A note on the products of the terms of linear recurrences

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Abstract. For an integer \( \nu > 1 \) let \( G^{(i)} \) \((i=1,\ldots,\nu)\) be linear recurrences defined by

\[
G^{(i)}_n = A_1^{(i)} G^{(i)}_{n-1} + \cdots + A_{k_i}^{(i)} G^{(i)}_{n-k_i} \quad (n \geq k_i).
\]

In the paper we show that the equation

\[
d G^{(1)} \cdots G^{(\nu)} = sw^q,
\]

where \( d,s,w,q,x_i \) are positive integers satisfying some conditions, implies the inequality \( q < q_0 \) with some effectively computable constant \( q_0 \). This result generalizes some earlier results of Kiss, Pethő, Shorey and Stewart.

1. Introduction

Let \( G^{(i)} = \{G^{(i)}_n\}_{n=0}^{\infty} \) \((i=1,2,\ldots,\nu)\) be linear recurrences of order \( k_i \) \((k_i \geq 2)\) defined by

\[
G^{(i)}_n = A_1^{(i)} G^{(i)}_{n-1} + \cdots + A_{k_i}^{(i)} G^{(i)}_{n-k_i} \quad (n \geq k_i),
\]

where the initial values \( G^{(i)}_j \) \((j=0,1,\ldots,k_i-1)\) and the coefficients \( A^{(i)}_l \) \((l=1,2,\ldots,k_i)\) of the sequences are rational integers. We suppose, that \( A^{(i)}_{k_i} \neq 0 \) and there is at least one non-zero initial value for any recurrences.

By \( \alpha^{(i)}_1 = \gamma_i, \alpha^{(i)}_2, \ldots, \alpha^{(i)}_{t_i} \) we denote the distinct roots of the characteristic polynomial

\[
p^{(i)}_t(x) = x^{k_i} - A_1^{(i)} x^{k_i-1} - \cdots - A_{k_i}^{(i)}
\]

of the sequence \( G^{(i)} \), and we assume that \( t_i > 1 \) and \( |\gamma_i| > |\alpha^{(i)}_j| \) for \( j > 1 \). Consequently \( |\gamma_i| > 1 \). Suppose that the multiplicity of the roots \( \gamma_i \) are 1. Then the terms of the sequences \( G^{(i)} \) \((i=1,2,\ldots,\nu)\) can be written in the form

\[
G^{(i)}_n = a_i \gamma_i^n + p^{(i)}_2(n) \left(\alpha^{(i)}_2\right)^n + \cdots + p^{(i)}_{t_i}(n) \left(\alpha^{(i)}_{t_i}\right)^n \quad (n \geq 0),
\]
where $a_i \neq 0$ are fixed numbers and $p_j^{(i)} (j = 1, 2, \ldots, t_i)$ are polynomials of

\[ Q(\gamma_i, \alpha_2^{(i)}, \ldots, \alpha_{i_i}^{(i)})[x] \]

(see e.g. [8]).

A. Pethő [4,5,6], T. N. Shorey and C. L. Stewart [7] showed that a sequence $G(= G^{(i)})$ does not contain $q$-th powers if $q$ is large enough. Similar result was obtained by P. Kiss in [2]. In [3] we investigated the equation

\[ G_x H_y = w^q \]

where $G$ and $H$ are linear recurrences satisfying some conditions, and showed that if $x$ and $y$ are not too far from each other then $q$ is (effectively computable) upper bounded: $q < q_0$.

2. Theorem

Now we shall investigate the generalization of equation (3). Let $d \in \mathbb{Z}$ be a fixed non-zero rational integer, and let $p_1, \ldots, p_t$ be given rational primes. Denote by $S$ the set of all rational integers composed of $p_1, \ldots, p_t$:

\[ S = \{ s \in \mathbb{Z} : s = \pm p_1^{e_1} \cdots p_t^{e_t}, e_i \in \mathbb{N} \} . \]

In particular $1 \in S (e_1 = \cdots = e_t = 0)$. Let

\[ G(x_1, \ldots, x_\nu) = G^{(1)}_{x_1} \cdots G^{(\nu)}_{x_\nu} \]

be a function defined on the set $\mathbb{N}^\nu$. By the definitions of the sequences $G^{(i)}$’s $G$ takes integer values. With a given $d$ let us consider the equation

\[ dG(x_1, \ldots, x_\nu) = sw^q \]

in positive integers $w > 1$, $q$, $x_i$ ($i = 1, 2, \ldots, \nu$) and $s \in S$. We will show under some conditions for $G$ that $q < q_0$ is also fulfilled if $q$ satisfies the equation above. Exactly, using the Baker-method, we will prove the following

**Theorem.** Let $G(x_1, \ldots, x_\nu)$ be the function defined in (5). Further let $0 \neq d \in \mathbb{Z}$ be a fixed integer, and let $\delta$ be a real number with $0 < \delta < 1$. Assume that $G(x_1, \ldots, x_\nu) \neq \prod_{i=1}^{\nu} a_i^{x_i}$ if $x_i > n_0$ ($i = 1, 2, \ldots, \nu$). Then the equation

\[ dG(x_1, \ldots, x_\nu) = sw^q \]
in positive integers \( w > 1, q, x_1, \ldots, x_\nu \) and \( s \in S \) for which \( x_j > \delta \max_i \{x_i\} \) \((j = 1, 2, \ldots, \nu)\), implies that \( q < q_0 \), where \( q_0 \) is an effectively computable number depending on \( n_0, \delta, G^{(1)}, \ldots, G^{(\nu)} \).

### 3. Lemmas

In the proof of our Theorem we need a result due to A. Baker \([1]\).

**Lemma 1.** Let \( \pi_1, \pi_2, \ldots, \pi_r \) be non-zero algebraic numbers of heights not exceeding \( M_1, M_2, \ldots, M_r \) respectively \((M_r \geq 4)\). Further let \( b_1, b_2, \ldots, b_{r-1} \) be rational integers with absolute values at most \( B \) and let \( b_r \) be a non-zero rational integer with absolute value at most \( B' \) \((B' \geq 3)\). Suppose, that \( \sum_{i=1}^r b_i \log \pi_i \neq 0 \). Then there exists an effectively computable constant \( C = C(r, M_1, \ldots, M_{r-1}, \pi_1, \ldots, \pi_r) \) such that

\[
\left| \sum_{i=1}^r b_i \log \pi_i \right| > e^{-C(\log M_r \log B' + \frac{B}{M_r})},
\]

where logarithms have their principal values.

We need the following auxiliary result.

**Lemma 2.** Let \( c_1, \ldots, c_k \) be positive real numbers and \( 0 < \delta < 1 \) be an arbitrary real number. Further let \( x_1, \ldots, x_k \) be natural numbers with maximum value \( x_m = \max_i \{x_i\} \) \((m \in \{1, \ldots, k\})\). If \( x_j > \delta x_m \) \((j = 1, \ldots, k)\) and \( x_m > x_0 \) then there exists a real number \( c > 0 \), which depends on \( k, \delta, \max_i \{c_i\} \) and \( x_0 \), for which

\[
\sum_{i=1}^k e^{-c_i x_i} < e^{-c(x_1 + \cdots + x_k)} = e^{-cx},
\]

where \( x = x_1 + \cdots + x_k \).

**Proof of Lemma 2.** Using the conditions of the lemma we have

\[
\sum_{i=1}^k e^{-c_i x_i} < \sum_{i=1}^k e^{-c_i \delta x_m} = \sum_{i=1}^k e^{-d_i x_m},
\]

where \( d_i = \delta c_i \). If \( d_m = \min_i \{d_i\} \) then

\[
\sum_{i=1}^k e^{-d_i x_m} \leq ke^{-d_m x_m} = e^{\log k - d_m x_m}.
\]
Since $x_m \geq x_0$, it follows that
\[
e^{\log k - d_m x_m} \leq e^{-d_m^* x_m} = e^{-ck x_m} \leq e^{-cx}
\]
with a suitable constant $d_m^*$ and $c = \frac{d_m^*}{k}$.

4. Proof of the Theorem

By $c_1, c_2, \ldots$ we denote positive real numbers which are effectively computable. We may assert, without loss of generality, that the terms of the recurrences $G^{(i)}$ are positive, $d > 0$, $s > 0$ and the inequality
\[
|\gamma_1| \geq |\gamma_2| \geq \cdots \geq |\gamma_\nu|
\]
also holds.

Let us observe that it is sufficient to consider the case $x_i > n_0$ ($i = 1, 2, \ldots, \nu$). Otherwise, if we suppose that some $x_j \leq n_0$ ($j \in \{1, 2, \ldots, \nu\}$) then $x_m = \max_i \{x_i\}$ cannot be arbitrary large because of the assertion $x_j > \delta x_m$. It means that we have finitely many possibilities to choose the $\nu$-tuples $(x_1, \ldots, x_\nu)$, and the range of $G(x_1, \ldots, x_\nu)$ is finite. So with a fixed $d$, if inequality (6) is satisfied then $q$ must be bounded.

In the sequel we suppose that $x_i > n_0$ ($i = 1, 2, \ldots, \nu$). Let $x_1, \ldots, x_\nu$, $w$, $q$ and $s \in S$ be integers satisfying (6). We may assume that if
\[
s = p_1^{e_1} \cdots p_t^{e_t}
\]
then $e_j < q$, else a part of $s$ can be joined to $w^q$. Using (2), from (6) we have
\[
sw^q = d \prod_{i=1}^\nu a_i (\gamma_i)^{x_i} \left(1 + \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i}\right)^{x_i} + \cdots\right).
\]
A consequence of the assumptions $|\gamma_i| > |\alpha_j^{(i)}|$ ($1 < j \leq t_i$) is that
\[
\left(1 + \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i}\right)^{x_i} + \cdots\right) \longrightarrow 1 \quad \text{whenever} \quad x_i \longrightarrow \infty.
\]
Hence there exist real constants $0 < \varepsilon_1, \ldots, \varepsilon_\nu < 1$ such that
\[
d \prod_{i=1}^\nu |a_i| \gamma_i |x_i| (1 - \varepsilon_i) < sw^q < d \prod_{i=1}^\nu |a_i| \gamma_i |x_i| (1 + \varepsilon_i),
\]
and
\[ c_1 \prod_{i=1}^{\nu} |\gamma_i|^{x_i} < sw^q < c_2 \prod_{i=1}^{\nu} |\gamma_i|^{x_i}. \]

As before, let \( x = x_1 + \cdots + x_\nu \) and applying (9) we may write
\[ \log c_1 + x \log |\gamma_\nu| < \log s + q \log w < \log c_2 + x \log |\gamma_1|. \]

Since \( \log s \geq 0 \), we have
\[ \log c_3 + x \log |\gamma_\nu| < q \log w < \log c_2 + x \log |\gamma_1| \]
with \( c_3 = \frac{c_1}{s} \). From (13) it follows that
\[ c_4 \frac{x}{q} < \log w < c_5 \frac{x}{q} \]
with some positive constants \( c_4, c_5 \). Ordering the equality (11) and taking logarithms, by the definition of \( \varepsilon_i \) we obtain
\[ Q = \left| \log \frac{sw^q}{d \prod_{i=1}^{\nu} |a_i| |\gamma_i|^{x_i}} \right| = \left| \log \prod_{i=1}^{\nu} \frac{1 + \frac{p_2^{(i)}(x_i)}{a_i}}{\left( \frac{\alpha_2^{(i)}}{\gamma_i} \right)^{x_i}} + \cdots \right| < \]
\[ < \sum_{i=1}^{\nu} \log |1 + \varepsilon_i| \leq \sum_{i=1}^{\nu} e^{-c_i^{*}x_i}, \]
where \( Q \neq 0 \) if we assume, that \( x_i > n_0 \) for every \( i = 1, 2, \ldots, \nu \), and \( c_i^{*} \) is a suitable positive constant \( (i = 1, 2, \ldots, \nu) \). Applying Lemma 2 and using the notation \( x = x_1 + \cdots + x_\nu \), it yields that
\[ Q < e^{-c_6(x_1+\cdots+x_\nu)} = e^{-c_6 x}. \]

On the other hand
\[ Q = \left| \log s + q \log w \log d - \log \prod_{i=1}^{\nu} |a_i| - x_1 \log |\gamma_1| - \cdots - x_\nu \log |\gamma_\nu| \right|, \]
where \( \log s = e_1 \log p_1 + \cdots + e_t \log p_t \) (see (10)). Now we may use Lemma 1 with \( \pi_r = w = M_r \), since the ordinary heights of \( p_j \) \( (j = 1, 2, \ldots, t) \), \( d \), \( \prod_{i=1}^{\nu} |a_i| \) and \( |\gamma_i| \) \( (i = 1, 2, \ldots, \nu) \) are constants. So \( B' = q \). In comparison
the absolute values of the integer coefficients of the logarithms in (16), we can choose $B$ as $B = x$. So by (16) and Lemma 1 it follows that

\begin{equation}
Q > e^{-c_7 \left( \log w \log q + \frac{x}{q} \right)}.
\end{equation}

Combining (15) and (17) it yields the following inequality:

\begin{equation}
c_6 x < c_7 \left( \log w \log q + \frac{x}{q} \right),
\end{equation}

and by (14) it follows that

\begin{equation}
c_6 x < c_7 \left( \log w \log q + \frac{1}{c_4} \log w \right) < c_8 \log w \log q
\end{equation}

with some $c_8 > 0$. Applying (14) again, we conclude that $\frac{1}{c_5} q \log w < x$ and so by (19)

\begin{equation}
c_9 q < \log q
\end{equation}

follows. But (20) implies that $q < q_0$, which proves the theorem.

References


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