)$ is generated as an $R$-module by elements $g - 1$ ($g \in G$), i.e. by the values of the polynomial $x - 1$.

We also use the following

**Lemma 2.1.** ([4], Proposition 1.4., page 2.) Let $f \in ZF$. Then $f$ defines a polynomial ideal $A_f(RG)$ in every group ring $RG$. Further, if $\theta: RG \to KH$
is a ring homomorphism induced by a group homomorphism \( \phi: G \to H \) and a ring homomorphism \( \psi: R \to K \), then
\[
\theta(A_f(RG)) \subseteq A_f(KH).
\]

(It is assumed here that \( \psi(1_R) = 1_K \), where \( 1_R \) and \( 1_K \) are identities of rings \( R \) and \( K \) respectively.)

For every natural number \( n \) \( A^{[n]}(RG) \) is a polynomial ideal (see in particular [4], Corollary 1.9., page 6.) and by Lemma 2.1.
\[
\overline{\phi}(A^{[n]}(RG)) \subseteq A^{[n]}(RG/L)
\]
for every \( n \). From this inclusion it can be obtained easily that
\[
(1) \quad \overline{\phi}(A^{[\omega]}(RG)) \subseteq A^{[\omega]}(RG/L).
\]

If \( \mathcal{K} \) denotes a class of groups we define the class \( \mathbf{R}\mathcal{K} \) of residually-\( \mathcal{K} \) groups by letting \( G \in \mathbf{R}\mathcal{K} \) if and only if: whenever \( 1 \neq g \in G \), there exists a normal subgroup \( H_g \) of the group \( G \) such that \( G/H_g \in \mathcal{K} \) and \( g \notin H_g \). It is easy to see that \( G \in \mathbf{R}\mathcal{K} \) if and only if there exists a family \( \{H_i\}_{i \in I} \) of normal subgroups \( G \) such that \( G/H_i \in \mathcal{K} \) for every \( i \in I \) and \( \cap_{i \in I} H_i = \langle 1 \rangle \).

A group \( G \) is said to be discriminated by \( \mathcal{K} \) if for every finite set \( g_1, g_2, \ldots, g_n \) of distinct elements of \( G \), there exists a group \( H \in \mathcal{K} \) and a homomorphism \( \phi: G \to H \) such that \( \phi(g_i) \neq \phi(g_j) \) if \( i \neq j \), \( (1 \leq i, j \leq n) \).

**Lemma 2.2.** Let a class of groups \( \mathcal{K} \) be closed with respect to forming subgroups and finite direct products and let \( G \) be a residually-\( \mathcal{K} \) group. Then \( G \) is discriminated by \( \mathcal{K} \).

The proof can be obtained easily.

It is easy to show that if \( G \) is discriminated by a class of groups \( \mathcal{K} \) and if \( x \) is a non-zero element of \( RG \), then there exists a group \( H \in \mathcal{K} \) and a homomorphism \( \phi \) of \( RG \) to \( RH \) such that \( \phi(x) \neq 0 \).

From this fact and from inclusion \( (1) \) we have

**Lemma 2.3.** If \( G \) is discriminated by a class of groups \( \mathcal{K} \) and for each \( H \in \mathcal{K} \) the equation \( A^{[\omega]}(RH) = 0 \) holds, then \( A^{[\omega]}(RG) = 0 \).

We use the following notations for standard group classes:

- \( \mathcal{D}_0 \) — the class of those nilpotent groups whose derived groups are torsion-free.
- \( \mathcal{D}_p \) — the class of nilpotent groups whose derived groups are \( p \)-groups of bounded exponent.
\( \mathcal{N}_0 \) — the class of torsion-free nilpotent groups.
\( \mathcal{N}_p \) — the class of nilpotent \( p \)-groups of bounded exponent.
\( \mathcal{N}_\Omega = \cup_{p \in \Omega} \mathcal{N}_p \) and
\( \mathcal{D}_\Omega = \cup_{p \in \Omega} \mathcal{D}_p \), where \( \Omega \) is a subset of the set of primes.

The ideal \( J_p(R) \) of a ring \( R \) is defined by \( J_p(R) = \cap_{n=1}^\infty p^n R \).

**Theorem 2.4.** ([4], Theorem 2.13., page 85.) Let \( G \) be a residually \( D_p \)-group and \( J_p(R) = 0 \). Then \( A^\omega(RG) = 0 \).

We shall use the following lemma, which gives some elementary properties of the Lie powers of \( A(RG) \).

**Lemma 2.5.** ([4], Proposition 1.7., page 4.) For arbitrary natural numbers \( n \) and \( m \) are true:

1. \( I(\gamma_n(G)) \subseteq A^n(RG) \),
2. \( [A^n(RG), A^m(RG)] \subseteq A^{n+m}(RG) \),
3. \( A^n(RG) \cdot A^m(RG) \subseteq A^{n+m-1}(RG) \),

where \( \gamma_n(G) \) is the \( n \)th term of the lower central series of \( G \).

We write \( D_{[n]}(RG) \) for the \( n \)th Lie dimension subgroup \( D_{[n]}(RG) \) of \( G \) over \( R \). That is

\[ D_{[n]}(RG) = \{ g \in G | g - 1 \in A^n(RG) \} \]

By Lemma 2.5. it follows that for every natural number \( n \) the inclusion

\[ \gamma_n(G) \subseteq D_{[n]}(RG) \]

holds.

We also use the following theorems

**Theorem 2.6.** ([1], Theorem 3.2.) Let a group \( G \) contain a non-trivial generalized torsion element. Then \( A(RG) \) is residually nilpotent if and only if there exists a non-empty subset \( \Omega \) of the set of primes such that \( \cap_{p \in \Omega} J_p(R) = 0 \), \( G \) is discriminated by the class \( \mathcal{N}_\Omega \) and for every proper subset \( \Lambda \) of the set \( \Omega \) at least one of the conditions

1. \( \cap_{p \in \Lambda} J_p(R) = 0 \)
2. \( G \) is discriminated by the class of groups \( \mathcal{N}_{\Omega \setminus \Lambda} \)

holds.

Let \( T(R^+) \) denote the torsion subgroup of the additive group \( R^+ \) of a ring \( R \) and let \( A^\omega(RG) = \cap_{i=1}^\infty A^n(RG) \), where \( A^n(RG) \) is the \( n \)th associative power of \( A(RG) \).
Theorem 2.7. ([4], Theorem 2.7., page 87.) If $G \in R N_0$ and $R$ is a ring with identity such that its additive group $R^+$ is torsion-free, then $A^\omega(RG) = 0$.

3. Residual Lie nilpotence

It is clear, that $A^{[2]}(RG) = 0$ if and only if $G$ is an Abelian group. Therefore we may assume that the derived group $G' = \gamma_2(G)$ of $G$ is non-trivial.

For a nilpotent group $G$ the following inclusion is true
\[(2) \quad A^{[\omega]}(RG) \subseteq A^\omega(RG')RG\]
(see in particular [4]). For every natural number $i > 1$ we define the normal subgroup
\[L_i = \{g \in G' | g^k \in \gamma_i(G) \text{ for a suitable } k \geq 1\}\]
of $G$. It is easy to see that $\gamma_i(G) \subseteq L_i$ and also that $G/L_i \in D_0$ for every $i > 1$.

An element $g$ of a group $G$ is called a generalized torsion element with respect to the lower central series of $G$ if for every $n$ the order of the elements $g\gamma_n(G)$ of the factor group $G/\gamma_n(G)$ is finite.

We recall that if the derived group $G'$ of $G$ contains no generalized torsion elements with respect to the lower central series of $G$, then $G'$ has no generalized torsion elements with respect to the lower central series of $G'$.

**Theorem A.** Let $R$ be a commutative ring with identity, $T(R^+) = 0$ and let $G'$ be with no generalized torsion elements with respect to the lower central series of $G$. Then $A^{[\omega]}(RG) = 0$ if and only if $G$ is a residually-$D_0$ group.

**Proof.** Since $G'$ is with no generalized torsion elements with respect to the lower central series of $G$, then $\cap_{i \geq 2} L_i = \langle 1 \rangle$ and so, $G \in RD_0$.

Conversely. Let $G \in RD_0$ and $T(R^+) = 0$. Since class $D_0$ is closed with respect to forming subgroups and finite direct products, by Lemmas 2.2. and 2.3. it is enough to show that $A^{[\omega]}(RG) = 0$ for all $G \in D_0$. So let $G \in D_0$. Then by (2)
\[A^{[\omega]}(RG) \subseteq A^\omega(RG')RG\]
Because $G'$ is a torsion-free nilpotent group, by Theorem 2.7. $A^\omega(RG') = 0$, and so, $A^{[\omega]}(RG) = 0$. The proof is completed.
Let $p$ be a prime and $n$ a natural number. Then $G^{p^n}$ is the subgroup of $G$ generated by all elements of the form $g^{p^n}$, $g \in G$.

For a prime $p$ and a natural number $k$ the normal subgroup $G_{[p,k]}$ of $G$ is defined by

$$G_{[p,k]} = \bigcap_{n=1}^{\infty} (G')^{p^n} \gamma_k(G).$$

We have the following sequence

$$G = G_{[p,1]} \supseteq G_{[p,2]} \supseteq \ldots \supseteq G_{[p]}$$

of normal subgroups $G_{[p,k]}$ of $G$, where

$$G_{[p]} = \bigcap_{k=1}^{\infty} G_{[p,k]}.$$

It is clear, that $G/(G')^{p^n} \gamma_k(G)$ are in $\mathcal{D}_p$, and $G/G_{[p,k]}$ and $G/G_{[p]}$ are residually-$\mathcal{D}_p$ groups for every $k$ and $n$.

**Lemma 3.1.** If $n \geq ks$ and $h \in (G')^{p^n} \gamma_k(G)$, then

$$h - 1 \equiv p^s X(k,h) \pmod{A[k](RG)}$$

for a suitable $X(k,h) \in A[2](RG)$.

**Proof.** Let $h \in (G')^{p^n} \gamma_k(G)$. We can write element $h$ as

$$h = h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} y_k$$

where $h_i \in G'$, $y_k \in \gamma_k(G)$. Using the identity

$$ab - 1 = (a - 1)(b - 1) + (a - 1) + (b - 1)$$

(3)

to $h - 1$ we have that

$$h - 1 = (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1)(y_k - 1) + (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) + (y_k - 1).$$

By Lemma 2.5. $I(\gamma_k(G)) \subseteq A[k](RG)$ and hence $y_k - 1 \in A[k](RG)$. Therefore

$$h - 1 \equiv (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) \pmod{A[k](RG)}.$$
Applying identity (3) repeatedly to \((h_1^{p_1} h_2^{p_2} \cdots h_m^{p_m} - 1)\) from the previous congruence it follows that

\[
h - 1 \equiv \sum_{i=1}^{m} (h_i^{p_i} - 1) b_i \equiv \sum_{i=1}^{m} \sum_{j=1}^{p_i} \binom{p_i}{j} (h_i - 1)^j b_i \pmod{A[k](RG)},
\]

where \(b_i \in RG\). Because \(h_i \in G' = \gamma_2(G)\), from Lemma 2.5. (cases 1 and 3) we obtain that \((h_i - 1)^j \in A[j+1](RG)\) for every \(i\) and \(j\). If \(n \geq sk\), then \(p^s\) divides \(\binom{p_i}{j}\) for every \(j = 1, 2, \ldots, k - 1\). Therefore

\[
h - 1 \equiv \sum_{i=1}^{m} (h_i^{p_i} - 1) b_i \equiv p^s \sum_{i=1}^{m} \sum_{j=1}^{k-1} d_j (h_i - 1)^j b_i \\
\equiv p^s X(k,h) \pmod{A[k](RG)},
\]

where \(X(k,h) = \sum_{i=1}^{m} \sum_{j=k}^{p_i} d_j (h_i - 1)^j b_i\), \(b_i \in RG\), \(p^s d_j = \binom{p_i}{j}\). The Lemma is proved.

It is easy to show that if \(g \in G'\) and \(g^{p^n} \in D[k](RG)\) then

\[
(4) \quad p^m (g - 1) \in A[k](RG)
\]

for a large enough \(m\).

**Lemma 3.2.** ([1], Lemma 3.6.) Let \(K\) be a class of groups and \(\{G_\alpha\}_{\alpha \in I}\) a family of normal subgroups of \(G\) such that for all \(\alpha\) (\(\alpha \in I\)) the conditions

1. \(G/G_\alpha \in K\)
2. \(G_\alpha\) is torsion-free

hold. If \(G\) is not discriminated by \(K\) then there exists a finite set of distinct elements \(g_1, g_2, \ldots, g_s\) from \(G\) such that the non-zero element \(y = (g_1 - 1)(g_2 - 1) \cdots (g_s - 1)\) lies in the ideal \(\cap_{\alpha \in I} I(G_\alpha)\).

The torsion subgroup \(T(R^+)\) of the additive group \(R^+\) of a ring \(R\) is the direct sum of its \(p\)-primary components \(S_p(R^+)\). Let \(\Pi\) be the set of those primes for which the \(p\)-primary components \(S_p(R^+)\) of \(T(R^+)\) are non-zero.

An element \(a\) of an additive Abelian group \(A\) is called an element of infinite \(p\)-height for a prime \(p\), if the equation \(p^n x = a\) has a solution in \(A\) for every natural number \(n\).

**Proposition 3.3.** ([1], Theorem 3.3.) Let \(T(R^+) \neq 0\), and suppose that for some \(p \in \Pi\) group \(T(R^+)\) has no element of infinite \(p\)-height.
Further let $G$ be a group with no generalized torsion elements. Then $A^\omega(RG) = 0$ if and only if $G$ is a residually-$N_p$ group for all $p \in \Pi$.

**Theorem B.** Let $T(R^+) \neq 0$. If $G'$ is with no generalized torsion elements with respect to the lower central series of $G$ and $T(R^+)$ is with no non-trivial elements of infinite $p$-height then $A^\omega(RG) = 0$ if and only if $G$ is a residually-$D_p$ group for all $p \in \Pi$.

**Proof.** Let $p$ an arbitrary prime of $\Pi$, $A^\omega(RG) = 0$, and let $p^s$ ($s \geq 1$) be the order of element $a \in T(R^+)$. Since the equation

$$G_{[p]} = \bigcap_{k=1}^{\infty} G[p, k] = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} (G')^{p^n} \gamma_k(G)) = \langle 1 \rangle$$

implies that $G \in RD_p$, it is enough to show, that $G_{[p]} = \langle 1 \rangle$.

Suppose that $g \in G_{[p]}$. Then $g \in (G')^{p^n} \gamma_k(G)$ for every $n$ and $k$ and by Lemma 3.1. we have that

$$g - 1 \equiv p^s X(k, g) \pmod{A^k(RG)}$$

for every $k$. From $p^s a = 0$ it follows that $a(g - 1) \in A^k(RG)$ for every $k$. Hence $a(g - 1) \in A^\omega(RG)$ and $a(g - 1) = 0$. This implies that $g = 1$. Consequently $G_{[p]} = \langle 1 \rangle$. This means that $G$ is a residually-$D_p$ group for all $p \in \Pi$.

Conversely. Let $G \in RD_p$ for $p \in \Pi$ and let $1 \neq g$ be an arbitrary element of $G'$. Then there exists a normal subgroup $H$ of $G$ such that $G/H \in D_p$ and $g \notin H$. Since $G/H \in D_p$ then $(G/H)' \in N_p$. By the isomorphism $G'H/H \cong G'/H \cap G'$ we have that $\overline{g} = g(H \cap G') \neq \overline{1}$. This means that if $G \in RD_p$ then $G' \in R N_p$. Using Proposition 3.3. we have that $A^\omega(RG') = 0$ and from (2) it follows that $A^\omega(RG) = 0$.

**Lemma 3.4.** Let

$$y \in \bigcap_{p \in \Gamma} \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} I((G')^{p^n} \gamma_j(G)).$$

Then for a prime $p \in \Gamma$ and arbitrary natural numbers $k$ and $s$

$$y \equiv p^s Y(p, k, s, y) \pmod{A^k(RG)},$$

where $Y(p, k, s, y) \in RG$ and $\Gamma$ is a subset of the set of prime numbers.

**Proof.** Let $p \in \Gamma$. For every natural $n$ we can express $y$ as

$$y = \sum_{i=1}^{l} \alpha_i z_i(h_i - 1),$$
where \( h_i \in (G')p^n \gamma_k(G) \), \( \alpha_i \in R \) and every \( z_i \) is from a set of coset representatives of \((G')p^n \gamma_k(G)\) in \( G \). For a large enough \( n \) by Lemma 3.1.

\[
 h_i - 1 \equiv p^s X(k, h_i) \pmod{A[k](RG)}
\]

for every \( i \) (\( i = 1, 2, \ldots, l \)) and the proof follow.

If \( g \in G' \) is a generalized torsion element of a group \( G \) then \( \Omega_g \) denotes the set of the prime divisors of the order of the elements \( g \gamma_k(G) \in G/\gamma_k(G) \) for every \( k = 2, 3, \ldots \).

**Lemma 3.5.** Let \( g \in G' \) be a generalized torsion element of a group \( G \), \( \Lambda \) an arbitrary subset of \( \Omega_g \), \( a \in \bigcap_{p \in \Lambda} J_p(R) \) and let

\[
 x \in \bigcap_{p \in \Omega_g \setminus \Lambda} \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{\infty} I((G')p^i \gamma_k(G)).
\]

Then one of the following statements

1. if \( \Lambda \) is a proper subset of \( \Omega_g \), then \( a(g - 1)x \in A[\omega](RG) \)
2. if \( \Lambda = \Omega_g \), then \( a(g - 1) \in A[\omega](RG) \)
3. if \( \Lambda = \emptyset \), then \( (g - 1)x \in A[\omega](RG) \)

holds.

**Proof.** It is enough to show that for an arbitrary natural number \( k \) the elements \( a(g - 1), (g - 1)x, a(g - 1)x \) are in the ideal \( A[k](RG) \).

If \( g \in \gamma_k(G) \) then by Lemma 2.5. \( (g - 1) \in A[k](RG) \), and the statements follow. Now let \( g \notin \gamma_k(G) \) and let

\[
 n_k = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}
\]

be the prime factorization of the order of the elements \( g \gamma_k(G) \) of the nilpotent group \( G/\gamma_k(G) \). It is clear that \( p_i \in \Omega_g \) for every \( i = 1, 2, \ldots, s \). Let \( \Lambda \) a subset of \( \Omega_g \). With loss of generality we may assume that \( p_1, p_2, \ldots, p_l \in \Lambda \) and \( p_i \notin \Lambda \) for \( i > l \).

Let \( g = g_1 g_2 \cdots g_s \gamma_k(G) \) be the decomposition of the element \( g \gamma_k(G) \) of the nilpotent group \( G/\gamma_k(G) \) in the product of \( p_i \)-elements \( g_i \gamma_k(G) \) (\( i = 1, 2, \ldots, s \)). Then

\[
 g = g_1 g_2 \cdots g_s y_k, \quad g_i \in G', i = 1, 2, \ldots, s
\]

for a suitable \( y_k \in \gamma_k(G) \). Then there exists \( m_i \) (\( i = 1, 2, \ldots, s \)) such that

\[
 g_i^{p_i^{m_i}} \in \gamma_k(G).
\]
Using identity (3) repeatedly to \((g - 1)\) we conclude that
\[
g - 1 \equiv v + w + (y_k - 1) \equiv v + w \pmod{A[k](RG)},
\]
where \(v = \sum_{i=1}^{l} (g_i - 1)x_i, \ w = \sum_{i=l+1}^{s} (g_i - 1)x_i\) and \(x_i \in RG\). In the case when \(\Lambda \cap \{p_1, p_2, \ldots, p_s\} = \emptyset\) we assume that \(v = 0\), and if \(\Lambda \cap \{p_1, p_2, \ldots, p_s\} = \{p_1, p_2, \ldots, p_s\}\) we put \(w = 0\). Because
\[
g_i^{p_i m_i} \in \gamma_k(G) \subseteq D[k](G)
\]
and \(g_i \in G'\) for every \(i = 1, 2, \ldots, s\), we conclude from (4) that there exists a natural number \(r_i\) \((i = 1, 2, \ldots, s)\) such that
\[
p_i^{r_i}(g_i - 1) \in A[k](RG).
\]
Also, since
\[
a \in \bigcap_{p \in \Lambda} J_p(R) \subseteq \bigcap_{i=1}^{l} J_p(R)
\]
we can express \(a\) as \(a = p_i^{r_i}a_i\) \((a_i \in R)\) for each \(i \leq l\). Then by (5)
\[
av = \sum_{i=1}^{l} a_i p_i^{r_i}(g_i - 1)x_i \equiv 0 \pmod{A[k](RG)}.
\]
Therefore
\[
a(g - 1) \equiv av + aw \equiv aw \pmod{A[k](RG)}.
\]
If \(\Lambda = \Omega_g\) then \(w = 0\) and case 2) is proved.

By Lemma 3.4.
\[
x \equiv p_i^{r_i}Y(p_i, k, r_i, x) \pmod{A[k](RG)},
\]
and so,
\[
wx \equiv \sum_{i=l+1}^{s} p_i^{r_i}(g_i - 1)x_i Y(p_i, k, r_i, x) \pmod{A[k](RG)}.
\]
Hence by (5)
\[
wx \equiv 0 \pmod{A[k](RG)}.
\]
If $\Lambda = \emptyset$, then $v = 0$, and so,
\[(g - 1)x \equiv vx + wx \equiv wx \equiv 0 \pmod{A^{[k]}(RG)}\]
and case 3) is proved.

Also, since
\[a(g - 1)x \equiv avx + awx \pmod{A^{[k]}(RG)}\]
from congruences (6) and (7) the proof (of case 1)) follows.

We recall that for a prime $p$ $N_p$ denotes the class of nilpotent groups whose derived groups are $p$-groups of bounded exponent, and if $\Omega$ a subset of the set of primes, then $N_\Omega = \cup_{p \in \Omega} N_p$ and $D_\Omega = \cup_{p \in \Omega} D_p$.

Let a group $G$ be discriminated by the class of groups $D_\Gamma$ ($\Gamma \neq \emptyset$) and let $g_1, g_2, \ldots, g_n$ be a finite set of distinct elements of $G'$. Then there exists a normal subgroup $H$ of $G$ such that $g_iH \neq g_jH$ if $i \neq j$ and $G/H \in D_\Gamma$. Therefore $(G/H)' \in N_p$ for any prime $p \in \Gamma$. By the isomorphism $G'H/H \cong G'/H \cap G'$ we have $g_iH(H \cap G') \neq g_jH(H \cap G')$ if $i \neq j$ ($i, j = 1, 2, \ldots, n$). This means, that if $G$ is discriminated by the class of groups $D_\Gamma$, then $G'$ is discriminated by the class of groups $N_\Gamma$.

**Lemma 3.6.** Let $\Omega$ be a non-empty subset of the set of primes such that
\[\cap_{p \in \Omega} J_p(R) = 0\]
and a group $G$ is discriminated by the class of groups $D_\Omega$. If for every proper subset $\Lambda$ of the set $\Omega$ at least one of the conditions
\[
(1) \cap_{p \in \Lambda} J_p(R) = 0
\]
\[
(2) G \text{ is discriminated by the class of groups } D_{\Omega \setminus \Lambda}
\]
holds, then $A^{[\omega]}(RG) = 0$.

**Proof.** Let
\[x = \sum_{i=1}^{n} \alpha_i g_i \in A^{[\omega]}(RG)\]

By Lemma 2.3. it is enough to show that $A^{[\omega]}(RG) = 0$ for all groups $G \in D_\Omega$. So let $G \in D_\Omega$. Then $G$ is a nilpotent group and by (2)
\[A^{[\omega]}(RG) \subseteq A^{[\omega]}(RG')RG.
\]
Clearly, $G' \in N_\Omega$. If $G$ is discriminated by the class of groups $D_\Gamma$, where $\Gamma$ is an arbitrary non-empty subset of $\Omega$, then $G'$ is discriminated by the class $N_\Gamma$, which was showed above. Then $G'$ satisfies Theorem 2.6. and so, $A^{[\omega]}(RG') = 0$. Consequently $A^{[\omega]}(RG) = 0$. 
Theorem C. Let the derived group \( G' \) contain a generalized torsion element of \( G \) with respect to the lower central series of \( G \). Then \( A(RG) \) is residually Lie nilpotent if and only if there exists a non-empty subset \( \Omega \) of the set of primes such that \( \cap_{p \in \Omega} J_p(R) = 0 \), \( G \) is discriminated by the class of groups \( D_\Omega \) and every proper subset \( \Lambda \) of the set \( \Omega \) at least one of the conditions

\[
(1) \cap_{p \in \Lambda} J_p(R) = 0 \\
(2) G \text{ is discriminated by the class of groups } D_{\Omega \setminus \Lambda}
\]

holds.

Proof. Let \( A^{[\omega]}(RG) = 0 \). Let us first consider the case when \( G' \) contains a non-trivial torsion element. Then there exists a \( p \)-element \( g \) in \( G' \) with \( p \in \Omega \). Then by (4) for every \( k \) there exists a natural number \( m \) such that

\[
p^m(g - 1) \in A^{[k]}(RG).
\]

If \( a \in J_p(R) \), then for each \( m \) we can write element \( a \) as \( a = p^m a_m \) (\( a_m \in R \)). Therefore \( a(g - 1) \in A^{[k]}(RG) \) for every \( k \), that is \( a(g - 1) \in A^{[\omega]}(RG) \).

Hence \( a(g - 1) = 0 \) and so, \( a = 0 \). Consequently \( J_p(R) = 0 \).

Now we show, that \( G \) is discriminated by \( D_{ \{ p \} } \). Let

\[
h \in \bigcap_{k=1}^\infty \bigcap_{i=1}^\infty (G')^p \gamma_k(G).
\]

Then

\[
h - 1 \in \bigcap_{k=1}^\infty \bigcap_{i=1}^\infty I((G')^p \gamma_k(G))
\]

and by Lemma 3.4. for every \( k \) and \( m \)

\[
h - 1 \equiv p^m Y(p,k,m,h - 1) \pmod{A^{[k]}(RG)}.
\]

By (8) and (9) we have that

\[
(g - 1)(h - 1) \equiv p^m(g - 1)(h - 1)Y(p,m,k,h - 1) \pmod{A^{[k]}(RG)}
\]

for every \( k \). This implies that

\[
(g - 1)(h - 1) \in A^{[\omega]}(RG) \text{ and so, } (g - 1)(h - 1) = 0.
\]

From this equation we have that the characteristic of \( R \) is \( p \) (= 2) and from (9) it follows that \( h - 1 \in A^{[\omega]}(RG) \). Therefore \( h = 1 \) and so

\[
\bigcap_{k=1}^\infty \bigcap_{i=1}^\infty (G')^p \gamma_k(G) = \langle 1 \rangle.
\]
For every $k$ and $i$, $G/(G')^{p^i} \gamma_k(G) \in D_{\{p\}}$. The class $D_{\{p\}}$ is closed with respect to forming subgroups and finite direct products, and by Lemma 2.2, $G$ is discriminated by $D_{\{p\}}$. Consequently we can choose the set $\Omega = \{p\}$.

Let us consider the case when $G'$ is a torsion-free group and $1 \neq g \in G'$ is a generalized torsion element of $G$. We put $\Omega = \Omega_g$. From Lemma 3.5. (case 2) it follows that

$$\bigcap_{p \in \Omega} J_p(R) = 0.$$  

From Lemma 3.2. (here we put $\{G_\alpha\}_{\alpha \in I} = \{(G')^{p^n} \gamma_k(G), k, n=1, 2, \ldots \}_{p \in \Omega}$) and Lemma 3.5. (case 3) we have that $G$ is discriminated by the class $D_\Omega$.

Let $\Lambda$ be an arbitrary subset of $\Omega$ and let $\bigcap_{p \in \Lambda} J_p(R) \neq 0$. If $G$ is not discriminated by the class of groups $D_{\Omega \setminus \Lambda}$, then by Lemma 3.2. there exists a set of elements $g_1, g_2, \ldots, g_n$ ($g_i \in G$) of infinite orders such that

$$0 \neq (g_1 - 1)(g_2 - 1) \cdots (g_n - 1) \in \bigcap_{p \in \Omega \setminus \Lambda} \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{\infty} I((G')^{p^i} \gamma_k(G)).$$

By Lemma 3.5. (case 1) for every element $a \in \bigcap_{p \in \Lambda} J_p(R)$

$$a(g - 1)(g_1 - 1)(g_2 - 1) \cdots (g_n - 1) \in A[\omega](RG).$$

Because $A[\omega](RG) = 0$ we have that

$$a(g - 1)(g_1 - 1)(g_2 - 1) \cdots (g_n - 1) = 0.$$  

Since element $g_i$ ($i = 1, 2, \ldots, n$) has infinite order and so has zero left (and right) annihilator in $RG$, then for $g_n$ we have

$$a(g - 1)(g_1 - 1)(g_2 - 1) \cdots (g_{n-1} - 1) = 0.$$  

Continuing this procedure for $i = n - 1, n - 2, \ldots, 1$ on the last step we get that

$$a(g - 1) = 0.$$  

Since the element $g$ has infinite order, its left annihilator is zero in $RG$, which implies $a = 0$. Consequently, if $G$ is not discriminated by the class of groups $D_{\Omega \setminus \Lambda}$, then $\bigcap_{p \in \Lambda} J_p(R) = 0$.

The sufficiency part is proved in Lemma 3.6.

**Corollary.** Let $R = \hat{\mathbb{Z}}_p$, the ring of $p$-adic integers. Then $A[\omega](\hat{\mathbb{Z}}_p G) = 0$ if and only if either
(1) \( G \) is discriminated by the class \( D_0 \) or
(2) \( G \) is discriminated by the class \( D_p \).

**Proof.** If \( G' \) is with no generalized torsion elements (with respect to the lower central series of \( G \)), then by Theorem A \( A^{[\omega]}(\hat{\mathbb{Z}}_p G) = 0 \) if and only if \( G \) is discriminated by the class \( D_0 \).

Let us consider the case when \( G' \) contains a generalized torsion element. Let \( A^{[\omega]}(\hat{\mathbb{Z}}_p G) = 0 \). By Theorem C there exists a non-empty subset \( \Omega \) of the set of primes, such that \( \cap_{q \in \Omega} J_q(\hat{\mathbb{Z}}_p) = 0 \). It is known that \( J_p(\hat{\mathbb{Z}}_p) = 0 \) and for a prime \( q \neq p \), \( J_q(\hat{\mathbb{Z}}_p) = \hat{\mathbb{Z}}_p \). Therefore \( p \in \Omega \). If \( \Omega = \{p\} \), then by the last theorem \( G \) is discriminated by \( D_p \). If \( \Omega \) contains a prime \( q \neq p \), then we choose \( \Lambda \subseteq \Omega \) such that \( \Omega \setminus \Lambda = \{p\} \). Then \( \cap_{q \in \Lambda} J_q(\hat{\mathbb{Z}}_p) \neq 0 \) and by Theorem C \( G \) is discriminated by the class \( D_p \).

Conversely. If \( G \) is discriminated by the class \( D_p \), we put \( \Omega = \{p\} \), and the proof follows from Theorem C.

From Theorem A and C we also get the results of I. Musson and A. Weiss ([2], Theorem A).

**References**


