Pure powers in recurrence sequences

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Abstract. Let $G$ be a linear recursive sequence of order $k$ satisfying the recursion $G_n = A_1 G_{n-1} + \cdots + A_k G_{n-k}$. In the case $k=2$ it is known that there are only finitely many perfect powers in such a sequence.

Ribenboim and McDaniel proved for sequences with $k=2$, $G_0=0$ and $G_1=1$ that in general for a term $G_n$ there are only finitely many terms $G_m$ such that $G_n G_m$ is a perfect square. P. Kiss proved that for any $n$ there exists a number $q_0$, depending on $G$ and $n$, such that the equation $G_n G_x = w^q$ in positive integers $x, w, q$ has no solution with $x>n$ and $q>q_0$. We show that for any $n$ there are only finitely many $x_1, x_2, \ldots, x_k, x, w, q$ positive integers such that $G_n G_{x_1} \cdots G_{x_k} G_x = w^q$ and some conditions hold.

Let $R = R(A, B, R_0, R_1)$ be a second order linear recursive sequence defined by

$$R_n = A R_{n-1} + B R_{n-2} \quad (n > 1),$$

where $A$, $B$, $R_0$ and $R_1$ are fixed rational integers. In the sequel we assume that the sequence is not a degenerate one, i.e. $\alpha/\beta$ is not a root of unity, where $\alpha$ and $\beta$ denote the roots of the polynomial $x^2 - Ax - B$.

The special cases $R(1, 1, 0, 1)$ and $R(2, 1, 0, 1)$ of the sequence $R$ is called Fibonacci and Pell sequence, respectively.

Many results are known about relationship of the sequences $R$ and perfect powers. For the Fibonacci sequence Cohn [2] and Wylie [23] showed that a Fibonacci number $F_n$ is a square only when $n = 0, 1, 2$ or 12. Pethő [12], furthermore London and Finkelstein [9,10] proved that $F_n$ is full cube only if $n = 0, 1, 2$ or 6. From a result of Ljunggren [8] it follows that a Pell number is a square only if $n = 0, 1$ or 7 and Pethő [12] showed that these are the only perfect powers in the Pell sequence. Similar, but more general results was showed by McDaniel and Ribenboim [11], Robbins [19,20] Cohn [3,4,5] and Pethő [15]. Shorey and Stewart [21] showed, that any non degenerate binary recurrence sequence contains only finitely many perfect powers which can be effectively determined. This results follows also from a result of Pethő [14].

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Another type of problems was studied by Ribenboim and McDaniel. For a sequence $R$ we say that the terms $R_m, R_n$ are in the same square-class if there exist non zero integers $x, y$ such that

$$R_m x^2 = R_n y^2,$$

or equivalently

$$R_m R_n = t^2,$$

where $t$ is a positive rational integer.

A square-class is called trivial if it contains only one element. Ribenboim [16] proved that in the Fibonacci sequence the square-class of a Fibonacci number $F_m$ is trivial, if $m \neq 1, 2, 3, 6$ or 12 and for the Lucas sequence $L(1, 1, 2, 1)$ the square-class of a Lucas number $L_m$ is trivial if $m \neq 0, 1, 3$ or 6. For more general sequences $R(A, B, 0, 1)$, with $(A, B) = 1$, Ribenboim and McDaniel [17] obtained that each square class is finite and its elements can be effectively computed (see also Ribenboim [18]).

Further on we shall study more general recursive sequences.

Let $G = G(A_1, \ldots, A_k, G_0, \ldots, G_{k-1})$ be a $k^{th}$ order linear recursive sequence of rational integers defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \quad (n > k - 1),$$

where $A_1, \ldots, A_k$ and $G_0, \ldots, G_{k-1}$ are not all zero integers. Denote by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ the distinct zeros of the polynomial $x^k - A_1 x^{k-1} - A_2 x^{k-2} - \cdots - A_k$. Assume that $\alpha, \alpha_2, \ldots, \alpha_s$ has multiplicity 1, $m_2, \ldots, m_s$ respectively and $|\alpha| > |\alpha_i|$ for $i = 2, \ldots, s$. In this case, as it is known, the terms of the sequence can be written in the form

$$(1) \quad G_n = a \alpha^n + r_2(n) \alpha_2^n + \cdots + r_s(n) \alpha_s^n \quad (n \geq 0),$$

where $r_i(i = 2, \ldots, s)$ are polynomials of degree $m_i - 1$ and the coefficients of the polynomials and $a$ are elements of the algebraic number field $Q(\alpha, \alpha_2, \ldots, \alpha_s)$. Shorey and Stewart [21] proved that the sequence $G$ does not contain $q^{th}$ powers if $q$ is large enough. This result follows also from [7] and [22], where more general theorems where showed.

Kiss [6] generalized the square-class notion of Ribenboim and McDaniel. For a sequence $G$ we say that the terms $G_m$ and $G_n$ are in the same $q^{th}$ power class if $G_m G_n = w^q$, where $w, q$ rational integers and $q \geq 2$.

In the above mentioned paper Kiss proved that for any term $G_n$ of the sequence $G$ there is no terms $G_m$ such that $m > n$ and $G_n, G_m$ are elements of the same $q^{th}$-power class if $q$ sufficiently large.
The purpose of this paper to generalize this result. We show that the under certain conditions the number of the solutions of equation

$$G_nG_{x_1}G_{x_2} \cdots G_{x_k}G_x = w^q$$

where $n$ is fixed, are finite.

We use a well known result of Baker [1].

**Lemma.** Let $\gamma_1, \ldots, \gamma_v$ be non-zero algebraic numbers. Let $M_1, \ldots, M_v$ be upper bounds for the heights of $\gamma_1, \ldots, \gamma_v$, respectively. We assume that $M_v$ is at least 4. Further let $b_1, \ldots, b_{v-1}$ be rational integers with absolute values at most $B$ and let $b_v$ be a non-zero rational integer with absolute value at most $B'$. We assume that $B'$ is at least three. Let $L$ defined by

$$L = b_1 \log \gamma_1 + \cdots + b_v \log \gamma_v,$$

where the logarithms are assumed to have their principal values. If $L \neq 0$, then

$$|L| > \exp(-C(\log B' \log M_v + B/B')),$$

where $C$ is an effectively computable positive number depending on only the numbers $M_1, \ldots, M_{v-1}, \gamma_1, \ldots, \gamma_v$ and $v$ (see Theorem 1 of [1] with $\delta = 1/B'$).

**Theorem.** Let $G$ be a $k^{\text{th}}$ order linear recursive sequence satisfying the above conditions. Assume that $a \neq 0$ and $G_i \neq a\alpha^i$ for $i > n_0$. Then for any positive integer $n, k$ and $K$ there exists a number $q_0$, depending on $n, G, K$ and $k$, such that the equation

$$(2) \quad G_nG_{x_1}G_{x_2} \cdots G_{x_k}G_x = w^q \quad (n \leq x_1 \leq \cdots \leq x_k < x)$$

in positive integer $x_1, x_2, \ldots, x_k, x, w, q$ has no solution with $x_k < Kn$ and $q > q_0$.

**Proof of the theorem.** We can assume, without loss of generality, that the terms of the sequence $G$ are positive. We can also suppose that $n > n_0$ and $n$ sufficiently large since otherwise our result follows from [20] and [7].

Let $x_1, x_2, \ldots, x_k, x, w, q$ positive integers satisfying (2) with the above conditions. Let $\varepsilon_m$ be defined by

$$\varepsilon_m := \frac{1}{a}r_2(m)\left(\frac{\alpha_2}{\alpha}\right)^m + \frac{1}{a}r_3(m)\left(\frac{\alpha_3}{\alpha}\right)^m + \cdots + \frac{1}{a}r_s(m)\left(\frac{\alpha_s}{\alpha}\right)^m \quad (m \geq 0).$$
By (1) we have

\[(1 + \varepsilon_n) (1 + \varepsilon_x) \prod_{i=1}^{k} (1 + \varepsilon_{x_i}) a^{k+2} \alpha^{n+x+x_1+\cdots+x_k} = w^q\]

from which

\[q \log w = (k + 2) \log a + \left( n + x + \sum_{i=1}^{k} x_i \right) \log \alpha + \log (1 + \varepsilon_n) \]

\[+ \log (1 + \varepsilon_x) + \sum_{i=1}^{k} \log (1 + \varepsilon_{x_i}) \]

(3)

follows. It is obvious that \(x < n + x + \sum_{i=1}^{k} x_i < (k + 2)x\). Using that \(\log |1 + \varepsilon_m|\) is bounded and \(\lim_{m \to \infty} \frac{1}{\alpha^i} (m) (\frac{\alpha_i}{\alpha})^m = 0\) \((i = 2, \ldots, s)\), we have

\[c_1 \frac{x}{q} < \log w < c_2 \frac{x}{q}\]

where \(c_1\) and \(c_2\) are constants.

Let \(L\) be defined by

\[L := \log \frac{w^q}{G_n G_{x_1} G_{x_2} \cdots G_{x_k} a \alpha^x} = |\log (1 + \varepsilon_x)|.\]

By the definition of \(\varepsilon_x\) and the properties of logarithm function there exists a constant \(c_3\) that

\[L < e^{-c_3 x}.\]

(5)

On the other hand, by the Lemma with \(v = k + 4, M_{k+4} = w, B' = q\) and \(B = x\) we obtain the estimation

\[L = \left| q \log w - \log G_n - \sum_{i=1}^{k} \log G_{x_i} - \log a - x \log \alpha \right| > e^{-C(\log q \log w + x/q)}\]

(6)

where \(C\) depends on heights. By \(x_k < Kn\) heights depend on \(G_n, \ldots, G_{Kn}\), i.e. on \(n, K, k\) and on the parameters of the recurrence. By (4), (5) and (6) we have \(c_3 x < C(\log q \log w + x/q) < c_4 \log q \log w\), i.e.

\[x < c_5 \log q \log w\]

(7)
with some $c_3, c_4, c_5$. Using (4) and (7) we get $c_6 q \log w < x < c_5 \log q \log w$, i.e. $q < c_7 \log q$, where $c_6$ and $c_7$ are constants. But this inequality does not hold if $q > q_0 = q_0(G, n, K, k)$, which proves the theorem.

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